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THEORETICAL STUDIES OF ULTRASHORT-SOLITON PROPAGATION IN NONLINEAR OPTICAL MEDIA FROM A GENERAL QUANTUM MODEL

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Abstract. We overview some recent theoretical studies of dynamical models beyond the framework of slowly varying envelope approximation, which adequately describe ultrashort-soliton propagation in nonlinear optical media. A general quantum model involving an arbitrary number of energy levels is considered. Model equations derived by rigorous application of the reductive perturbation formalism are presented, assuming that all transition frequencies of the nonlinear medium are either well above or well below the typical wave frequency. We briefly overview (a) the derivation of a modified Korteweg-de Vries equation describing the dynamics of few-cycle solitons in a centrosymmetric nonlinear optical Kerr (cubic) type material, (b) the analysis of a coupled system of Korteweg-de Vries equations describing ultrashort-soliton propagation in quadratic media, and (c) the derivation of a generalized double-sine-Gordon equation describing the dynamics of few-cycle solitons in a generic optical medium. The significance of the obtained results is discussed in detail.

Key words: few-cycle pulses, ultrashort solitons.

1. INTRODUCTION

Over the past three decades, the field of optical solitons (temporal, spatial and spatiotemporal ones) and related nonlinear optical phenomena has been substantially advanced; see, for example, two comprehensive historical overviews [1, 2] and a list of several relevant works in this extremely broad research area [3]-[56]. As is well known, a *temporal optical soliton* is a pulse that propagates in a dispersive medium in such a way that a nonlinear optical effect (self-phase modulation) exactly compensates the group-velocity dispersion; this exact balance of two counteracting physical phenomena implies that the pulse remains unchanged during propagation. A *spatial optical soliton* is a self-trapped light beam; the spreading of the optical beam due to diffraction is compensated by a self-induced lensing effect due to intensity-dependent refractive index change associated with the optical Kerr effect. A *spatiotemporal optical soliton* (alias a nonlinear ”light bullet”) is a spatially confined light pulse, *i.e.*, an optical wave packet that is self-trapped in both space and time physical dimensions. We only would like to point out the continuous theoretical and experimental activity in the area of nonlinear ”light bullets”; see the early seminal works [57]-[62] and a few recent review papers [2, 6, 40]. The formation of fully three-dimensional spatiotemporal optical solitons in two-dimensional photonic lattices was reported in recent quite complex experiments [63]-[65]. It is believed that such three-dimensional spatiotemporal optical solitons could be used as information carriers (in fact they are ideal information units) in future high-speed all-optical information processing systems for both serial and parallel data processing and transmission, see *e.g.*, recent works [66]-[67]. Actually, such nonlinear light bullets offer potential for ultrafast digital optical logic devices with switching rates of several 10^{12} Hz, *i.e.*, with terahertz switching speeds.

The nonlinear self-trapped spatiotemporal wave packets, in both conservative and dissipative media, have been extensively studied during the past few decades in diverse areas of research such as nonlinear optics and photonics, plasmas, condensed matter physics, fluid mechanics, physics of elementary particles, astro-particle physics, atomic or molecular Bose-Einstein condensate, and biology and medicine [68]-[69]. With the rapid advances in the creation of ultrashort optical pulses (only a few cycles long), the study of propagation of such pulses become a matter of intensive research over the past decade. It is worth noting that several pioneering papers on experimental generation and characterization of ultrashort optical pulses (two-cycle or even sub-two-cycle pulses) from Kerr-lens mode-locked Ti:sapphire lasers were published in 1999 by a few groups from USA, Europe, and Japan [70]-[73]. Since then the interest in intense ultrashort light pulses containing only a few optical cycles has steadily grown. Such ultrashort optical pulses possess extensive applications to the field of light-matter interactions, high-order harmonic generation, extreme [74] and single-cycle [75] nonlinear optics, materials science and processing (*e.g.*, femtosecond laser ablation) [76], and attosecond physics [77, 78]; see Ref. [79] for a review of earlier works in this exciting field. The shortest pulses that can be directly generated with modern lasers by using Kerr-lens mode-locking techniques can have a duration of about 5 fs, thus they have a spatial length of about 1500 nm in air (vacuum). Such ultrashort pulses have a large spectral bandwidth; a pulse with duration of 10 fs has a spectral bandwidth of the order of 30 THz, *i.e.*, the product between pulse duration and spectral bandwidth (the time-bandwidth product) is about 0.3. In order to obtain shorter pulses, the process of high harmonic generation is currently used; this technique allows the formation of either single (isolated) attosecond pulses or attosecond pulse trains. During the past few years, attosecond pulses with durations below 100 as (of the order of one atomic unit of time) were generated by different research groups [75, 80, 81]. Note that one atomic unit of time (the time scale of electron motion in atoms), is about 24 as.

