Romanian Reports in Physics, Vol. 65, No. 1, P. 94–102, 2013

SOME FRACTIONAL COMPARISON RESULTS AND STABILITY THEOREM FOR FRACTIONAL TIME DELAY SYSTEMS

S.J. SADATI[a], R. GHADERI, A. RANJBAR N.*

Faculty of Electrical and Computer Engineering,
Babol University of Technology, Babol, P.O. Box: 47135-484, Iran
E-mail[a]: jalil.sadati@gmail.com

Received March 31, 2012

Abstract. In this paper, we have investigated that boundedness criteria and Lyapunov stability for fractional order time delay systems (fractional order differential-difference equations) in Caputo's sense are unified with Lyapunov-like functions to establish comparison result. The qualitative behaviour of a fractional order time-delay differential equation with Caputo's derivative has been studied. We present some new comparison results that again give the null solution a central role in the comparison fractional order differential systems with delay when establishing boundedness criteria and Lyapunov stability of these systems in Caputo's sense.

Key words: comparison theorem, time-delay systems, fractional calculus, fractional functional differential equation, stability.

PACS: 02.30.Ks, 02.30.Yy

1. INTRODUCTION

The concept of noninteger-order derivative, popularly known as fractional derivative, goes back to the 17th century [1–3]. It is only a few decades ago, it was realized that the derivatives of arbitrary order provide an excellent framework for modelling the real-world problems in a variety of disciplines from physics, chemistry, biology and engineering such as viscoelasticity and damping, diffusion and wave propagation, electromagnetism, chaos and fractals, heat transfer, electronics, signal processing, robotics, system identification, traffic systems, genetic algorithms, percolation, modelling and identification, telecommunications, irreversibility, control systems as well as economy, and finance [4–7, 13, 20, 21].

Lyapunov's second method has been employed with great success in a wide variety of investigations to understand qualitative and quantitative properties of dynamic systems for many years. Lyapunov's direct method is a standard technique used in the study of the qualitative behaviour of differential systems along with a comparison result [8, 9] that allows the prediction of behaviour of a differential sys-

*Corresponding author: A. Ranjbar N., a.ranjbar@nit.ac.ir, Tel: +98 911 214 3879.

tem when the behaviour of the null solution of a comparison system is known. The application of Lyapunov's direct method in boundedness theory [8] has the advantage of not requiring knowledge of solutions.

Since 1950s different types of the Lyapunov functions have been proposed for the stability analysis of delay systems, see the pioneering works of Razumikhin [10] and Krasovskii [11]. Whereas Razumikhin [10] used the Lyapunov-type functions $V(x(t))$ depending on the current value $x(t)$ of the solution, Krasovskii [11] proposed to use functionals $V(x_t)$ depending on the whole solution segment x_t, that is, the true state of the delay system. The reader can see [12] for more details.

In the base of Lyapunov's second method, some work has been done in the field of stability of fractional-order nonlinear systems without delay [9, 14–16].

But recently the stability of fractional-order nonlinear time-delay systems has been considered by some researchers [17–19, 22].

In this work, we have extended and generalized some comparison theorems and stability theorem for fractional functional differential equation with definition a new Lyapunov-like function by fractional-order Dini derivatives in Caputo's sense.

The organization of the manuscript is given below.
In Section 2 some basic definitions of fractional calculus used in this paper are mentioned. Section 3 introduces briefly the fractional nonlinear time-delay systems.In section 4 we develop fractional comparison principle by generalization of Lyapunov-like function for fractional nonlinear time-delay systems. In Section 5, a Lyapunov stability theorem is proved by using comparison results of section 4 . Finally, Section 6 is devoted to our conclusions.

2. FRACTIONAL CALCULUS

Caputo's and Riemann-Liouville's definitions of fractional derivatives, are namely,

$$^{c}D^{q}x = \frac{1}{\Gamma(1-q)}\int_{t_0}^{t}(t-s)^{-q}x'(s)\,\mathrm{d}s \tag{1}$$

$$D^{q}x = \frac{1}{\Gamma(1-q)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{t_0}^{t}(t-s)^{-q}x(s)\,\mathrm{d}s \tag{2}$$

respectively, order of $0 < q < 1$, where Γ denotes the Gamma function.

