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NEWTONIAN MECHANICS ON FRACTALS SUBSET OF REAL-LINE

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Abstract. In this paper, we have studied the calculus on the fractals, meanwhile Newtonian mechanics on fractals subset of real-line has been suggested. Further, work and energy theorem on fractals with the examples has been explained. Finally Langevin F^{α}-Equation on fractals is derived.

Key words: Calculus on fractal, dynamics, staircase function, Cantor set, harmonic oscillator .

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1. INTRODUCTION

The fractals can be pattern of many natural phenomena. Fractal geometry play part in some of the branches of science, engineering and art. Fractals are useful for structures that ordinary calculus can't apply. For example: Lebesgue-Cantor staircase function is zero almost everywhere, therefore this function isn't a solution of an ordinary differential equations. Consequently ordinary calculus can't apply in dynamics on fractals, fields of factually distributed sources and *etc.* Fractional derivatives are nonlocal so is suitable for fractal functions. Several authors have recognized the need to use fractional derivatives and integrals to explore the characteristic features of fractal walks, anomalous diffusion, transport, etc [1–39].

During recent time analysis on fractal developed, and applied in many important cases as heat conduction, fractal-time diffusion equation, waves, *etc.* on fractals- see [28, 32, 33] and several references therein. There is a greater development using a measure-theoretical method [30, 31]. It consist of defining derivative as a inverse of integral with respect to a measure and defining other operators using the derivative. All these cases raise our understanding a simple method of fractional order operators on fractal sets is only a moderately survey.

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Riemann integration like procedures have their own place. They are better and more useful algorithmic method [40]. In this paper we have used this frame work to generalized Newtonian mechanics on fractal subset real line. So we can modelled motion of particle undergoing fraction in a fractal medium.

The organization of the paper is as follows: We begin in section 2 by reviewing the F^{α}-calculus. In section 3 Newtonian mechanics on fractal subset of real line has been introduced. In section 4 work and energy theorem on fractals subset of real line has been discussed. Section 5 we have presented Langevin F^{α}-Equation On fractals subset real line. Section 6 is dedicated to our conclusions.

2. SUMMARY OF FRACTIONAL F^{α}-CALCULUS

We begin by defining the integral staircase function [40].

2.1. THE MASS FUNCTION AND THE INTEGRAL STAIRCASE

Definition 1: F be a subset of real line ($\Re$). Let F be in the most cases a fractal. The flag function for a set F is denoted by $\theta(F,I)$ and define as [40].

$$\theta(F,I) = \begin{cases} 1 & \text{if} \quad F \cap I \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

where $I = [a,b]$ is a interval in $\Re$.

Definition 2: For a set F and a subdivision $P_[a,b], a < b$

$$\sigma^{\alpha}[F,I] = \sum_{i=1}^{n} \frac{(x_i - x_{i-1})^{\alpha}}{\Gamma(\alpha+1)} \, \theta(F,[x_{i-1},x_i]), \tag{2}$$

where $a < b$ and $0 < \alpha \leq 1$.

Definition 3: Given $\delta > 0$ and $a \leq b$ the coarse-grained mass $\gamma_{\delta}^{\alpha}(F,a,b)$ of $F \cap [a,b]$ is given by

$$\gamma_{\delta}^{\alpha}(F,a,b) = \inf_{P_{[a,b]}:|P|\leq\delta} \sigma^{\alpha}[F,I], \tag{3}$$

where $|P| = \max_{1\leq i\leq n}(x_i - x_{i-1})$. Taking infimum over all subdivisions P of $[a,b]$ satisfying $|P| \leq \delta$.