Ultrashort laser pulses with duration of only a few optical cycles are currently used to study chemical reactions, molecular vibrations, electron motion in atoms and molecules, etc. The availability of ultrashort and ultraintense laser pulses generated by the powerful technique of *chirped pulse amplification* [82] along with the development of high-fluence laser materials has opened up the field of relativistic optics [83]. Recent activity in the area of realization of future large laser facilities, namely *exawatt-class lasers* was overviewed by Mourou and Tajima [84]. Note that such huge power levels will be obtained by releasing a few kilojoules of energy into an ultrashort pulse with a duration of only 10 fs. It is worth noting that the possibility to increase the laser peak powers relies on three revolutionary experimental achievements. First, an efficient laser amplification technique, namely the chirped pulse amplification which had a great influence on a variety of laser applications, was introduced in 1985 by Strickland and Mourou [82]. A second important advance in this area has been the *optical parametric chirped pulse amplification* introduced in 1992 by Dubietis et al., see Ref. [85]. It is conceptually similar to the chirped pulsed amplification, however, it relies on the parametric amplification of light. A third important amplification technique, which was introduced in 1999 by Malkin et al. [86], is a new compression technique based on *backwards Raman scattering* and has the advantage of avoiding diffraction gratings. Mourou *et al.* [87] recently put forward a new amplification technique, the so-called *cascaded conversion compression*, which has the capability to compress with good efficiency nanosecond laser pulses with energy of about 10 kJ into femtosecond pulses having the same energy. Thus it is hoped that exawatt-zettawatt peak powers might be reachable within the next period of time. The possibility of generating zeptosecond and even yoctosecond electromagnetic pulses that may allow one to operate on nuclear as well as quark-gluon plasma time scales has been recently investigated by Kaplan [88]. We also mention a recent brief overview of different types of experiments in *nuclear photonics* and related areas by using the planned Extreme Light Infrastructure-Nuclear Physics (ELI-NP) facility [89]. These many theoretical and experimental developments in the fast developing area of ultrashort and ultraintense laser pulses are very promising both from the fundamental and the application points of view.

The continuing experimental progress in the study of the wave dynamics of few-cycle pulses (FCPs) in different kinds of nonlinear optical media has paved the way for the development of new theoretical approaches to model their propagation in a series of physical settings. It is worth noting that three main classes of dynamical models for FCPs have been put forward in the literature: (i) the quantum approach [90]-[91], (ii) the refinements within the framework of slowly varying envelope approximation (SVEA) of the nonlinear Schrödinger-type envelope equations [92]-[94], and (iii) the non-SVEA dynamical models [95]-[102]. Extremely short pulses can be described by solving directly the Maxwell-Bloch equations for a two-level system; soliton (sech-type) solutions have been derived in Ref. [103]. The propagation of FCPs in Kerr media can be described beyond the SVEA by using the modified Korteweg-de Vries (mKdV) [97]-[98], sine-Gordon (sG) [99]-[100], or mKdV-sG equations [101]. The nonlinear propagation of FCPs in quadratic optical media can be described by means of a Korteweg-de Vries (KdV) model, which can be rigorously derived by means of the reductive perturbation method [104]. Recent relevant works on FCPs deal with few-cycle light bullets created by femtosecond filaments [105], the study of ultrashort spatiotemporal optical solitons in quadratic nonlinear media [106,107], the ultrashort spatiotemporal optical pulse propagation in cubic (Kerr-like) media without the use of SVEA [108,109], single-cycle gap solitons generated in resonant two-level dense media with a subwavelength structure [110], and the possibility of generating few-cycle dissipative optical solitons [111–113]. A comprehensive theoretical study of the generation of single-cycle pulses from a passively mode-locked laser with inhomogeneously broadened active medium was reported by Kozlov and Rosanov [114]. In a recent work [115] it was demonstrated the compression of 35 fs pulses down to a duration of 3.8 fs in a single femtosecond filament; this ultrashort pulse corresponds to sub-1.5 optical cycles of the electric field. Half-optical-cycle damped solitons in quadratic nonlinear media have been also investigated [116]. By using a classical model of the radiation-matter interaction, it was shown in Ref. [116] that the propagation of (1+1)-dimensional few-optical-cycle pulses in quadratic nonlinear media, taking moderate absorption into account, can be adequately described, in the long-wave approximation regime, by a generic Korteweg-de Vries-Burgers equation without using the SVEA. An efficient approach to obtain soliton compression to few-cycle pulses with a high quality factor by engineering cascaded quadratic nonlinearities was advanced in Ref. [117]. Numerical results reported in Ref. [117] show that compressed pulses with less than three-cycle duration can be achieved. It is worth noting that in contrast to standard soliton compression, these compressed pulses have minimal pedestal and high quality factor, see Ref. [117] for more details. Propagation of subcycle pulses (containing less than a single period of oscillations of electric field) in a two-level medium, beyond both SVEA and rotating-wave approximation, was investigated in Ref. [118]. It was shown that for such ultrashort pulses, a breakdown of the area theorem occurs for pulses of large enough area. Such deviations from the area theorem appear to be strongly dependent on the pulse shape and cannot be observed for longer few-cycle pulses, see Ref. [118] for a detailed study of these important issues. The phenomenon of self-focusing of femtosecond surface plasmon polaritons was investigated numerically by Pusch *et al.* [119] by using the finite-difference time-domain method. It was put forward the effect of self-focusing of plasmon pulses in the case of defocusing Kerr-like nonlinearity of the dielectric medium due to normal group-velocity dispersion [119]; the self-focusing effect is similar to the corresponding one that occurs in optical fibers [3].