The most important advantage for fractional-order differential equations with Caputo's derivative is the initial conditions that are the same form as that of ordinary differential equations with integer derivatives. Another difference is that the Caputo derivative for a constant c is zero, while the Riemann-Liouville fractional derivative for a constant c is not zero but equals to $D^{q}c = c(t-t_0)^{-q}/\Gamma(1-q)$, By using (2.1), therefore,

$$
\begin{aligned}
{}^{c}D^{q}x(t) &= D^{q}\left[x(t)-x(t_0)\right], \\
&= D^{q}x(t)-\frac{x(t_0)}{\Gamma(1-q)}(t-t_0)^{-q}
\end{aligned} \tag{3}
$$

In particular, if $x(t_0)=0$, we obtain

$$
{}^{c}D^{q}x(t) = D^{q}x(t) \tag{4}
$$

Hence, we can see that Caputo's derivative is defined for functions for which Riemann-Liouville fractional-order derivative exists.

3. FRACTIONAL FUNCTIONAL DIFFERENTIAL EQUATIONS

Let $C([a,b],\mathbb{R}^n)$ be the set of continuous functions mapping the interval $[a,b]$ to $\mathbb{R}^n$. In many situations, one may wish to identify a maximum time delay τ of a system. In this case, we are often interested in the set of continuous function mapping $[-\tau,0]$ to $\mathbb{R}^n$, for which we simplify the notation to $\mathscr{C}=C([-\tau,0],\mathbb{R}^n)$. For any $A>0$ and any continuous function of time $\psi\in C([t_0-\tau,t_0+A],\mathbb{R}^n)$, $t_0\le t\le t_0+A$, let $x_t(\theta)\in\mathscr{C}$ be a segment of function x defined as $x_t(\theta)=x(t+\theta)$, $-\tau\le\theta\le 0$.

Consider Caputo fractional nonlinear time-delay system

$$
{}_{t_0}^{c}D_t^{q}x(t) = f(t,x_t). \tag{5}
$$

where $x(t)\in\mathbb{R}^n$, $0<q\le 1$ and $f:\mathbb{R}\times\mathscr{C}\to\mathbb{R}^n$. Equation (5) indicates Caputo derivatives of the state variable x on $[t_0,t]$ and $x(\xi)$ for $t-\tau\le\xi\le t$. As such, to determine the future evolution of the state, it is necessary to specify the initial state variables $x(t)$ in a time interval of length τ, say, from $t_0-\tau$ to t_0, *i.e.*,

$$
x_{t_0}=\phi, \tag{6}
$$

where $\phi\in\mathscr{C}$ is given. In other words we have $x(t_0+\theta)=\phi(\theta), -\tau\le\theta\le 0$. Throughout the manuscript we will use the Euclidean norm for vectors denoted by $\|\cdot\|$. The space of continuous initial functions $C([-\tau,0],\mathbb{R}^n)$is provided with the *supremum* norm

$$
\|\phi\|_0=\max_{\theta\in[-\tau,0]}\|\phi(\theta)\|. \tag{7}
$$

Let $\rho>0$ be a given constant, and let

$$
\mathscr{C}_\rho=\left\{\phi\in\mathscr{C}:\|\phi\|_0<\rho\right\} \tag{8}
$$

and

$$
S_\rho=\left\{x\in\mathbb{R}^n:\|x\|\le\rho\right\} \tag{9}
$$

4. GENERALIZATION OF SOME COMPARISON RESULTS IN FRACTIONAL FUNCTIONAL DIFFERENTIAL EQUATIONS

In this section we will develop and generalize some basic comparison results in theory of functional differential equation to the fractional case. The integer order derivative version of these theorems can be found in [8].

Let $0 < q < 1$ and $p = 1 - q$. Denote by $C_p([t_0,T],\mathbb{R})$, the function space as follows:

$$C_p([t_0,T],\mathbb{R}) = \{u \in C((t_0,T],\mathbb{R}) \text{ and } (t-t_0)^p u(t) \in C([t_0,T],\mathbb{R})\} \quad (10)$$

Lemma 4.1. *Let $m \in C_p([t_0,T],\mathbb{R})$ be locally Hölder continuous with exponent $\lambda > q$ and for any $t_1 \in (t_0,T]$, we have*

$$m(t_1) = 0 \quad \textit{and} \quad m(t) \leq 0 \quad \textit{for} \quad t_0 \leq t \leq t_1. \quad (11)$$

Then it follows that,

$$D^q m(t_1) \geq 0. \quad (12)$$

For proof of Lemma 4.1, please see in [16].