Definition 4: The mass function $\gamma^{\alpha}(F,a,b)$ is given by [40]

$$\gamma^{\alpha}(F,a,b) = \lim_{\delta\to 0} \gamma_{\delta}^{\alpha}(F,a,b) \tag{4}$$

Definition 5: Let a_0 be an arbitrary but fixed real number. The integral staircase function $S_F^\alpha(x)$ of order α for a set F is given by [40]

$$S_F^\alpha(x) = \begin{cases} \gamma^\alpha(F,a_0,x) & \text{if} \quad x \geq a_0 \\ -\gamma^\alpha(F,a_0,x) & \text{otherwise} \end{cases} \tag{5}$$

Definition 6: We say that a point x is a point of change of a function f if f is not constant over any open interval (a,d) containing x. The set of all points of change of f is called the set of change of f and is denoted by $\mathrm{Sch}\, f$ [40].

Definition 7: The γ-dimension of $F \cap [a,b]$ denoted by $\dim_\gamma(F \cap [a,b])$ is

$$\begin{aligned} \dim_\gamma(F \cap [a,b]) &= \inf\{\alpha : \gamma^\alpha(F,a,b) = 0\} \\ &= \sup\{\alpha : \gamma^\alpha(F,a,b) = \infty\} \end{aligned} \tag{6}$$

Definition 8: Let $F \subset R$ be such that $S_F^\alpha(x)$ is finite for all $x \in R$ for $\alpha = \dim_\gamma F$. Then the $\mathrm{Sch}(S_F^\alpha)$ is said to be α-perfect (Closed and every point of $\mathrm{Sch}(S_F^\alpha)$ is its limit point).

2.2. F-LIMIT AND F- CONTINUITY

Definition 9: Let $F \subset R$, $f : R \to R$ and $x \in F$. A number l is said to be the limit of f through the points of F or simply F-limit of f as $y \to x$ if given any $\epsilon > 0$ there exists $\delta > 0$ such that [40]

$$y \in F \quad \text{and} \quad |y - x| < \delta \Rightarrow |f(y) - l| < \epsilon \tag{7}$$

If such a number exists then it is denoted by

$$l = F - \lim_{y \to x} f(y) \tag{8}$$

Definition 10: A function $f : R \Rightarrow R$ is said to be F-continues at $x \in F$ if

$$f(x) = F - \lim_{y \to x} f(x) \tag{9}$$

2.3. F^α-INTEGRATION

Definition 11: Let f be bounded function on F and I be a closed interval [40]. Then

$$\begin{aligned} M[f,F,I] &= \sup_{x \in F \cap I} f(x) \quad \text{if} \ \ F \cap I \neq \emptyset \\ &= 0 \quad \text{otherwise} \end{aligned} \tag{10}$$

and similarly

$$\begin{aligned} m[f,F,I] &= \inf_{x \in F \cap I} f(x) \quad \text{if} \ \ F \cap I \neq \emptyset \\ &= 0 \quad \text{otherwise} \end{aligned} \tag{11}$$

Definition 12: Let $S_F^\alpha(x)$ be finite for $x \in [a,b]$. Let P be a subdivision of $[a,b]$ with points $x_0,...,x_n$ The upper F^α-sum and the lower F^α-sum for function f over the subdivision P are given respectively by [40]

$$U^\alpha[f,F,P] = \sum_{i=1}^{n} M[f,F,[x_i,x_{i-1}]](S_F^\alpha(x_i,x_{i-1})) \tag{12}$$

and

$$L^\alpha[f,F,P] = \sum_{i=1}^{n} m[f,F,[x_i,x_{i-1}]](S_F^\alpha(x_i,x_{i-1})) \tag{13}$$

Definition 13: If f be a bounded function on F. we say that f is F^α-integrable on $[a,b]$ if [40]

$$\underline{\int_a^b} f(x) d_F^\alpha x = \sup_{P_{[a,b]}} L^\alpha[f,F,P] = \overline{\int_a^b} f(x) d_F^\alpha x = \inf_{P_{[a,b]}} U^\alpha[f,F,P] \tag{14}$$

In that case the F^α-integral of f on $[a,b]$ denoted by $\int_a^b f(x) d_F^\alpha x$ is given by the common value [40].

