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SOLVING MULTI-TERM ORDERS FRACTIONAL DIFFERENTIAL EQUATIONS BY OPERATIONAL MATRICES OF BPs WITH CONVERGENCE ANALYSIS

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Abstract. In this paper, we present a numerical method for solving a class of fractional differential equations (FDEs). Based on Bernstein Polynomials (BPs) basis, new matrices are utilized to reduce the multi-term orders fractional differential equation to a system of algebraic equations. Convergence analysis is shown by several theorems. Illustrative examples are included to demonstrate the validity and applicability of this method.

Key words: Bernstein polynomials, fractional differential equations, operational matrix, Caputo derivative, convergence analysis.

1. INTRODUCTION

Fractional differential equations (FDEs) are generalizations of ordinary differential equations to an arbitrary (non-integer) order. These equations have attracted considerable interest because of their ability to model complex phenomena. Also, we observe that these equations capture nonlocal relations in space and time with power-law memory kernels. There are several (non-equivalent) definitions of the fractional derivative in widespread use and we choose to focus on one particular form (the so-called Caputo version) in this paper [1]. The use of fractional orders differential operators and integral operators in mathematical models has become increasingly widespread in recent years (see for example Refs. [2, 3] and the references therein). Several forms of fractional differential equations have been proposed in standard models, and there has been significant interest in developing numerical schemes for their solution [2–12].

In this paper, we focus on the multi-term orders fractional differential equations as follows:

$$D^{\alpha} y(t)=\sum_{j=1}^{k} a_j(t) D^{\beta_j} y(t)+a_0(t) y(t)+f(t), \tag{1}$$

with the initial conditions

$$y^{(i)}(0)=b_i, \; i=0,\; 1,\; \ldots ,\; \lceil \alpha \rceil -1, \tag{2}$$

where $\alpha>\beta_1>\beta_2>\ldots>\beta_k>0$ are constant and $\lceil \alpha \rceil$ denotes the smallest integer greater than or equal to α. Also, $\left\{a_j(t)\right\}_{j=1}^{k}$, $f(t)$ (as an input signal) are known functions and unknown function $y(t)$ is as an output response.

Now, we use the initial conditions to reduce problem (1) and (2) to a problem with zero initial conditions. Therefore, we define

$$y(t)=\hat{y}(t)+z(t), \tag{3}$$

where $\hat{y}(t)$ is a known function that satisfied the initial conditions (2) and $z(t)$ is a new unknown function.

Substituting (3) in (1) and (2), we have the following initial-value problem where $g(t)$ is known function:

$$D^{\alpha} z(t)=\sum_{j=1}^{k} a_j(t) D^{\beta_j} z(t)+a_0(t) z(t)+g(t), \tag{4}$$

with the initial conditions

$$z^{(i)}(0)=0, \quad i=0,\; 1,\; \ldots ,\; \lceil \alpha \rceil -1. \tag{5}$$

The rest of this paper is as follows. In Section 2, we present some definitions and preliminaries in fractional calculus and BPs. Then we approximate functions by using BPs and we discuss convergence analysis. Also, we obtain BPs operational matrix for product in Section 3. We present an operational matrix for fractional integration by BPs in Section 4. In Section 5, we apply BPs for solving linear multi-order fractional differential equation. In Section 6, we discuss on the convergence of the proposed method. In Section 7, numerical examples are simulated to demonstrate the high performance of the proposed method. Finally, Section 8 concludes our work in this paper.

2. BASIC DEFINITIONS

In this section, some basic definitions and properties of the fractional calculus and BPs are briefly.

Definition 2.1 [1]. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_\mu, \quad \mu \geq -1$, is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x) \mathrm{d}x, \quad \alpha > 0, \quad t > 0, \\ I^0 f(t) = f(t). \tag{6}$$

Definition 2.2 [1]. The fractional derivative of $f(t)$ in the Caputo sense is defined as

$$D^\alpha f(t) = I^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-x)^{n-\alpha-1} f^{(n)}(x) \mathrm{d}x, \tag{7}$$

for $n-1 < \alpha \leq n, \quad n \in \mathrm{N}, \quad t > 0, \quad f \in C_{-1}^n$.

If $n-1 < \alpha \leq n, \quad n \in \mathrm{N}$ and $f \in C_\mu^n, \quad \mu \geq -1$, then [13–15])

$$1. \quad D^\alpha I^\alpha f(t) = f(t), \tag{8}$$

$$2. \quad I^\alpha D^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \ \frac{x^k}{k!}, \quad t > 0. \tag{9}$$

In the following, we define the Bernstein polynomials (BPs) of *m*-th degree on the interval $[0,1]$ as follows:

$$B_{i,m}(x) = \binom{m}{i} x^i (1-x)^{m-i}, \quad i = 0, \ 1, \ \cdots, \ m, \tag{10}$$

Moreover, we know that set $\{B_{0,m}(x), \ B_{1,m}(x), \ \cdots, \ B_{m,m}(x)\}$ is a complete basis in Hilbert space $L^2[0, \ 1]$. Therefore, for each polynomial of degree m, we can write

$$P(x) = \sum_{i=0}^{m} c_i B_{i,m}(x) \tag{11}$$

We recall that [16]

$$B_{i,m}(x) = \sum_{k=0}^{m-i} (-1)^k \binom{m}{i} \binom{m-i}{k} \ x^{i+k},$$

for $i = 0,\ 1,\ \dots,\ m$. Defining
$\Phi_m(x) = \left[B_{0\,,\,m}(x),\ B_{1\,,\,m}(x),\ \dots,\ B_{m\,,\,m}(x)\right]^{\mathrm{T}}$ and $T_m(x) = \left[1,\ x,\ \dots,\ x^m\right]^{\mathrm{T}}$,
we obtain

$$\Phi_m(x) = AT_m(x), \tag{12}$$

where the elements of the invertible matrix $A = \left(a_{i\,,\,j}\right)_{i,j=1}^{m+1}$ are defined as

$$a_{i+1\,,\,j+1} = \begin{cases} (-1)^{j-i} \dbinom{m}{i}\dbinom{m-i}{j-i} & i \le j, \\ 0 & i > j. \end{cases} \qquad i, j = 0,\ 1,\ \dots,\ m \tag{13}$$

