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INTEGRABLE COUPLINGS OF THE BURGERS EQUATION AND THE SHARMA-TASSO-OLVER EQUATION: MULTIPLE KINK SOLUTIONS

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Abstract. In this work, we present construction of the integrable couplings of the Burgers equation (BE) and the Sharma-Tasso-Olver (TSO) equation. We use the algebra of coupled scalars to develop the two classes of couplings. The Bäcklund transformation and the simplified Hirota's method will be used to study the developed couplings. We show that these couplings possess multiple kink solutions the same as the multiple kink solutions of the BE and the STO equations, but differ only in the coefficients of the Bäcklund transformation. This difference exhibits kink solutions for some equations and anti-kink solutions for others.

Key words: Burgers couplings; Sharma-Tasso-Olver Couplings; Bäcklund transformations; multiple kink solutions.

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1. INTRODUCTION

The ubiquitous KdV equation [1–6] in dimensionless variables

$$u_t + 6uu_x + u_{xxx} = 0, \tag{1}$$

models a variety of nonlinear wave phenomena such as shallow water waves, acoustic waves in a harmonic crystal, and ion-acoustic waves in plasmas. The KdV equation admits multiple-soliton solutions and exhibits an infinite number of conservation laws of energy [7–12].

The theory of nonlinear integrable couplings of ordinary soliton systems was presented in [1–3] and further studied by many others such as in [4–6]. In [2, 3], Ma et. al proposed the perturbation method for establishing integrable couplings. Zhang et. al. [4] presented the enlarged Lie algebra method to obtain integrable couplings. Particularly noteworthy are the constructions of integrable couplings based on the non-semi simple Lie algebras [1]. In fact, there are several methods adopted to construct integrable couplings, such as perturbations, enlarging the spectral problem, creating new loop algebras, and semi direct sums of Lie algebra. It is now known that for an integrable system, we can construct a new integrable differential equation system, called integrable couplings, which includes the given integrable equation as a sub-system.

In [1], a very natural triangular nonlinear couplings of integrable systems were developed. The construction in [1] was made on the level of evolution equations by a modification of the algebra of dynamical fields. The n-coupled KdV (nc-KdV) was given in the form

$$\begin{aligned}
&(u_1)_t+(u_1)_{xxx}+6u_1(u_1)_x=0,\\
&(u_2)_t+(u_2)_{xxx}+6u_2(u_2)_x+6(u_1u_2)_x=0,\\
&(u_3)_t+(u_3)_{xxx}+6u_3(u_3)_x+6(u_1u_3)_x+6(u_2u_3)_x=0,\\
&\qquad\vdots\\
&(u_n)_t+(u_n)_{xxx}+6u_n(u_n)_x+6\sum_{k=1}^{n-1}(u_ku_n)_x=0.
\end{aligned} \tag{2}$$

The algebra of coupled scalars was introduced in [1] and was shown to be unital, commutative and associative. The integrability of the couplings (2) was examined in [1].

The viscous Burgers equation reads

$$u_t+uu_x+\nu u_{xx}=0, \tag{3}$$

where $\nu>0$ is the constant of viscosity. This is the simplest partial differential equation combining both nonlinear propagation effects uu_x and dispersive effects u_{xx}. Eq. (3) is the lowest order approximation for the one-dimensional propagation of weak shock waves in a fluid [12]. Burgers equation occurs in various areas of applied mathematics, such as modelling of gas dynamics and traffic flow.

The Sharma-Tasso-Olver (STO) equation reads

$$u_t+\alpha(u^3)_{xxx}+\frac{3}{2}\alpha(u^2)_{xx}+\alpha u_{xxx}=0. \tag{4}$$

The STO equation appears in many scientific applications. It appears as an evolution equation that possesses an infinitely many symmetries. It has been shown that this equation possesses the bi-Hamiltonian formulation. Both the BE and the STO equations are integrable and give rise to multiple kink solutions and an infinite number of conserved laws.

Many reliable methods are used in the solitary waves theory to investigate solitons, and in particular multiple soliton solutions of completely integrable equations. The algebraic-geometric method, the inverse scattering method, the Bäcklund transformation method, the Darboux transformation method, the Hirota bilinear method, and other methods are used to make progress and new developments in this filed. In this work we aim to apply the Bäcklund transformations and the simplified Hirota's method [7–18] for a reliable study.

Our aim from this work is two fold. The first goal is to employ the newly developed algebra of coupled scalars [1] to construct nonlinear integrable couplings for the Burgers equation and for the Sharma-Tasso-Olver equation. We aim second to study these couplings and show that each equation possesses multiple kink solutions the same as the BE and the STO equations, but differ only in the Bäcklund transformations. This difference exhibits kink solutions for some equations and anti-kink solutions for others. This conclusion holds for both sets of couplings.