Properties [40]:

$$\int_a^b f(x) d_F^\alpha x = \int_a^c f(x) d_F^\alpha x + \int_c^b f(x) d_F^\alpha x \tag{15}$$

$$\int_a^b \lambda f(x) d_F^\alpha x = \lambda \int_a^b f(x) d_F^\alpha x, \quad \text{where } \lambda \text{ is constant} \tag{16}$$

$$\int_a^b (f(x)+g(x)) d_F^\alpha x = \int_a^b f(x) d_F^\alpha x + \int_a^b g(x) d_F^\alpha x \tag{17}$$

$$\int_a^b f(x)\ d_F^\alpha x \geq \int_a^b g(x) d_F^\alpha x \quad \text{if } f(x) \geq g(x) \tag{18}$$

$$\int_b^a f(x)\ d_F^\alpha x = -\int_a^b f(x)\ d_F^\alpha x \tag{19}$$

2.4. F^α-DIFFERENTIATION

Definition 14: If F is an α-perfect set then the F^α-derivative of f at x is [40]

$$D_F^\alpha f(x) = \begin{cases} F - \lim\limits_{y \to x} \dfrac{f(y)-f(x)}{S_F^\alpha(y)-S_F^\alpha(x)} & \text{if} \quad x \in F \\ 0 & \text{otherwise} \end{cases} \tag{20}$$

if the limit exists.

Properties:

$$D_F^{\alpha}\lambda f(x) = \lambda D_F^{\alpha} f(x)$$
$$D_F^{\alpha}(f(x)+g(x)) = D_F^{\alpha} f(x) + D_F^{\alpha} g(x)$$
$$D_F^{\alpha} c = 0 \qquad \text{if } c \text{ is constant}$$
$$D_F^{\alpha}(u(x)v(x)) = (D_F^{\alpha} u(x))v(x) + u(x)(D_F^{\alpha} v(x)).$$

2.5. FUNDAMENTAL THEOREM OF F^{α}-CALCULUS

Theorem 1: Let $F \subset R$ be an α-perfect set. If f is bounded on F is an F-continuous function on $F \cap [a,b]$ and [40]

$$g(x) = \int_a^x f(y) d_F^{\alpha} y \tag{21}$$

for all $x \in [a,b]$ then

$$D_F^{\alpha} g(x) = f(x)\chi_F(x) \tag{22}$$

Theorem 2: Let $f : R \to R$ be a continuous F^{α}-differentiable function such that $\mathrm{Sch}(f)$ is contained in an α-perfect set F and $h : R \to R$ be F-continuous such that

$$D_F^{\alpha} f(x) = h(x)\chi_F(x) \tag{23}$$

then

$$\int_a^b h(x) d_F^{\alpha} x = f(b) - f(a) \tag{24}$$

Theorem 3: Let the functions $u : R \to R$ $v : R \to R$ be such that
1.$u(x)$ is continuous on $[a,b]$ and $\mathrm{Sch}(u) \subset F$
2.$D_F^{\alpha} u(x)$ exists and is F-continuous on $[a,b]$
3.$v(x)$ is F-continuous on $[a,b]$ [40].
Then

$$\int_a^b u(x)v(x) d_F^{\alpha} x = [u(x)\int_a^x v(x') d_F^{\alpha} x']_a^b - \int_a^b D_F^{\alpha} u(x) \int_a^x v(x') d_F^{\alpha} d_F^{\alpha} x. \tag{25}$$

Remarks [40]:

$$\int_0^y (S_F^{\alpha}(x))^n d_F^{\alpha} x = \frac{1}{n+1}(S_F^{\alpha}(y))^{n+1}$$
$$D_F^{\alpha}(S_F^{\alpha}(x))^n = n(S_F^{\alpha}(x))^{n-1}\chi_F(x)$$