Suppose that $S_m = \operatorname{Span}\left\{B_{0,\,m},\ B_{1,\,m},\ \cdots,\ B_{m,\,m}\right\}$ and $y \in L^2\left[0,\ 1\right]$. Since S_m is a finite dimensional and closed subspace, then S_m is a complete subset of $L^2\left[0,1\right]$. Therefore, y has the unique best approximation out of S_m such as $s_0 \in S_m$, that is we have the unique coefficients $\left\{c_i\right\}_{i=0}^{m}$ such that (see [17])

$$y(x) \approx s_0(x) = \sum_{i=0}^{m} c_i B_{i,\,m}(x) = c^T \Phi_m(x), \tag{14}$$

where $c = Q^{-1}\int_0 y(x)\Phi_m(x)\mathrm{d}x,$ and $Q = \left(Q_{i\,,\,j}\right)_{i\,,\,j=1}^{m+1}$,

$$Q_{i+1\,,\,j+1} = \int_0^1 B_{i\,,\,m}(x) B_{j\,,\,m}(x)\mathrm{d}x = \frac{\dbinom{m}{i}\dbinom{m}{j}}{(2m+1)\dbinom{2m}{i+j}}, \quad i, j = 0, 1, \cdots m. \tag{15}$$

LEMMA 2.3. *Suppose that the function* $y : \left[0,\ 1\right] \to R$ *is* $m+1$ *times continuously differentiable* $\left(\text{i.e. } y \in C^{m+1}\left(\left[0\ ,\ 1\right]\right)\right)$, *and* $S_m = \operatorname{Span}\left\{B_{0,\,m}, B_{1,\,m},\ \cdots, B_{m,\,m}\right\}$. *If* $c^{\mathrm{T}}B$ *be the best approximation* y *out of* S_m *then*

$$\left\| y - c^T B \right\|_{L^2[0,1]} \le \frac{\hat{K}}{(m+1)!\sqrt{2m+3}}, \tag{16}$$

where $\hat{K} = \max_{x\in[0\ ,\ 1]} \left|y^{(m+1)}(x)\right|$. Also, if $y \in C^{\infty}\left(\left[0,\ 1\right]\right)$ then the error bound vanishes.

Proof. See [18].

3. BPS PRODUCT OPERATIONAL MATRIX

In this section, similar to [16], we obtain a formula for BPs operational matrix of product. Moreover, we survey the error distribution in this approximation.

Suppose that $c = c_{(m+1)\times 1}$ is an arbitrary vector. Now, we obtain the matrix $\hat{C} = \hat{C}_{(M+1)\times(m+1)}$ where

$$c^{\mathrm{T}}\Phi_m(x)\Phi_m(x)^{\mathrm{T}} \approx \Phi_M(x)^{\mathrm{T}}\hat{C}. \tag{17}$$

From (12) we have [16]

$$\begin{aligned} c^T\Phi_m(x)\ \ \Phi_m(x)^{\mathrm{T}} &= c^{\mathrm{T}}\Phi_m(x)\left(T_m(x)^{\mathrm{T}}A^{\mathrm{T}}\right) = \\ &= \left[\sum_{i=0}^{m} c_i B_{i\ ,\ m}(x),\ \sum_{i=0}^{m} c_i x B_{i\ ,\ m}(x),\ \ldots,\ \sum_{i=0}^{m} c_i x^m B_{i\ ,\ m}(x)\right]A^{\mathrm{T}}. \end{aligned} \tag{18}$$

Thus we define $e_{k\ ,\ i} = \left[e^0_{k\ ,\ i},\quad e^1_{k\ ,\ i},\ \ \ldots,\ \ e^M_{k\ ,\ i}\right]^{\mathrm{T}}$, then by (14) we can write

$$x^k B_{i,m}(x) = e_{k,i}{}^T\ \Phi_M(x) + E^M_{i,k},\qquad i,k = 0,1,\ldots,m, \tag{19}$$

where $E^M_{i,k}$ is the approximation error. So, we get [16]

$$\begin{aligned} e_{k\ ,\ i} &= Q^{-1}\left(\int_0^1\left(x^k B_{i\ ,\ m}(x)\right)\Phi_M(x)\mathrm{d}x\right) = \\ &= \frac{Q^{-1}\binom{m}{i}}{M+m+k+1}\left[\frac{\binom{M}{0}}{\binom{M+m+k}{i+k}},\ \frac{\binom{M}{1}}{\binom{M+m+k}{i+k+1}},\ \ldots,\ \frac{\binom{M}{M}}{\binom{M+m+k}{i+k+M}}\right]^{\mathrm{T}}, \end{aligned}$$