The propagation of few-cycle optical solitons in nonlinear dispersive media with anomalous dispersion and cubic nonlinearity was described by Amiranashvili *et al.* in a recent work [55]. The nonenvelope ultrashort solitons were obtained numerically using the so-called spectral renormalization method originally developed for envelope solitons; for details about this powerful numerical method, see [55]. In a recent work, Drozdov *et al.* [120] performed a comprehensive study of self-phase modulation and frequency generation with few-cycle optical pulses in nonlinear dispersive media. A detailed study of ultrashort pulses and short-pulse equations in (2+1) dimensions was also performed by Shen *et al.* [121]. Kolesik *et al.* [122] quantified the limits of unidirectional ultrashort optical pulse propagation, explored the limits of unidirectional pulse propagation equation in general nonlinear media, and investigated under which physical conditions two-way propagation becomes significant, and leads to a breakdown of unidirectional approximation. Whalen *et al.* [123] studied optical shock and blow-up of ultrashort pulses in transparent media and examined various ultrashort pulse propagation models and their relative effectiveness in explaining these phenomena. The propagation of few-cycle pulses inside nonlinear Kerr media and the generation of odd-order harmonics in these media with cubic (Kerr) nonlinearity have been investigated in a recent work [124]. The formation of robust ultrashort (only two-cycles long) spatiotemporal optical solitons in carbon nanotube arrays has also been considered in a recent paper [47]. A short-wave approximation was used in order to derive a generic two-dimensional sine-Gordon equation, describing ultrashort-soliton evolution in such nanomaterials. The governing model was derived by using a rigorous application of the multiscale analysis for the Maxwell equations and for the corresponding Boltzmann kinetic equation for the distribution function of electrons in arrays of carbon nanotubes. Diffractionless and dispersionless robust propagation over a few millimeters (thus the propagation distance is very long with respect to the wavelength) of two-cycle spatiotemporal optical solitons in the form of two-dimensional breathers was reported in Ref. [47].

Next we draw the reader's attention to some earlier theoretical studies performed beyond the SVEA. Thus to the best of our knowledge, the necessity of using the non-SVEA approach for the adequate description of FCPs was put forward in a paper by Akhmediev, Mel'nikov and Nazarkin published in 1989 [125]. In a subsequent work, Belenov and Nazarkin [95] obtained exact solutions of nonlinear Maxwell-Bloch equations, outside the approximation of slowly varying amplitudes and phases, which describe ultrashort light pulses (only a few wavelengths long) for very large power densities. We also mention that Farnum and Nathan Kutz [126], in a comprehensive study of ultrafast pulse propagation in mode-locked laser cavities, clearly stated that the standard approach based on nonlinear Schrödinger-type envelope equations should be abandoned in the study of few femtosecond pulses/solitons. However, to the best of our knowledge, the necessity of using models beyond the traditional SVEA in describing the phenomenon of *self-induced transparency* for ultrashort pulses propagating in optical media was advanced in 1987 by Kujawski [127]; see also the subsequent works by Andreev [128] and by Parkhomenko and Sazonov [129].