2. COUPLINGS OF THE BURGERS EQUATION: MULTIPLE KINK SOLUTIONS

In a like manner to the approach introduced in [1], where the algebra of coupled scalars was developed, we set a one field soliton system

$$u_t = K[u] \equiv K[u, u_x, u_{xx}, u_{xxx}, \cdots], \tag{5}$$

that can be extended to the system of coupled PDEs of the form [1]

$$\mathbf{u}_t = K[\mathbf{u}] \equiv K[\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, \cdots], \tag{6}$$

where

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}. \tag{7}$$

Accordingly, the system (6) takes the form [1]

$$\begin{aligned} &(u_1)_t = K[u_1], \\ &(u_k)_t = K\left[\sum_{i=1}^{k} u_i\right] - K\left[\sum_{i=1}^{k-1} u_i\right], k = 2, 3, \cdots n. \end{aligned} \tag{8}$$

Using (3), we can set

$$\mathbf{u}_t = -\mathbf{u}\mathbf{u}_x - \nu \mathbf{u}_{xx}. \tag{9}$$

Inserting (9) into (8), we develop the n-coupled Burgers equation (nc-BE), given in the form

$$\begin{aligned}
(u_1)_t &= -\nu(u_1)_{xx} - u_1(u_1)_x, \\
(u_2)_t &= -\nu(u_2)_{xx} - u_2(u_2)_x - (u_1u_2)_x, \\
(u_3)_t &= -\nu(u_3)_{xx} - u_3(u_3)_x - ((u_1+u_2)u_3)_x, \\
(u_4)_t &= -\nu(u_4)_{xx} - u_4(u_4)_x - ((u_1+u_2+u_3)u_4)_x, \\
&\vdots \\
(u_n)_t &= -\nu(u_n)_{xx} - u_n(u_n)_x - \left[\left(\sum_{k=1}^{n-1} u_k\right) u_n\right]_x, n \geq 2.
\end{aligned} \tag{10}$$

2.1. MULTIPLE KINK SOLUTIONS

Substituting

$$u_i(x,t) = e^{k_i x - \omega_i t}, 1 \leq i \leq n, \tag{11}$$

into the linear terms of each equation in (10) gives the dispersion relation by

$$\omega_i = \nu k_i^2, \tag{12}$$

and as a result we obtain the following phase variables

$$\theta_i = k_i x - \nu k_i^2 t. \tag{13}$$

The multiple kink solutions of the couplings (10) are assumed to be

$$u_i(x,t) = R_i \ln(f(x,t))_x, \tag{14}$$

where the auxiliary function $f(x,t)$ for the single kink solution is given by

$$f(x,t) = 1 + e^{k_1 x - \nu k_1^2 t}. \tag{15}$$

Substituting (14) into (10) and solving for R_i we obtain

$$R_i = (-1)^{i+1} 2\nu, 1 \leq i \leq n. \tag{16}$$

Combining the obtained results gives the following set of single soliton solutions

$$u_i(x,t) = (-1)^{i+1} \frac{2\nu k_1 e^{k_1 x - \nu k_1^2 t}}{1 + e^{k_1 x - \nu k_1^2 t}}, 1 \leq i \leq n. \tag{17}$$

In other words we obtain kink solutions for i odd, and anti-kink solutions with the same amplitude for i even.

For the two kink solutions we set the auxiliary function by

$$f(x,t) = 1 + e^{k_1 x - \nu k_1^2 t} + e^{k_2 x - \nu k_2^2 t}. \tag{18}$$

Substituting (18) into (14) we obtain a set of kink solutions and another set of anti-kink solutions with the same amplitude, given by

$$u_i(x,t) = (-1)^{i+1}\,\frac{2\nu(k_1 e^{k_1 x-\nu k_1^2 t} + k_2 e^{k_2 x-\nu k_2^2 t})}{1+e^{k_1 x-\nu k_1^2 t}+e^{k_2 x-\nu k_2^2 t}}, 1 \le i \le n. \tag{19}$$

For the three kink solutions, we set

$$f(x,t) = 1+e^{k_1 x-\nu k_1^2 t}+e^{k_2 x-\nu k_2^2 t}+e^{k_3 x-\nu k_3^2 t}. \tag{20}$$

Proceeding as before, we find the following sets of three kink solutions and three anti-kink solutions, with the same amplitude, given by

$$u_i(x,t) = (-1)^{i+1}\,\frac{2\nu(k_1 e^{k_1 x-\nu k_1^2 t} + k_2 e^{k_2 x-\nu k_2^2 t} + k_3 e^{k_3 x-\nu k_3^2 t})}{1+e^{k_1 x-\nu k_1^2 t}+e^{k_2 x-\nu k_2^2 t}+e^{k_3 x-\nu k_3^2 t}}, 1 \le i \le n. \tag{21}$$