$i,k = 0,\ 1,\ \ldots,\ m.$

Then, we have

$$\sum_{i=0}^{m} c_i x^k B_{i,\ m}(x) = \sum_{i=0}^{m} c_i\left(\sum_{j=0}^{M}\left(e^j_{k,\ i}\quad B_{j,\ M}(x)\right) + E^M_{i,k}\right) = \Phi_M(x)^T V_{k+1}c + \sum_{i=0}^{m} c_i E^M_{i,k}, \tag{20}$$

where $V_{k+1}(k=0,\ 1,\ldots m)$ is an $(M+1)\times(m+1)$ matrix that has vectors $e_{k\ ,\ i}\ (i=0,\ 1,\ \ldots,\ m)$ for each column's. If we define $\bar{C}=\left[V_1c,\ V_2c,\ \ldots,\ V_{m+1}c\right]$, from (18) and (20) we can write

$$c^{\mathrm{T}}\Phi_m(x)\Phi_m(x)^{\mathrm{T}}=\Phi_M(x)^{\mathrm{T}}\bar{C}A^{\mathrm{T}}+E_P^M,$$

where

$$E_P^M=\left[\sum_{i=0}^{m}c_iE_{i,0}^M,\ \sum_{i=0}^{m}c_iE_{i,1}^M,\ \ldots,\ \sum_{i=0}^{m}c_iE_{i,m}^M\right]A^{\mathrm{T}}. \tag{21}$$

Therefore we obtain the operational matrix of product $\hat{C}=\bar{C}A^{\mathrm{T}}$.

LEMMA 3.1. *If* E_P^M *is the approximation error for product in* (17), *then we have* $E_P^M\to 0$ *as* $M\to\infty$.

Proof. From (19) and Lemma 2.3, we have $E_{i,k}^M\to 0$ as $M\to\infty$ for $i,k=0,\ 1,\ \ldots,\ m$. Therefore, from (21) it is clear that $E_P^M\to 0$ *as* $M\to\infty$.

4. BPS OPERATIONAL MATRIX FOR THE FRACTIONAL INTEGRATION

Now, we want to the operational matrix for the fractional integration. We can write:

$$I^{\alpha}\Phi_m(t)=\frac{1}{\Gamma(\alpha)}t^{\alpha-1}*\Phi_m(t),\quad 0\le t\le 1, \tag{22}$$

where $*$ denotes the convolution product. From (12) we have

$$t^{\alpha-1}*\Phi_m(t)=t^{\alpha-1}*\left(AT_m(t)\right)=A\left(t^{\alpha-1}*T_m(t)\right). \tag{23}$$

We can get

$$\begin{aligned} t^{\alpha-1}*T_m(t)&=\left[t^{\alpha-1}*1,\ t^{\alpha-1}*t,\ \ldots,\ t^{\alpha-1}*t^m\right]^{\mathrm{T}}=\Gamma(\alpha)\left[I^{\alpha}1,\ I^{\alpha}t,\ \ldots,\ I^{\alpha}t^m\right]^{\mathrm{T}}=\\ &=\Gamma(\alpha)\left[\frac{0!}{\Gamma(\alpha+1)}t^{\alpha},\ \frac{1!}{\Gamma(\alpha+2)}t^{\alpha+1},\ \ldots,\ \frac{m!}{\Gamma(\alpha+m+1)}t^{\alpha+m}\right]^{\mathrm{T}}=\\ &=\Gamma(\alpha)D\bar{T}, \end{aligned} \tag{24}$$

where $D_{(m+1)\times(m+1)}$ and $\bar{T}_{(m+1)\times 1}$ are as follows:

$$D_{i,j}=\begin{cases}\dfrac{i!}{\Gamma(\alpha+i+1)} & i=j,\\ 0 & i\neq j,\end{cases}\quad i,j=0,\ 1,\ \ldots,\ m,\ \ \overline{T}=\left[t^{\alpha},\ t^{\alpha+1},\ \ldots,\ t^{\alpha+m}\right]^{\mathrm{T}}. \quad (25)$$

Now, we need to approximate $t^{\alpha+i}$ $(i=0,\ 1,\ \ldots\ ,\ m)$ with respect to BPs by using (14). Therefore, we have

$$t^{\alpha+i}=E_i^T\Phi_M(t)+\tilde{E}_i^M, \quad (26)$$

where $\tilde{E}_i^M$ is the approximation error for $i=0,\ 1,\ \ldots,\ m$, and $E_i=Q^{-1}\overline{E}_i$, where

$$\overline{E}_i=[\overline{E}_{i\ ,\ 0},\ \overline{E}_{i\ ,\ 1},\ \ldots,\ \overline{E}_{i\ ,\ M}]^{\mathrm{T}},$$

$$\overline{E}_{i,\ j}=\int_0^1 t^{\alpha+i}B_{j,\ M}(t)\mathrm{d}t=\frac{M!\Gamma(i+j+\alpha+1)}{j!\Gamma(i+M+\alpha+2)},$$

$$i=0,\ldots,\ m,\quad \text{and}\quad j=0,\ 1,\ \ldots,\ M.$$

Now, we suppose E is an $(M+1)\times(m+1)$ matrix that has vector E_i $(i=0,\ 1,\ \ldots,\ m)$ for ith column's. Therefore, we can write

$$I^{\alpha}\Phi_m(t)\ =ADE^{\mathrm{T}}\Phi_M(t)+\tilde{E}_I^M,$$

where

$$\tilde{E}_I^M=AD\left[\tilde{E}_0^M,\tilde{E}_1^M,\ \cdots\ ,\ \tilde{E}_m^M\right]^{\mathrm{T}}. \quad (27)$$

Finally, we obtain

$$I^{\alpha}\Phi_m(t)\approx F_{\alpha}\Phi_M(t). \quad (28)$$

where $F_{\alpha}=ADE^{\mathrm{T}}$ is called the Bernstein polynomials operational matrix of fractional integration.