Definition 4.2. A function $a(r)$ is said to belong to the class $\mathscr{K}$, if $\alpha \in C([0,\rho),\mathbb{R}_+)$, $a(0) = 0$ and $a(r)$ is strictly monotone increasing in r.

Definition 4.3. A function $V(t,x)$ with $V(t,0) \equiv 0$ is said to positive definite if there exist a function $b \in \mathscr{K}$ such that

$$V(t,x) \geq b(\|x\|), \quad (13)$$

is satisfied for $(t,x) \in \mathscr{R}_+ \times S_\rho$, and it is said to be decrescent if a function $a \in \mathscr{K}$ exist such that

$$V(t,x) \leq a(\|x\|), \quad (t,x) \in \mathbb{R}_+ \times S_\rho. \quad (14)$$

Definition 4.4 (Lyapunov-like function)**.** Let $V \in C([-\tau,\infty) \times S_\rho, \mathbf{R}_+)$, and let $\phi \in \mathscr{C}_\rho$. We define the fractional-order Dini derivatives in Caputo's sense ${}^cD^q_+V(t,\phi(0),\phi)$ and ${}^cD^q_-V(t,\phi(0),\phi)$ with respect to the functional differential system (5) as follows:

$${}^cD^q_+V(t,\phi(0),\phi) = \limsup_{h\to 0^+} \frac{1}{h^q}[V(t,\phi(0)) - V(t-h,\phi(0) - h^q f(t,\phi))] \quad (15)$$

$${}^cD^q_-V(t,\phi(0),\phi) = \liminf_{h\to 0^-} \frac{1}{h^q}[V(t,\phi(0)) - V(t-h,\phi(0) - h^q f(t,\phi))] \quad (16)$$

we need, consequently, the following subsets of $\mathscr{C}$ defined by

$$\Omega_1 = \left\{\phi \in \mathscr{C}_\rho : |V_t|_0 = V(t,\phi(0)), t \in [t_0,\infty]\right\} \quad (17)$$

$$\Omega_0 = \left\{\phi \in \mathscr{C}_\rho : V(t+\theta,\phi(\theta)) < L(V(t,\phi(0))), t \in [t_0,\infty]\right\} \quad (18)$$

where $L(u)$ is continuous on $\mathbb{R}_+$, non-decreasing in u, and $L(u) < u$, for $u > 0$; and

$$|V_t|_0 = \sup_{-\tau \le \theta \le 0} V\left(t+\theta, \phi(\theta)\right) \tag{19}$$

In fact, these Lyapunov-like functions act like a transformation of (5) into a relatively simple fractional differential equation, the properties of solutions of this simple system can be transferred back to the original more complicated system. This is known as the comparison principle, in general.We can now state some fractional comparison results via the Lyapunov-like function.

Theorem 4.5. *Let $V \in C\left([-\tau,\infty) \times S_\rho, \mathbf{R}_+\right)$ and $V(t,x)$ be locally Lipschitzian in x. Assume that the functional ${}^cD^q_-V\left(t,\phi(0),\phi\right)$, defined by* (16). *verifies the inequality*

$${}^cD^q_-V\left(t,\phi(0),\phi\right) \le g\left(t, V\left(t,\phi(0)\right)\right), \quad t > t_0, \quad \phi \in \Omega_1, \tag{20}$$

where $g \in C\left([t_0,\infty) \times \mathbf{R}_+, \mathbf{R}_+\right)$, and $r\left(t,t_0,u_0\right)$ is the maximal solution of the scalar differential equations of

$${}^cD^q = g(t,u), \quad u(t_0) = u_0 \ge 0, \tag{21}$$

existing on $[t_0,\infty)$. Let $x(t_0,\phi_0)$ be any solution of (5) *defined in the future, satisfying*

$$\sup_{-\tau \le s \le 0} V\left(t_0, \phi_0(s)\right) \le u_0 \tag{22}$$

Then,

$$V\left(t, x\left(t_0, \phi_0(t)\right)\right) \le r\left(t,t_0,u_0\right), \quad t \ge t_0. \tag{23}$$

