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# FARADAY WAVES IN CIGAR-SHAPED BOSE-EINSTEIN CONDENSATES WITH RADIALLY INHOMOGENEOUS SCATTERING LENGTHS

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*Abstract.* In this short note, we investigate by variational means the emergence of Faraday waves in cigar-shaped Bose-Einstein condensates with radially inhomogeneous scattering lengths subject to periodic modulations of the radial confining potential. Our main result is that for weakly inhomogeneous scattering lengths, the Faraday waves follow a dispersion relation identical to that observed in condensates with spatially homogeneous scattering lengths.

*Key words*: Bose-Einstein condensates, density waves.

Pattern formation in quantum fluids outside dynamic equilibrium is a very active research topic (see Refs. [1–3] and references therein) which attracts experimental and theoretical physicists, nonlinear dynamicists and computer scientists. It has a long-honoured history in systems as different as bacterial colonies, chemical reactions, granular materials, Newtonian and non-Newtonian fluids (see Ref. [4] for a substantial review), but Bose-Einstein condensates (BECs) [5] now emerge as the ideal play ground for nonlinear phenomena in general and pattern formation in particular due to the large flexibility of the experimental setups and the accuracy of the mean field description governed by the Gross-Pitaevskii equation (see Ref. [6] for efficient numerical recipes) that describes the dynamics of the condensates in the vicinity of $T = 0$ K. A similar equation was used in nonlinear optics in order to study a wide family of nonlinear localized optical waveforms in various physical settings (see, *e.g.*, a few recent overviews of current theoretical and experimental activity in this fast growing area [7, 8]). Complementary investigations have been focused on the dynamics of quantum gases using fast converging path integrals and exact diagonalization of the space-discretized evolution operator [9].

Most experimental investigations are done in approximately one- and two-

dimensional setups where the physical observables are most clear and BECs are conveniently described by effectively one- and two-dimensional equations [10]. Pattern formation, in particular, has been proposed in a two-dimensional setting through periodic modulations of the two-body scattering length, but has been observed experimentally in a one-dimensional setup where the modulation was on the strength of the radial component of the magnetic trap [2]. In both setups, the numerically-predicted and experimentally-observed patterns were showing density oscillations on a frequency equal to half the frequency of the drive, therefore representing Faraday patterns [11]. Similar patterns have been observed in binary BECs [12], BECs with dipolar (*i.e.*, long range) interactions [13], ultra-cold Fermi gases [14] and mixture of bosons and fermions [15]. Complementary, it has been shown that by adding a longitudinal optical lattice potential in a quasi-one-dimensional setup, one can suppress the emergence of Faraday waves [16].

In this short communication, we show by variational means [17] that the dispersion relation of Faraday waves in longitudinally-homogeneous quasi-one-dimensional cigar-shaped BECs subject to radially anisotropic two-body interactions is identical to that of Faraday waves in condensates with homogeneous two-body interactions. To this end, we use the standard Gross-Pitaevskii (GP) Lagrangian density (with $\hbar = m = 1$) of a cylindrically-symmetric condensate [5, ch. 7], namely

$$\mathcal{L}[\psi; r, z, t] = \frac{i}{2}\left(\psi\frac{\partial\psi^*}{\partial t} - \psi^*\frac{\partial\psi}{\partial t}\right) + \frac{1}{2}\left|\nabla\psi\right|^2 + +V(\mathbf{r},t)\left|\psi\right|^2 + \frac{g(r)N}{2}\left|\psi\right|^4, \quad (1)$$

where $r^2 = x^2 + y^2$, $\psi = \psi(\mathbf{r},t)$, $V(\mathbf{r},t) = 1/2 \cdot \Omega^2(t) r^2$ and

$$g(r) = g_0 \exp\left(-\frac{r^2}{2b^2}\right), \quad (2)$$

using the hybrid trial wave function

$$\psi = f\left[k, w(t), u(t), v(t)\right] \exp\left[-\frac{r^2}{2w^2(t)} + ir^2\alpha(t)\right]\left[1 + (u(t) + iv(t))\cos kz\right]. \quad (3)$$

The wave function accounts for the bulk part of the condensate through a radial Gaussian envelope of width $w(t)$ and includes a cosine-like longitudinal density wave of wave number $k$. Imposing the normalization of the wave function over one period of the density waves, we have

$$f\left[k, w(t), u(t), v(t)\right] = \frac{1}{\pi w(t)}\sqrt{\frac{k}{2 + u^2(t) + v^2(t)}}. \quad (4)$$