LEMMA 4.1. *If $\tilde{E}_I^M$ is the approximation error for fractional integration in (28), then we have $\tilde{E}_I^M\to 0$ as $M\to\infty$.*

Proof. From (26) and Lemma 2.3, we have $\tilde{E}_i^M\to 0$ as $M\to\infty$ for $i=0,\ 1,\ \ldots,\ m$. Therefore, from (27) it is clear that $\tilde{E}_I^M\to 0$ as $M\to\infty$.

5. BPS FOR LINEAR MULTI-ORDER FRACTIONAL DIFFERENTIAL EQUATION

By using (14), the input signal $g(t)$, $D^{\alpha}z(t)$ and $a_j(t)$, $(j=0,\ 1,\ \ldots,\ k)$ in (4) may be expanded as follows:

$$g(t) \approx G^{\mathrm{T}} \Phi_m(t), \tag{29}$$

$$D^{\alpha} z(t) \approx C^{\mathrm{T}} \Phi_m(t), \tag{30}$$

$$a_j(t) \approx A_j^{\mathrm{T}} \Phi_m(t), \tag{31}$$

where G, A_j are known $(m+1)\times 1$ column vectors and C is an unknown $(m+1)\times 1$ column vector. From (30) and (28), we have

$$\begin{aligned} D^{\beta j} z(t) = I^{\alpha-\beta j} D^{\alpha} z(t) \approx I^{\alpha-\beta j} \left(C^{\mathrm{T}} \Phi_m(t) \right) \\ = C^{\mathrm{T}} I^{\alpha-\beta j} \Phi_m(t) \approx C^{T} F_{\alpha-\beta_j} \Phi_m(t). \end{aligned} \tag{32}$$

Now, by substituting (29)-(31) and (32) into (4), we obtain

$$C^{\mathrm{T}} \Phi_m(t) = \sum_{j=1}^{k} \left(A_j^{\mathrm{T}} \Phi_m(t) \Phi_m(t)^{\mathrm{T}} F_{\alpha-\beta_j}{}^{\mathrm{T}} C \right) + A_0^{\mathrm{T}} \Phi_m(t) \Phi_m(t)^{\mathrm{T}} F_{\alpha}^{\mathrm{T}} C + G^{\mathrm{T}} \Phi_m(t).$$

Then, from (17) we have

$$A_j^{T} \Phi_m(x) \Phi_m(x)^{\mathrm{T}} = \Phi_m(x)^{\mathrm{T}} \hat{A}_j, \tag{33}$$

where $\hat{A}_j$, $(j = 0,\ 1,\ \ldots,\ m)$ is a $(m+1)\times(m+1)$ matrix. Therefore by (33) we get

$$C^{\mathrm{T}} \Phi_m(t) = \sum_{j=1}^{k} \left(\Phi_m(t)^{\mathrm{T}} \hat{A}_j F_{\alpha-\beta_j}{}^{\mathrm{T}} C \right) + \Phi_m(t)^{\mathrm{T}} \hat{A}_0 F_{\alpha}^{\mathrm{T}} C + G^{\mathrm{T}} \Phi_m(t). \tag{34}$$

Finally, we have the following linear system:

$$\left(\mathrm{I} - \left(\sum_{j=1}^{k} \left(\hat{A}_j F_{\alpha-\beta_j}{}^{T} \right) + \hat{A}_0 F_{\alpha}{}^{T} \right) \right) C = G, \tag{35}$$

that by solving this linear system we can obtain the vector C. Then, we can get

$$z(t) = I^{\alpha} D^{\alpha} z(t) \approx C^{\mathrm{T}} I^{\alpha} \Phi_m(t) \approx C^{\mathrm{T}} F_{\alpha} \Phi_m(t). \tag{36}$$

Therefore from (3) we have

$$y(t) \approx \hat{y}(t) + C^{\mathrm{T}} F_{\alpha} \Phi_m(t). \tag{37}$$

6. CONVERGENCE ANALYSIS

In this section, we investigate the convergence analysis for the method presented in section 5.

By using (9), problem (4) change to the following problem

$$D^{\alpha} z(t) = \sum_{j=1}^{k} a_j(t) I^{\alpha-\beta j} D^{\alpha} z(t) + a_0(t) I^{\alpha} D^{\alpha} z(t) + g(t). \tag{38}$$

By taking $u(t) = D^{\alpha} z(t)$ we obtain the following fractional integral equation

$$u(t) = \sum_{j=1}^{k} a_j(t) I^{\alpha-\beta j} u(t) + a_0(t) I^{\alpha} u(t) + g(t). \tag{39}$$

If we use the approximation $u(t) \approx C^{\mathrm{T}} \Phi_m(t)$, then the problem (39) from space $C^1[0,1]$ reduce to the following problem in space S_m

$$C^T \Phi_m(t) = \sum_{j=1}^{k} a_j(t) \left(C^T I^{\alpha-\beta j} \Phi_m(t) \right) + a_0(t) \left(C^T I^{\alpha} \Phi_m(t) \right) + g(t). \tag{40}$$

Now, similar to Theorem 1 in [6], we can propose the following Theorem.

THEOREM 6.1. *Suppose that* $u^*(t) \in C^1[0,\ 1]$ *is the exact solution of the Eq. (39) and* $\mu_m = J[u_m] = \underset{u \in S_m}{Min}\, J[u]$, *where*

$$J[u] = \left\| u(t) - \sum_{j=1}^{k} a_j(t) I^{\alpha-\beta j} u(t) - a_0(t) I^{\alpha} u(t) - g(t) \right\|. \tag{41}$$

Then we have $\mu_m \to 0$ as $m \to \infty$ $\left(i.e.\ \ u_m(t) \to u^*(t) \ \text{ as } \ m \to \infty \right)$.