Generally, we can set the following generalized kink and anti-kink solutions, for $i \ge 1$,

$$u_i(x,t) = (-1)^{i+1}\,\frac{2\nu \sum_{r=1}^{i} k_r e^{k_r x-\nu k_r^2 t}}{1+\sum_{r=1}^{i} e^{k_r x-\nu k_r^2 t}}, 1 \le i \le n. \tag{22}$$

3. COUPLINGS OF THE SHARMA-TASSO-OLVER EQUATION: MULTIPLE KINK SOLUTIONS

Using the algebra of coupled scalars leads to a one field soliton system

$$u_t = K[u] \equiv K[u, u_x, u_{xx}, u_{xxx}, \cdots], \tag{23}$$

that can be extended to the system of coupled PDEs of the form [1]

$$\mathbf{u}_t = K[\mathbf{u}] \equiv K[\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, \cdots], \tag{24}$$

where

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}. \tag{25}$$

Accordingly, the system (24) takes the form [1]

$$\begin{aligned} (u_1)_t &= K[u_1], \\ (u_k)_t &= K\left[\sum_{i=1}^{k} u_i\right] - K\left[\sum_{i=1}^{k-1} u_i\right], k = 2, 3, \cdots n. \end{aligned} \tag{26}$$

Using (4), we can set

$$\mathbf{u}_t = -\alpha(\mathbf{u^3})_x - \frac{3}{2}\alpha(\mathbf{u^2})_{xx} - \alpha(\mathbf{u^3})_{xxx}. \tag{27}$$

Inserting (27) into (26), we establish the n-coupled Sharma-Tasso-Olver equation (nc-STO), given in the system

$$
\begin{aligned}
(u_1)_t &= -\alpha(u_1^3)_x - \alpha(u_1)_{xxx} - \frac{3}{2}\alpha(u_1^2)_{xx},\\
(u_2)_t &= -\alpha(u_2^3)_x - \alpha(u_2)_{xxx} - 3\alpha(u_1^2u_2 + u_1u_2^2)_x - \frac{3}{2}\alpha(u_2^2 + 2u_1u_2)_{xx},\\
(u_3)_t &= -\alpha(u_3^3)_x - \alpha(u_3)_{xxx} - 3\alpha((u_1+u_2)^2u_3 + (u_1+u_2)u_3^2)_x\\
&\quad - \frac{3}{2}\alpha(u_3^2 + 2(u_1+u_2)u_3)_{xx},\\
(u_4)_t &= -\alpha(u_4^3)_x - \alpha(u_4)_{xxx} - 3\alpha((u_1+u_2+u_3)^2u_4 + (u_1+u_2+u_3)u_4^2)_x\\
&\quad - \frac{3}{2}\alpha(u_4^2 + 2(u_1+u_2+u_3)u_4)_{xx},\\
&\;\;\vdots\\
(u_n)_t &= -\alpha(u_n^3)_x - \alpha(u_n)_{xxx} - 3\alpha\left[\left(\sum_{i=1}^{n-1}u_i\right)^2 u_n + \left(\sum_{i=1}^{n-1}u_i\right)u_n^2\right]_x\\
&\quad - \frac{3}{2}\alpha\left[u_n^2 + 2\left(\sum_{i=1}^{n-1}u_i\right)u_n\right]_{xx},\quad n \geq 2.
\end{aligned}
\tag{28}
$$

3.1. MULTIPLE KINK SOLUTIONS

Substituting

$$u_i(x,t) = e^{k_ix-\omega_it}, 1 \leq i \leq n, \tag{29}$$

into the linear terms of each equation in (28) gives the dispersion relation by

$$\omega_i = \alpha k_i^3, \tag{30}$$

and hence the following phase variables are defined by

$$\theta_i = k_ix - \alpha k_i^3 t. \tag{31}$$

The multiple kink solutions of the couplings (28) are assumed to be

$$u_i(x,t) = R_i \ln(f(x,t))_x, \tag{32}$$

where the auxiliary function $f(x,t)$ for the single kink solution IS given by

$$f(x,t) = 1 + e^{k_1x-\alpha k_1^3 t}. \tag{33}$$

Substituting (32) into (28) and solving for R_i we obtain

$$R_i = (-1)^{i+1}, 1 \leq i \leq n. \tag{34}$$

Combining the obtained results gives the following set of single soliton solutions

$$u_i(x,t) = (-1)^{i+1}\,\frac{k_1 e^{k_1 x-\alpha k_1^3 t}}{1+e^{k_1 x-\alpha k_1^3 t}}, 1 \le i \le n, \tag{35}$$

In other words we obtain kink solutions for i odd, and anti-kink solutions with the same amplitude for i even.