Proof. Let $x(t_0,\phi_0)$ be any solution of (5) with an initial function $\phi_0 \in \mathscr{C}_\rho$ at $t = t_0$. Define the function

$$m(t) = V(t, x(t_0,\phi_0)(t)).$$

For $\varepsilon > 0$ sufficiently small, consider the fractional differential equation

$${}^cD^q u = g(t,u) + \varepsilon, \quad u(t_0) = u_0 \ge 0, \tag{24}$$

whose solutions $u(t,\varepsilon) = u(t,t_0,u_0,\varepsilon)$ exists as far as $r(t,t_0,u_0)$ to the right of t_0. Since

$$\lim_{\varepsilon \to 0} u(t,\varepsilon) = r(t,t_0,u_0) \tag{25}$$

it is enough to show that

$$m(t) \le u(t,\varepsilon), \quad t \ge t_0. \tag{26}$$

If this inequality is not true, let t_1 be greatest lower bound of numbers $t > t_0$ for which (26) is false. The continuity of the functions $m(t)$ and $u(t,\varepsilon)$ implies that

(i) $m(t) \le u(t,\varepsilon), \quad t_0 \le t \le t_1;$

(ii) $m(t_1) = u(t_1,\varepsilon), \quad t = t_1$

Now by Lemma 4.1 we have

$$D^q m(t_1) \geq D^q u(t_1,t_0,u_0,\varepsilon) = g\left(t_1, u(t_1,t_0,u_0,\varepsilon)\right) + \varepsilon \tag{27}$$

Since $g(t,u)+\varepsilon \geq 0$, $u(t,t_0,u_0,\varepsilon)$ is non-decreasing in t; and this implies, from (i) and (ii) that

$$|m_{t_1}|_0 = u(t_1,t_0,u_0,\varepsilon) = m(t_1) \tag{28}$$

Setting $\phi = x_{t_1}(t_0,\phi_0)$ and noting that $\phi(0) = x(t_0,\phi_0)(t_1)$, it follows that

$$|V_{t_1}|_0 = V(t_1,\phi(0)). \tag{29}$$

This means that $\phi \in \Omega_1$, and, consequently. using the Lipschitzian character of $V(t,x)$ in x and the relation (20), we obtain the inequality

$${}^cD^q m(t_1) \leq g(t_1, m(t_1)). \tag{30}$$

which is incompatible with (27). Hence (23) is valid and the proof is complete. □

Corollary 4.6. *Let $V \in C\left([-\tau,\infty)\times S_\rho, \mathbb{R}_+\right)$ and $V(t,x)$ be locally Lipschitz in x. Assume that, for $t > t_0$, $\phi \in \Omega_0$,*

$${}^cD^q_+ V\left(t,\phi(0),\phi\right) \leq 0. \tag{31}$$

Let $x(t_0,\phi_0)$ be any solution of (5) *such that $x(t_0,\phi_0)(t) \in S_\rho$ for $t \in [t_0,t_1]$. Then*

$$V\left(t, x(t_0,\phi_0)(t)\right) \leq \sup_{-\tau \leq s \leq 0} V\left(t_0,\phi_0(s)\right), \quad t \in [t_0,t_1]. \tag{32}$$

Proof. If we set $g \equiv 0$ in Theorem 4.5, we obtain the inequality

$$V\left(t, x(t_0,\phi_0)(t)\right) \leq V\left(t_2, x(t_0,\phi_0)(t_2)\right), \tag{33}$$

where $t_2 \in (t_0,t_1)$. Since $V\left(t_2, x(t_0,\phi_0)(t_2)\right) > 0$, the assumption on $L(u)$ implies that

$$V(t_2+s, x(t_0,\phi_0)(t_2+s)) < L(V(t_2, x(t_0,\phi_0)(t_2))), \tag{34}$$

which shows that $x_t(t_0,\phi_0) \in \Omega_0$, $t_0 \leq t \leq t_2$. The remain of the proof is similar to the proof of Theorem 4.5. □

The next comparison theorem gives a better estimate.