Following some simple algebra, one finds that

$$L = \frac{1}{2}\left\{ \frac{b^2 g_0 \rho \left(8+3u(t)^2+8v(t)^2+3v(t)^4+6u(t)^2\left(4+v(t)^2\right)\right)}{\pi\left(2+u(t)^2+v(t)^2\right)^2 w(t)^2\left(4b^2+w(t)^2\right)} \right.$$
$$+ \frac{k^2\left(u(t)^2+v(t)^2\right)}{\left(2+u(t)^2+v(t)^2\right)} + \frac{1}{w(t)^2} + 4w(t)^2\alpha(t)^2 \tag{5}$$
$$\left. + w(t)^2\Omega(t)^2 + 2w(t)^2\dot{\alpha}(t) + \frac{2\left(u(t)\dot{v}(t)-v(t)\dot{u}(t)\right)}{2+u(t)^2+v(t)^2} \right\},$$

which immediately yields the Euler-Lagrange equations for $u(t)$, $v(t)$, $w(t)$ and $\alpha(t)$, namely

$$\dot{w}(t) = 2w(t)\alpha(t), \tag{6}$$

$$\dot{\alpha}(t) = \frac{1}{2w(t)^4} + \frac{b^2 g_0\rho}{\pi w(t)^2\left(4b^2+w(t)^2\right)^2} + \frac{b^2 g_0\rho}{\pi w(t)^4\left(4b^2+w(t)^2\right)} - 2\alpha(t)^2 - \frac{\Omega(t)^2}{2}, \tag{7}$$

$$\dot{u}(t) = \frac{1}{2}k^2 v(t), \tag{8}$$

$$\dot{v}(t) = -u(t)\left[\frac{2b^2k^2}{4b^2+w(t)^2} + \frac{4b^2 g_0\rho}{\pi w(t)^2\left(4b^2+w(t)^2\right)} + \frac{k^2 w(t)^2}{2\left(4b^2+w(t)^2\right)}\right]. \tag{9}$$

Equations (6) and (7) can be written under the convenient form of a nonlinear driven harmonic oscillator, namely

$$\ddot{w}(t) = \frac{1}{w(t)^3}\left[1 + \frac{4b^2 g_0\rho\left(2b^2+w(t)^2\right)}{\pi\left(4b^2+w(t)^2\right)^2} - w(t)^4\Omega(t)^2\right], \tag{10}$$

which indicates that the "equilibrium" [18] radial width of the condensate (*i.e.*, $\ddot{w}(t) = 0$) is given for large values of $b$ by

$$w^* = \frac{2b\left(2\pi+g_0\rho\right)^{1/4}}{\left(g_0\rho+32b^4\pi\Omega(t)\right)^{1/4}}, \tag{11}$$

where the corrections are of the order $1/b^5$ and higher. Considering now that $\Omega(t) = \Omega\cdot(1+\epsilon\sin\omega t)$ and using the equilibrium width of the condensate, we can recast equations (8) and (9) as

$$\ddot{u}(\tau) = -\frac{k^2}{\omega^2}\left(k^2 + \frac{8b^2 g_0\rho}{4b^2\pi w^{*2}+\pi w^{*4}}\right)u(\tau), \tag{12}$$

where $\omega t = 2\tau$. This last equation represents the parametric equation which governs the emergence of Faraday waves in cigar-shaped condensates with radially inhomogeneous scattering lengths. Expanding the right hand term in equation (12) in series

of $\epsilon$, we arrive at the general Hill equation

$$\ddot{u}(t) = \frac{k^2}{\omega^2}\left[\left(k^2 + \frac{2g_0\rho\Omega}{\sqrt{\pi^2 + \frac{\rho g_0 \pi}{2}}}\right) + \epsilon\frac{2g_0\rho\Omega\sin 2\tau}{\sqrt{\pi^2 + \frac{\rho g_0 \pi}{2}}} + \epsilon^2 T(\sin 2\tau)^2 + \mathcal{O}(\epsilon^3)\right], \quad (13)$$

where

$$T = \frac{8b^2 g_0\rho\,(2\pi + \rho g_0)\,\Omega^2 E\left[4\pi\sqrt{E} + 3g_0\rho\left(\sqrt{2\pi + g_0\rho} + \sqrt{E}\right)\right]}{\left[2\pi\left(16b^4\Omega^2\sqrt{2\pi + g_0\rho} + \sqrt{E}\right) + g_0\rho\left(\sqrt{2\pi + g_0\rho} + \sqrt{E}\right)\right]^3} \quad (14)$$

and $E = g_0\rho + 32b^4\pi\Omega^2$. Please note that $T$ (and, in fact, all the prefactors of terms of the order $\mathcal{O}(\epsilon^3)$) vanish in the limit $b \to \infty$. Provided that $\epsilon \ll 1$, it has been show that the most unstable solution of the previous Hill equation (13) corresponds to (see Ref. [19])

$$\frac{k^2}{\omega^2}\left(k^2 + \frac{2g_0\rho\Omega}{\sqrt{\pi^2 + \frac{\rho g_0 \pi}{2}}}\right) = 1,$$

which is precisely the dispersion relation of Faraday waves that was derived in Ref. [20] (see, in particular, equation (15)). This is the main result of our paper and it indicates that for weakly inhomogeneous scattering lengths, *i.e.*, $b$ sufficiently large such that the approximate solution in equation (11) holds, the dispersion relation of the Faraday waves is identical to that in condensates with homogeneous scattering lengths.

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