3. NEWTONIAN MECHANICS ON FRACTALS

3.1. KINEMATICS OF MOTION

Generalized average velocity: Suppose $x_F^{\alpha}(t)$ is position function of a particle. Such that $\mathrm{Sch}(x_F^{\alpha}(t))$is α-perfect set then we can define the generalized average velocity between $[t_1, t_2]$ as follows:

$$\overline{v_F^\alpha}(t) = \frac{x_F^\alpha(t_2) - x_F^\alpha(t_1)}{S_F^\alpha(t_2) - S_F^\alpha(t_1)} \tag{26}$$

Generalized velocity: If $x_F^\alpha(t)$ is position function of a particle. Let $\mathrm{Sch}(x_F^\alpha(t))$ is α-perfect set. The velocity of particle on a fractal set F is defined

$$v_F^\alpha(t) = D_F^\alpha x(t) \tag{27}$$

Generalized average acceleration: Let $x_F^\alpha(t)$ be velocity function of a particle. Therefore, we define average acceleration of a particle as follows

$$\overline{a_F^\alpha}(t) = \frac{v_F^\alpha(t_2) - v_F^\alpha(t_1)}{S_F^\alpha(t_2) - S_F^\alpha(t_1)} \tag{28}$$

Here $\mathrm{Sch}(v_F^\alpha(t))$ is α-perfect set.
Generalized acceleration: Consider $x_F^\alpha(t)$ as position function of a particle. So we can define the generalized acceleration as

$$a_F^\alpha(t) = (D_F^\alpha)^2 x_F^\alpha(t) = D_F^\alpha v_F^\alpha(t) \tag{29}$$

Where $\mathrm{Sch}(x_F^\alpha(t)),\ v_F^\alpha(t)$ are α-perfect set.

Example 1. Suppose a particle is moving with constant acceleration such $k\chi_F$. Then the position and velocity function of particle can be derived as follows

$$a_F^\alpha(t) = D_F^\alpha v_F^\alpha(t) = k\chi_F \tag{30}$$

then by applying F^α-integration in both side we have

$$v_F^\alpha(t) = v_F^\alpha(t_1) + k\chi_F(S_F^\alpha(t) - S_F^\alpha(t_1)) \tag{31}$$

likewise using F^α-integration in both side we obtain

$$x_F^\alpha(t) = k\frac{\chi_F}{2}(S_F^\alpha(t) - S_F^\alpha(t_1))^2 + x_F^\alpha(t_1) + \chi_F\ v_F^\alpha(t_1)\ (S_F^\alpha(t) - S_F^\alpha(t_1)) \tag{32}$$

3.2. DYNAMICS OF MOTION

Generalized momentum: If $x_F^\alpha(t)$ is position function of particle such that $\mathrm{Sch}(x_F^\alpha(t))$ is α-perfect set. Let $v_F^\alpha(t)$

$$p_F^\alpha = m v_F^\alpha \tag{33}$$

Generalized Newton's second law: Suppose a particle with mass m whose position function $x_F^\alpha(t)$ is such that $\mathrm{Sch}(x_F^\alpha(t)$ is α-perfect set. Then generalized Newton law is

$$f_F^\alpha = m\ a_F^\alpha(t) \tag{34}$$

or

$$f_F^\alpha = D_F^\alpha \, p_F^\alpha \tag{35}$$

Example 2. Suppose the force applied to a particle is $f_F^\alpha = (S_F^\alpha(t))^2$ then equation of the motion as follows:

$$a_F^\alpha(t) = \frac{1}{m}(S_F^\alpha(t))^2 \tag{36}$$

then velocity will be

$$v_F^\alpha(t) = v_F^\alpha(t_1) + \frac{1}{3m}(S_F^\alpha(t) - S_F^\alpha(t_1))^3 \tag{37}$$

position function

$$x_F^\alpha(t) = x_F^\alpha(t_1) + v_F^\alpha(t_1)\,(S_F^\alpha(t) - S_F^\alpha(t_1)) + \frac{1}{12m}((S_F^\alpha(t) - S_F^\alpha(t_1))^4 \tag{38}$$