Proof. Since $u^*(t)$ is solution of the problem (39), we have $J[u^*] = 0$. Therefore we can write

$$\underset{u \in C^1[0\ ,\ 1]}{\text{Min}} \ J[u] = 0 .$$

Also, $\mu_{m+1} \le \mu_m$, because $S_m \subset S_{m+1}$. For $\varepsilon_m = \dfrac{1}{m}$, exist $k_m \in C^1[0,\ 1]$, such that

$$J[k_m] < \varepsilon_m . \tag{42}$$

Since $J[u]$ is the continuous functional on $C^1[0,\ 1]$, for all $\varepsilon_m > 0$, then exist a $\delta(\varepsilon_m)$ such that

$$\|u-k_m\|<\delta(\varepsilon_m)\ \Rightarrow\ |J[u]-J[k_m]|<\varepsilon_m. \tag{43}$$

Since polynomials space is dense in the space $C^1[0,\ 1]$, the set of Bernstein polynomials on [0, 1] form a basis for the Space $C^1[0,\ 1]$ and for m sufficiently large, $v_m \in S_m$ exist such that $\|v_m - k_m\| < \delta(\varepsilon_m)$, therefore from (43) we have

$$|J[v_m]-J[k_m]|<\varepsilon_m. \tag{44}$$

From (44) and (42) we get

$$J[v_m]<J[k_m]+\varepsilon_m<2\varepsilon_m. \tag{45}$$

On the other hand, we have

$$J[v_m]\ge\mu_m\ge 0. \tag{46}$$

Therefore, using (45) and (46) we obtain

$$0\le\mu_m<2\varepsilon_m. \tag{47}$$

Now, obviously such $\lim_{m\to\infty}\mu_m=0$. Therefore the proof is complete.

THEOREM 6.2. *Suppose that $u^*(t)\in C^1[0,\ 1]$ is the exact solution of the Eq. (39) and $u_m(t)\in S_m$ is the obtained solution of Eq. (34). Then we have $u_m(t)\to u^*(t)$ as $m\to\infty$.*

Proof. Substituting (29), (31) and (28) in (40) we have

$$\begin{aligned}C^T\Phi_m(t)=\sum_{j=1}^{k}\Bigg(\left(A_j^T\Phi_M(t)+e_j^M\right)\left(\left(F_{\alpha-\beta_j}\Phi_M(t)+E_j^M\right)^{\mathrm{T}}C\right)\Bigg)+\\+\left(A_0^T\Phi_M(t)+e_0^M\right)\left(\left(F_\alpha\Phi_M(t)+E_0^M\right)^{\mathrm{T}}C\right)+\left(G^{\mathrm{T}}\Phi_M(t)+e_g^M\right).\end{aligned} \tag{48}$$

From Lemma 2.3 $e_j^M\,(j=0,1,\ldots,k)$, $e_g^M\to 0$ as $M\to\infty$ and using Lemma 4.1 $E_j^M\to 0\ \ (j=0,1,\ldots,k)$ as $M\to\infty$. So we can observe that, as M increases Eq. (48) gets close to Eq. (40). Now, by taking $M=m$ we propose the following problem that is gets close to (40) as m increases

$$C^T\Phi_m(t)=\sum_{j=1}^{k}\left(A_j^{\mathrm{T}}\Phi_m(t)\ \ \Phi_m(t)^{\mathrm{T}}F_{\alpha-\beta_j}{}^{\mathrm{T}}C\right)+A_0^T\Phi_m(t)\Phi_m(t)^{\mathrm{T}}F_\alpha^{\mathrm{T}}C+G^{\mathrm{T}}\Phi_m(t). \tag{49}$$

Then by (17) the Eq. (49) reduce to the following equation

$$C^{\mathrm{T}}\Phi_m(t)=\sum_{j=1}^{k}\left(\left(\Phi_M(t)^{\mathrm{T}}\hat{A}_j+\tilde{E}_j^M\right)F_{\alpha-\beta_j}{}^{\mathrm{T}}C\right)+\left(\Phi_M(t)^{\mathrm{T}}\hat{A}_0+\tilde{E}_0^M\right)F_\alpha^{\mathrm{T}}C+G^{\mathrm{T}}\Phi_m(t). \quad (50)$$

The Eq. (50) gets to (49) as $M\to\infty$, because from Lemma 3.1 $\tilde{E}_j^M\,(j=0,1,\ldots,k)\to 0$ as $M\to\infty$. Then by taking $M=m$ and deleting $\tilde{E}_j^M\,(j=0,1,\ldots,k)$ in (50), we get the Eq. (34). Obviously, if $\tilde{u}_m(t)$ is solution of Eq. (40) then we have $\tilde{u}_m-u_m\to 0$ as $m\to\infty$.

On the other hand, from Theorem 6.1 we obtained $\tilde{u}_m\to u^*$ as $m\to\infty$. Therefore we can write $u_m\to u^*$ as $m\to\infty$ and proof is complete.

7. NUMERICAL EXAMPLES

In this section, we apply our method to solve the following examples. We define $y_m(t)$ and $y(t)$ for the approximate solution and the exact solution in problem (1), respectively.