For the two kink solutions we set the auxiliary function by

$$f(x,t) = 1 + e^{k_1 x-\alpha k_1^3 t} + e^{k_2 x-\alpha k_2^3 t}. \tag{36}$$

Substituting (36) into (32) we obtain a set of two kink solutions and another set of two anti-kink solutions with the same amplitude, given by

$$u_i(x,t) = (-1)^{i+1}\,\frac{(k_1 e^{k_1 x-\alpha k_1^3 t} + k_2 e^{k_2 x-\alpha k_2^3 t})}{1+e^{k_1 x-\alpha k_1^3 t}+e^{k_2 x-\alpha k_2^3 t}}, 1 \le i \le n. \tag{37}$$

For the three kink solutions, we set

$$f(x,t) = 1 + e^{k_1 x-\alpha k_1^3 t} + e^{k_2 x-\alpha k_2^3 t} + e^{k_3 x-\alpha k_3^3 t}. \tag{38}$$

Proceeding as before, we find the following sets of three kink solutions and three anti-kink solutions, with the same amplitude, given by

$$u_i(x,t) = (-1)^{i+1}\,\frac{(k_1 e^{k_1 x-\alpha k_1^3 t} + k_2 e^{k_2 x-\alpha k_2^3 t} + k_3 e^{k_3 x-\alpha k_3^3 t})}{1+e^{k_1 x-\alpha k_1^3 t}+e^{k_2 x-\alpha k_2^3 t}+e^{k_3 x-\alpha k_3^3 t}}, 1 \le i \le n. \tag{39}$$

Generally, we can set the following generalized kink and anti-kink solutions, for $i \ge 1$,

$$u_i(x,t) = (-1)^{i+1}\,\frac{\sum_{r=1}^{i} k_r e^{k_r x-\alpha k_r^3 t}}{1+\sum_{r=1}^{i} e^{k_r x-\alpha k_r^3 t}}, 1 \le i \le n. \tag{40}$$

4. DISCUSSION

In this work, we constructed nonlinear integrable couplings for the viscous Burgers equation and for the Sharma-Tasso-Olver equation. The algebra of coupled scalars developed in [1] was mainly employed to achieve the goal of couplings. We derived multiple kink and multiple anti-kink solutions for the couplings of the BE and the STO equations. We showed that each equation of the two couplings possesses the same properties as the BE equation or the STO equation: the same phase variable, the same phase shift, and the same amplitude. However, the only difference is that some equations give kink solutions whereas others give anti-kink solutions. The proposed scheme is reliable and will be used for constructing other couplings of other integrable equation.

REFERENCES

1. M. Błaszak, B. Szablikowski, B. Silindir, Appl. Math. Comput **219**, 1866 (2012).
2. W.X. Ma, B. Fuchssteiner, Chaos, Solitons and Fractals **7**, 1227 (1996).
3. W.X. Ma, Z.-N. Zhu, Comput. Math. Applications **60**, 2601 (2010).
4. Y. Zhang, H. Tam, J. Math. Phys. **51**, 043510 (2010).
5. Y. Fa-Jun, L. Li, Phys. Lett. A **373**, 1632 (2009).
6. Y. Fa-Jun, Chin. Phys. B **21**(1), 010201 (2012).
7. C.M. Khalique, Filomat **26**(5), 957 (2012).
8. C.M. Khalique, A. Biswas, International Journal of Theoretical Physics **48** (11), 3110 (2009).
9. C.M. Khalique, A. Biswas, Phys. Lett. A **373**(23/24), 2047 (2009).
10. B.B. Kadomtsev, V.I. Petviashvili, Sov. Phys. Dokl. **15**, 539 (1970).
11. R. Hirota, Progress of Theoretical Physics, **52** (5), 1498 (1974).
12. A.M. Wazwaz, *Partial Differential Equations and Solitary Waves Theorem* (Springer and HEP, Berlin and Peking, 2009).
13. A.M. Wazwaz, Communications in Nonlinear Science and Numerical Simulations **14**, 2962 (2009).
14. A.M. Wazwaz, Physica Scripta **86**, 035007 (2012).
15. A.M. Wazwaz, Central European Journal of Physics **10**(4), 1013 (2012).
16. A.M. Wazwaz, Journal of the Franklin Institute **347**, 618 (2010).
17. A.M. Wazwaz, Physica Scripta **82**, 025001 (2010).
18. A.M. Wazwaz, Applied Mathematics and Computation **188**, 1205 (2007).