4. WORK AND ENERGY THEOREM ON FRACTALS

Generalized kinetic energy: If $x_F^\alpha(t)$ is position function of particle such that $\mathrm{Sch}(x_F^\alpha(t))$ is α-perfect set. Let $v_F^\alpha(t)$ be velocity of particle

$$k_F^\alpha = \frac{1}{2} m \, (v_F^\alpha)^2 \tag{39}$$

Generalized potential energy: Let f_F^α be generalized conservative force that means F^α-integral of f_F^α in the closed curve be zero. Then we can derive the generalized potential u_F^α as follows

$$u_F^\alpha(t) = u_F^\alpha(a) - \chi_F \int_a^t f_F^\alpha d_F^\alpha x \tag{40}$$

or

$$f_F^\alpha = -D_F^\alpha \, u_F^\alpha(t) \tag{41}$$

Example 3. Let the conservative force $f_C^\alpha = x\chi_C(x)$ here C stand for Cantor set. then the corresponding potential will be

$$\begin{aligned} u_C^\alpha(t) &= u_C^\alpha(a) - \chi_C \int_a^t x d_C^\alpha x \\ &= u_C^\alpha(a) - \chi_C \sum_{n=1}^{\infty} I_n(t), \end{aligned} \tag{42}$$

where C stand for Cantor set [40].

$$I_n(t) = \begin{cases} 0 & \text{if} \quad y_n(t) = 0 \quad \text{or} \quad y_i(t) = 1 \quad \text{for some} \quad i < n \\ \frac{1}{\Gamma(\alpha+1)}\left[\frac{T_{n-1}(y)}{2^n} + \frac{1}{2.6^n}\right] & \text{otherwise} \end{cases} \tag{43}$$

An approximation of $y \in [0,1]$ by a finite number of digits is denoted by

$$T_0(t) = 0 \qquad \text{and} \qquad T_n(t) = \sum_{i=1}^{n} \frac{y_i(t)}{3^i} \tag{44}$$

Generalized work and energy theorem on fractals: If $x_F^\alpha(t)$ is position function of particle such that $\mathrm{Sch}(x_F^\alpha(t))$ is α-perfect set. Let $v_F^\alpha(t)$ be velocity of particle

$$\frac{1}{2}m\ (v_F^\alpha(t_2))^2 - \frac{1}{2}m\ (v_F^\alpha(t_1))^2 = \int_{t_1}^{t_2} f_F^\alpha d_F^\alpha x \tag{45}$$

Example 4. Suppose we have a particle having simple harmonic oscillator the conservation of energy is as follows:

$$\frac{1}{2}m\ (v_F^\alpha(t_2))^2 - \frac{1}{2}m\ (v_F^\alpha(t_1))^2 = \chi_C \sum_{n=1}^{\infty} I_n(t) \tag{46}$$

5. LANGEVIN F^α-EQUATION ON FRACTALS

The theory of Brownian motion is the simplest way to treat the dynamics of non-equilibrium system. The fundamental equation is called the Langevin equation, it contains both frictional forces and random forces. The fluctuation-dissipation theorem relates these forces to each other. While the motion of a dust particle performing Brownian motion appears to be quite random, it must nevertheless be describable by the same equation of motion as is any other dynamical system. The equations of motion for the Brownian particle are Langevinian equations. Langevinian equations on the fractals subset real line is introduced. Generalized Langevinian F^α-equation of a Brownian particle of mass m, with the surrounding medium represented by a generalized force $-\epsilon\chi_F v(t)$ and random density fluctuations in the fluid is

$$\frac{m}{2}(D_F^\alpha)^2 x(t) = -\epsilon\chi_F v(t) + \zeta\ \xi(t), \tag{47}$$

where $\xi(t)$ random force and ζ is constant.

6. CONCLUSION

In this work we have generalized Newtonian mechanics on the fractals subset of real line. Subsequently one can model one dimensional motion of a particle such that underling medium is a fractal set *e.g.* cloud.

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