EXAMPLE 7.1. We consider the composite fractional oscillation equation $D^2y(t)+D^\alpha y(t)+y(t)=8,\quad 0<t\leq 1,$ with the initial conditions $y(0)=0,\quad y'(0)=0.$

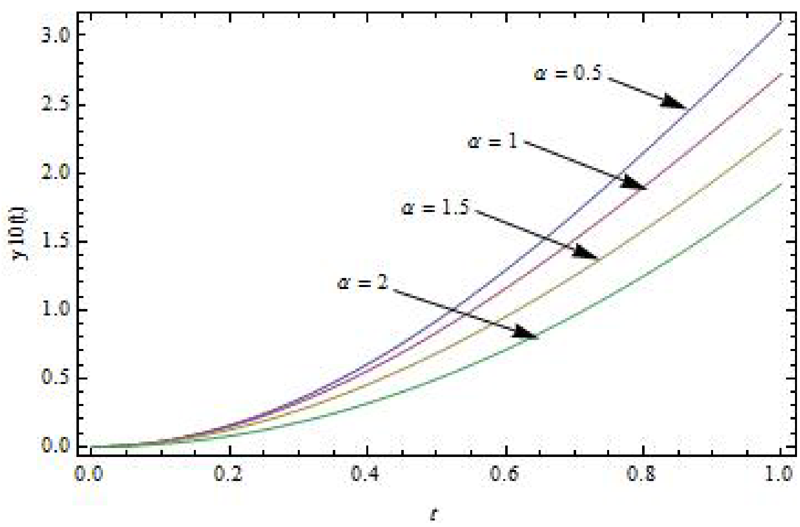

Fig. 1 – Plot of $y_{10}(t)$ for different α in Example 7.1.

Table 1

Numerical results for $\alpha = 0.5$ in Example 7.1 with comparison to Refs. [9,11,12]

t	BPs (ours)	BPFs [12]	ADM [11]	FDTM [9]	Exact
0.1	0.039750178860887476	0.039754	0.039874	0.039750	0.039750
0.2	0.1570357303495698	0.157043	0.158512	0.157036	0.157036
0.3	0.3473700100695401	0.347373	0.353625	0.347370	0.347370
0.4	0.6046951645035491	0.604699	0.622083	0.604695	0.604695
0.5	0.921767632166414	0.921768	0.960047	0.921768	0.921768
0.6	1.2904565421305207	1.290458	1.363093	1.290457	1.290457
0.7	1.7020079734684939	1.702007	1.826257	1.702008	1.702008
0.8	2.147286954798279	2.147286	2.344224	2.147287	2.147287
0.9	2.617000952366722	2.616998	2.911278	2.617001	2.617001
1	3.1019054661977234	3.101902	3.521462	3.101906	3.101906

Table 2

Numerical results for $\alpha = 1.5$ in Example 7.1 with comparison to Refs. [9,11,12]

t	BPs (ours)	BPFs [12]	ADM [11]	FDTM [9]	Exact
0.1	0.033497508377282555	0.033510	0.036478	0.033507	0.033507
0.2	0.12522700622852032	0.125226	0.140640	0.125221	0.125221
0.3	0.2676092519829363	0.267611	0.307485	0.267609	0.267609
0.4	0.455431094577048	0.455439	0.533284	0.455435	0.455435
0.5	0.6843377923655005	0.684336	0.814757	0.684335	0.684335
0.6	0.9503942211892613	0.950395	1.148840	0.950393	0.950393
0.7	1.2499556250467097	1.249959	1.532571	1.249959	1.249959
0.8	1.579558787681416	1.579558	1.963033	1.579557	1.579557
0.9	1.9358325798163138	1.935832	2.437331	1.935832	1.935832
1	2.315534398939192	2.315526	2.952567	2.315526	2.315526

This problem was solved in [9,11,12] for $\alpha = 0.5$ and $\alpha = 1.5$. We compare the numerical results with Refs. [9,11,12] that are given in Table 1 and 2, (where the exact solution refers to the closed form series solution presented in [11] by taking $N = 30$). Also, Fig. 1 shows the plot of $y_{10}(t)$ for different α. We see that our method is very effective and accuracy of approximate solutions in this method are in high agreement with results obtained using the FDTM and better than those obtained using the BPFs and ADM.

EXAMPLE 7.2. Consider the equation $D^{1.5} y(t) = t^{1.5} y(t) + 4\sqrt{\frac{t}{\pi}} - t^{3.5}$, $0 < t \le 1$, with the following initial conditions $y(0) = 0, \quad y'(0) = 0.$

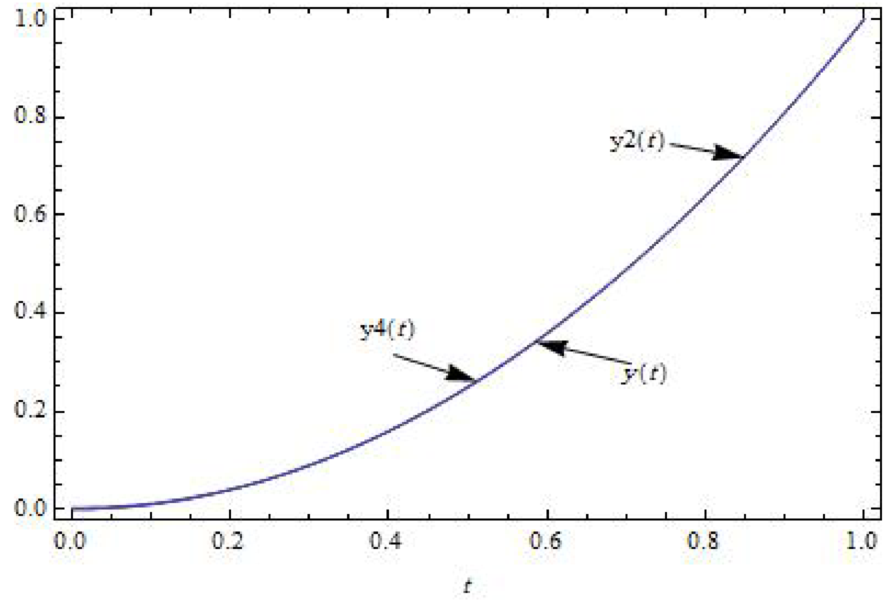

Fig. 2 – Plot of $y_2(t)$, $y_4(t)$ and $y(t)$ for Example 7.2.