Theorem 4.7. *Let the assumptions of Theorem 4.5 hold except that the inequality* (20) *is replaced by*

$${}^cD^q_+ V\left(t,\phi(0),\phi\right) + \mathrm{d}(\|\phi(0)\|) \leq g\left(t, V\left(t,\phi(0)\right)\right), \tag{35}$$

for $t \geq t_0$, $\phi \in \mathscr{C}_\rho$, where the function $\mathrm{d} \in \mathscr{K}$. Assume further that $g(t,u)$ is monotone non-decreasing in u for each t. Then (22) *implies*

$$V(t, x(t_0,\phi_0)(t)) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} \mathrm{d}(\|x(t_0,\phi_0)(s)\|)\, \mathrm{d}s \leq r(t) \tag{36}$$

Proof. Consider

$$m(t) = V(t, x(t_0, \phi_0)(t)) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} \mathrm{d}(\|x(t_0, \phi_0)(s)\|) \, \mathrm{d}s, \tag{37}$$

for $t \geq t_0$ and set $\phi = x_t(t_0, \phi_0)$ so that $\phi(0) = x(t_0, \phi(0))(t)$. We then obtain, using the condition (35), the inequality

$${}^c D_+^q m(t_1) \leq g(t_1, m(t_1)) \tag{38}$$

Here, we have used the monotonicity of $g(t,u)$ in u and the fact that

$$V(t, x(t_0, \phi_0)(t)) \leq m(t), \tag{39}$$

while applying the assumption (36). Therefore, it is easy to prove the result of Theorem 4.7, following the statements in the proof of Theorem 4.5. □

5. STABILITY

Let us consider the fractional functional differential system (5). We will assume that $f(t, 0) \equiv 0$, so that the system (5) possesses the trivial solution ($x = 0$). Let us also suppose that the solutions $x(t_0, \phi_0)$ of (5) exist in the future.

Definition 5.1. The trivial solution of (5) is said to be *stable* if, for each $\varepsilon > 0$, t_0, there exists a positive function $\delta = \delta(t_0, \varepsilon)$ that is continuous in t_0 for each ε, such that, whenever

$$\|\phi_0\| \leq \delta, \tag{40}$$

we have

$$\|x(t_0, \phi_0)(t)\| < \varepsilon \quad t \geq t_0 \tag{41}$$

Definition 5.2. The solution $x = 0$ is said to be *uniformly sable* if the number δ in the previous definition is independent of t_0.

Theorem 5.3. *Assume that there exists a function $V(t,x)$ satisfying the following conditions:*

(i) $V \in C([-\tau, \infty) \times S_\rho, \mathbb{R}_+)$, *$V(t,x)$ is positive definite, decrescent, and locally Lipschitzian in x;*

(ii) *for $t > t_0$, $\phi \in \Omega_0$,*

$${}^c D_+^q V(t, \phi(0), \phi) \leq 0. \tag{42}$$

Then, the trivial solution of (5) *is uniformly stable.*

Proof. Since V is positive definite and decrescent, there exist functions $a, b \in \mathscr{K}$ satisfying

$$b(\|x\|) \leq V(t,x) \leq a(\|x\|), \quad (t,x) \in [t_0, \infty) \times S_\rho. \tag{43}$$

Let $0 < \varepsilon < \rho$, $t_0 \in \mathbb{R}_+$ be given. Choose $\delta = \delta(\varepsilon) > 0$ such that

$$a(\delta) < b(\varepsilon) \tag{44}$$

We claim that, if $\|\phi_0\| \leq \delta$, then $\|x\,(t_0, \phi_0)\| < \varepsilon$, $t \geq t_0$. Suppose that this is not true. Then, there exists a solution $x(t_0, \phi_0)$ of (5) with $\|\phi_0\|_0 \leq \delta$ such that

$$\|x\,(t_0, \phi_0)\,(t_2)\| = \varepsilon \tag{45}$$

and

$$\|x\,(t_0, \phi_0)\,(t)\| \leq \varepsilon, \quad t \in [t_0, t_2]\,, \tag{46}$$

so that

$$V\,(t_2, x\,(t_0, \phi_0)\,(t_2)) \geq b(\varepsilon), \tag{47}$$

because of (43). Furthermore, this means that $x\,(t_0, \phi_0)\,(t_2) \in S_\rho$, $t \in [t_0, t_2]$. Hence, the choice $u_0 = a\,(\|\phi_0\|_0)$ and the condition

$$^{c}D^{q}_{-}V(t, \phi(0), \phi) \leq 0, \quad t \in [t_0, t_2]\,, \quad \phi \in \Omega_0, \tag{48}$$

give the estimate

$$V\,(t, x(t_0, \phi_0)(t)) \leq a\,(\|\phi_0\|_0)\,, \quad t \in [t_0, t_2]\,, \tag{49}$$

because of Corollary 4.6. Now the relations (47), (49), and (44) lead to contradiction

$$b(\varepsilon) \leq V\,(t_2, x\,(t_0, \phi_0)\,(t_2)) \leq a\,(\|\phi_0\|_0) \leq a(\delta) < b(\varepsilon). \tag{50}$$