Table 3

Absolute error for different m in Example 7.2

t / m	0.1	0.3	0.5	0.7	0.9
2	0.00175729	0.000303298	0.00129982	0.00123228	0.000100674
4	0.000100836	0.000122769	0.000183192	0.0000327544	0.000142422
10	0.00001264	1.58535×10^{-6}	9.04761×10^{-6}	6.1136×10^{-6}	8.63576×10^{-6}

We know that the exact solution is $y(t)=t^2$. The obtained results of BPs for $m=2,4,10$ are reported in Table 3 and are plotted in Fig. 2. We observe that our method is very effective.

EXAMPLE 7.3. We consider the following linear fractional differential equation
$$D^2 y(t) - t^2 D^{1.5} y(t) - \sqrt{t} D^{0.5} y(t) - \sqrt[3]{t}\, y(t) = 6\sqrt{\pi} t - 8\sqrt{t^7} - \frac{16}{5} t^3 - \sqrt[3]{t^{10}}\sqrt{\pi},$$
$0 < t \le 1$, subject to $y(0)=0$, $y'(0)=0$. With the exact solution $y(t)=\sqrt{\pi} t^3$. Table 4 is reported the absolute errors for different values of t and Fig. 3 shows the absolute error for our method.

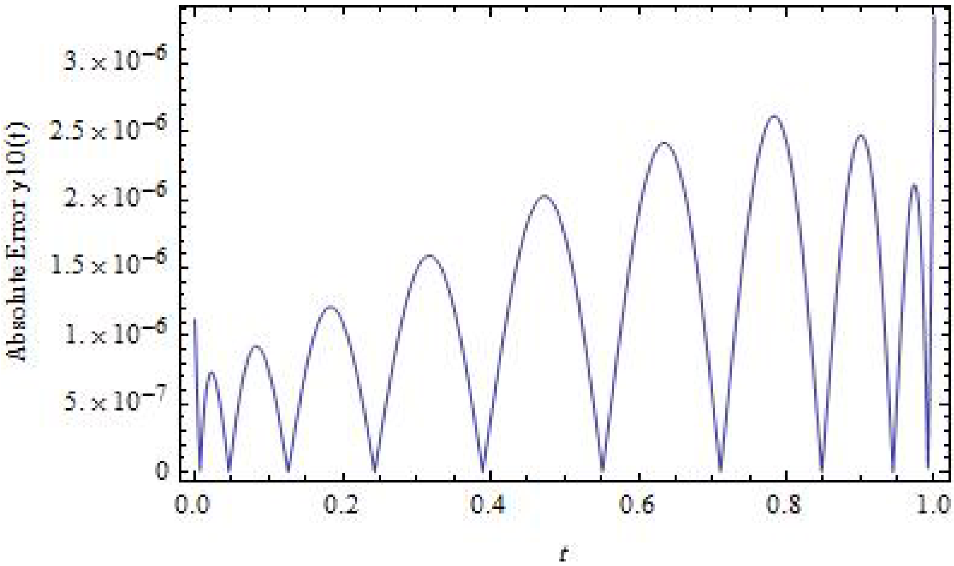

Fig. 3 – Plot of absolute error function for $m = 10$ in Example 7.3.

Table 4

Our results for $m = 10$ in Example 7.3

t	0.1	0.3	0.5	0.7	0.9
Absolute error	7.45873×10^{-7}	1.48329×10^{-6}	1.74701×10^{-6}	5.51155×10^{-7}	2.47276×10^{-6}

EXAMPLE 7.4 [10]. Consider the equation $aD^2 y(t) + b(t)D^{\alpha 2} y(t) + c(t)Dy(t) + e(t)D^{\alpha 1} y(t) + k(t)y(t) = f(t), \quad 0 < t \le 1,$ and $f(t) = -a - \frac{b(t)}{\Gamma(3-\alpha_2)} t^{2-\alpha 2} - c(t)t - \frac{e(t)}{\Gamma(3-\alpha_1)} t^{2-\alpha 1} + k(t)\left(2 - \frac{t^2}{2}\right),$ with the initial conditions $y(0) = 2, \quad y'(0) = 0.$

We know that the exact solution is $y(t) = 2 - \frac{t^2}{2}$. For $\alpha_1 = 0.5, \quad \alpha_2 = 1.5$ and $a = 1, \ b(t) = \sqrt[2]{t}, \ c(t) = \sqrt[3]{t}, \ e(t) = \sqrt[4]{t}, \ k(t) = \sqrt[5]{t}$, the obtained results of BPs are reported in Table 5 and are plotted in Fig 4. We observe that our solutions are in perfect agreement with the exact solutions.

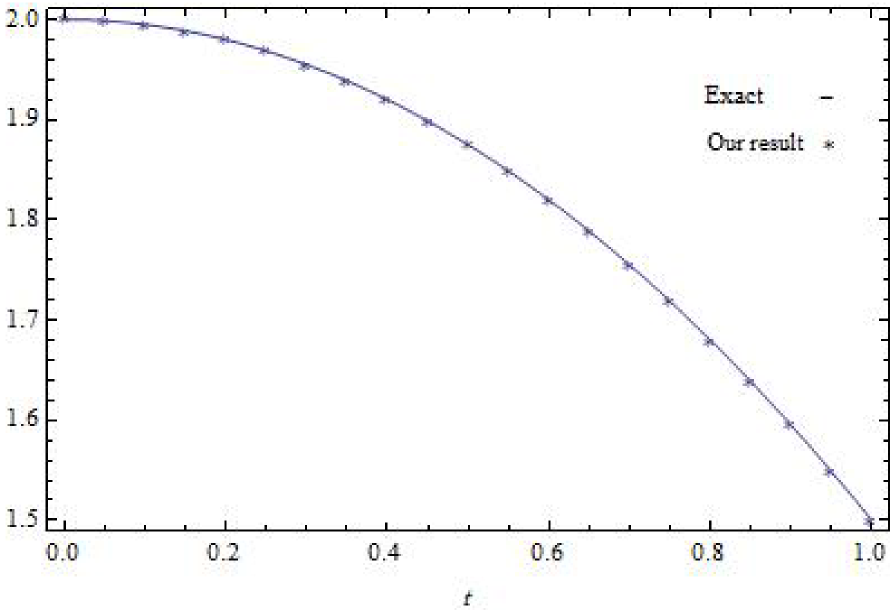

Fig. 4 – Plot of $y(t)$ and $y_{10}(t)$ in Example 7.4.

Table 5

Our results for $m = 10$ in Example 7.4

t	0.1	0.3	0.5	0.7	0.9
Absolute error	1.59646×10^{-6}	7.08929×10^{-7}	3.65487×10^{-6}	5.19327×10^{-8}	8.59565×10^{-7}

8. CONCLUSION

In this paper, we proposed a numerical solution for the linear multi-term orders fractional differential equation by the operational matrices of BPs. We get operational matrices of the product and fractional integration. Then by using these matrices, we reduced the linear multi-term orders fractional differential equation to a system of algebraic equations that can be solved easily. Finally, numerical examples are simulated to demonstrate the high performance of proposed method. We see that the obtained results are in good agreement with the existing ones in open literature and it is shown that the technique introduced here is robust, accurate and easy to apply.

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REFERENCES

1. Caputo M., *Linear models of dissipation whose Q is almost frequency independent II*, Geophys. J. Roy. Astronom. Soc., **13**, 529–539 (1967).
2. Baleanu D., Diethelm K., Scalas E. and J.J. Trujillo J.J., *Fractional Calculus Models and Numerical Methods. Series on Complexity, Nonlinearity and Chaos*, Boston, World Scientific, 2012.
3. Podlubny I., *Fractional Differential Equations*, San Diego, Academic Press, 1999.
4. Diethelm K., Ford N. J., *Numerical solution of the Bagley-Torvik equation*, *BIT*, **42**, 490–507 (2002).
5. Jafari H., Kadem A., Baleanu D. and Yılmaz T., Rom. Rep. Phys., **64**, *2*, 337–346 (2012).
6. Jarad F., Abdeljawad T., Gündoğdu E. and Baleanu D., Proc. Rom. Acad. A, **12**, 309–314 (2011).
7. Baleanu D., Golmankhaneh Alireza K. and Golmankhaneh Ali K., *Solving of the fractional non-linear and linear Schrodinger equations by homotopy perturbation method*, Rom. J. Phys., **54**, *9–10*, 823–832 (2009).
8. Daftardar-Gejji V. and Jafari H., *Solving a multi-order fractional differential equation using adomian decomposition*, J. Math. Anal. Appl., **189**, 541–548 (2007).
9. Shawagfeh N. T., *Analytical approximate solutions for nonlinear fractional differential equations*, Appl. Math. Comput., **131**, 517–529 (2002).
10. Arikoglu A. and Ozkol I., *Solution of fractional differential equations by using differential transform method*, Chaos Solit. Fract., **34**, 1473–1481 (2007).
11. El-Mesiry A., El-Sayed A. and El-Saka H., *Numerical methods for multiterm fractional (arbitrary) orders differential equations*, Appl. Math. Comput., **160**, 683–699 (2005).
12. Momani S. and Odibat Z., *Numerical comparison of methods for solving linear differential equations of fractional order*, Chaos Solit. Fract., **31**, 1248–1255 (2007).
13. Li Y. and Sun N., *Numerical solution of fractional differential equations using the generalized block pulse operational matrix*, Comput. Math. Appl., **62**, 2011, 1046–1054.
14. Luchko Y. and Gorneflo R., *The initial value problem for some fractional differential equations with the Caputo derivative*, Preprint series A08-98, Fachbereich Mathematik und Informatik, Freie Universitat Berlin, 1998.
15. Miller K. S. and Ross B., *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons, Inc., New York, 1993.
16. Oldham K.B. and Spanier J., *The Fractional Calculus*, Academic Press, New York, 1974.
17. Alipour M., Rostamy D. and Baleanu D., *Solving multi-dimensional FOCPs with inequality constraint by BPs operational matrices*, J. Vibr. Contr., in press.
18. Kreyszig E., Introduction Functional Analysis with Applications, John Wiley and Sons Incorporated, 1978.
19. Alipour M. and Rostamy D., *Bernstein polynomials for solving Abel's integral equation*, J. Math. Comput. Sci., **3**, *4*, 403–412 (2011).