This proves that the trivial solution of (5) is uniformly stable. □

6. CONCLUSION

Lyapunov's second method is a standard technique used in the study of the qualitative behaviour of fractional order differential systems with Caputo derivatives along with a comparison results that allows the prediction of behaviour of a differential system when the behaviour of the null solution of a comparison system is known. In this work we generalized some comparison theorems in fractional differential-difference equations and used this results to prove a stability theorem for a class of fractional nonlinear time-delay systems.

REFERENCES

1. S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications* (Gordon and Breach, Yverdon, 1993).
2. A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations* (North-Holland, Amsterdam, The Netherlands, 2006).
3. I. Podlubny, *Fractional Differential Equations, vol. 198 of Mathematics in Science and Engineering* (Academic Press, San Diego, California, USA, 1999).

4. M. Caputo, Geophysical Journal of the Royal Astronomical Society **13**, 529-539 (1967).
5. P. J. Torvik and R. L. Bagley, Journal of Applied Mechanics **51**(2), 294-298 (1984).
6. M. Nakagava and K. Sorimachi, IEICE Transactions, Fundamentals, **E75-A**(12), 1814-1818 (1992).
7. R. L. Magin, *Fractional Calculus in Bioengineering* (Begell House, Redding, Conn, USA, 2006).
8. V. Lakshmikantham and S. Leela, *Differential and Integral Inequalities, vol. 1 & 2* (Academic Press, New York, NY, USA, 1969).
9. Y. Li, Y.Q. Chen, I. Podlubny, Comput. Math. Appl. **59**(5 SI), 1810-1821 (2010).
10. B. S. Razumikhin, Prikladnoi Matematiki i Mekhaniki **20**, 500-512 (1956).
11. N. N. Krasovskii, Prikladnaja Matematika i Mehanika **20**, 315-327 (1956).
12. J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional-Differential Equations* in *Applied Mathematical Sciences*, vol. **99** (Springer, New York, NY, USA, 1993).
13. S.H. Hosseinnia, R. Ghaderi, A. Ranjbar N., M. Mahmudian and S. Momani, Comput. Math. Appl. **2010**, 762857, 16 pp. (2010).
14. S. Momani and S. Hadid, International Journal of Mathematics and Mathematical Sciences **47**(45-48), 2503-2507 (2004).
15. C. Yakar, Abstract and Applied Analysis **2010**, 762857, 16 pp. (2010).
16. V. Lakshmikantham, S. Leela, and J. Vasundhara Devi, *Theory of Fractional Dynamic Systems* (Cambridge Scientific, Cambridge, UK, 2009).
17. S. J. Sadati, D. Baleanu, A. Ranjbar, R. Ghaderi,and T. Abdeljawad (Maraaba), Abstract and Applied Analysis **2010**, 124812, 9 pp. (2010).
18. D. Baleanu, S. J. Sadati, R. Ghaderi, A. Ranjbar, T. Abdeljawad (Maraaba),and F. Jarad, Abstract and Applied Analysis **2010**, 108651, 7 pp. (2010).
19. D. Baleanu, A. Ranjbar N., S.J. Sadati R., H. Delavari, T. Abdeljawad (Maraaba), V. Gejji, Rom. Journ. Phys. **56**(5-6), 636-643 (2011).
20. D. Baleanu, K. Diethelm, E. Scalas, and J. J. Trujillo, *Fractional Calculus Models and Numerical Methods (Series on Complexity, Nonlinearity and Chaos)* (World Scientific, 2012).
21. Alireza K. Golmankhaneh, Ali K. Golmankhaneh; Dumitru Baleanu, Rom. Rep. Phys. **63**(3), 609-623 (2011).
22. Fahd Jarad, Thabet Abdeljawad, Emrah Gündoğdu, Dumitru Baleanu, Proceedings of the Romanian Academy - Series A **12**(4), 309-314 (2011).