Most of the past research interest in the study of propagation of FCPs was confined to a generic two-level atomic system, which is a very simple and a rather academic model. However, a more realistic description should take into account an arbitrary number of atomic levels. Basically, the study of FCPs dynamics in a multilevel system, which might be, in our opinion, the most relevant correction to the two-level model is an exciting and important task. In this situation, we expect to get rather complicated evolution equations for FCPs when more than two levels are involved. Accordingly, a general Hamiltonian, with an arbitrary number of energy levels is used to describe the dynamics of atoms in the considered optical medium. In a recent work [56] we have reviewed several models of few-cycle optical solitons beyond the SVEA. We systematically used the powerful *reductive expansion method* in order to derive rather simple either integrable or nonintegrable evolution models describing both nonlinear wave propagation and interaction of ultrashort optical solitons. We concentrated in Ref. [56] on the adequate description of a collection of two-level atoms and we performed the multiple-scale analysis on the Maxwell-Bloch equations and the corresponding Schrödinger-von Neumann equation for the corresponding density matrix. However, in a series of recent papers [48]-[50] some of the main results concerning the systematic use of the reductive expansion method beyond the SVEA in the simplest case of two-level atoms were relatively easily extended to few-cycle optical solitons in media described by a generic atomic Hamiltonian, *i.e.*, we considered a general quantum model involving an arbitrary number of energy levels.

In the present work we intend to briefly overview those recent results concerning the derivation and the analysis of different non-SVEA models, which adequately describe ultrashort-soliton propagation in nonlinear optical media by using a general atomic Hamiltonian, with an arbitrary number of energy levels. In the presentation of both theoretical models and numerical results, we chiefly focus on those models which are closest to current experiments, *i.e.*, the ones with $\chi^{(2)}$ and $\chi^{(3)}$ nonlinearities. More precisely, we consider both centrosymmetric nonlinear optical Kerr (cubic) media and noncentrosymmetric quadratically nonlinear optical media, see [48]-[50]. It is also relevant to mention that the governing dynamical equations are derived using the reductive perturbation method which is a very powerful way of

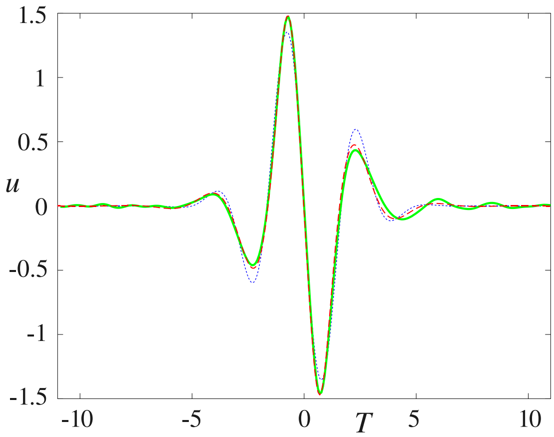

Fig. 1 – Propagation of a FCP according to the mKdV equation. Blue dotted line: the input with Gaussian envelope. Green thick solid line: the FCP soliton observed after some propagation distance ($Z = 79.72$). Dashed red line: fit of the output soliton by the analytic mKdV breather (after [48]).

obtaining simplified models describing nonlinear wave propagation and interaction.

This paper is organized as follows. In the next Section we briefly discuss the derivation of a mKdV model for ultrashort-soliton propagation (few-cycles long) from a general Hamiltonian, see Ref. [48]. In Sect. 3 we briefly consider a coupled system of KdV equations describing ultrashort soliton propagation in quadratic media by using a general Hamiltonian [49] and the reduction of that cumbersome system to a single KdV equation for linear polarizations in the degenerate case (when the two possible linear polarizations have the same refraction index n). Then, in Sect. 4 we obtain and analyze a generalized double-sine-Gordon equation describing ultrashort-soliton propagation in a generic optical medium [50]. Finally, in Sec. 5 we present our conclusions and we give a brief discussion of a few open problems in this research area.

2. MODIFIED KORTEWEG-DE VRIES MODEL FOR DESCRIBING ULTRASHORT-SOLITON PROPAGATION BY USING A GENERAL HAMILTONIAN: CUBIC (KERR) NONLINEAR OPTICAL MEDIA

We consider the electromagnetic wave equation for a set of identical atoms. The evolution of the electric field $\vec{E}$ is governed by the equation

$$\Delta\vec{E} - \nabla\left(\nabla\cdot\vec{E}\right) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\vec{E} + \frac{1}{\varepsilon_0}\vec{P}\right), \tag{1}$$

where c is the light velocity in vacuum, Δ is the Laplacian operator, ε_0 is the dielectric permittivity of vacuum, and $\vec{P}$ is the polarization density. The considered medium consists of an assembly of identical atoms with Hamiltonian H_0, and the density matrix will be denoted by ρ. The atomic Hamiltonian H_0 in the general case is given in diagonal form

$$H_0 = \hbar \mathrm{diag}(\omega_1, \omega_2, \ldots, \omega_N). \tag{2}$$

In order to simplify notations, we assume a finite number N of non-degenerated energy levels. The light-matter coupling is described by

$$\vec{P} = \mathcal{N} \mathrm{Tr}\,(\rho \vec{\mu}), \tag{3}$$

where $\vec{\mu}$ is the dipolar momentum operator and $\mathcal{N}$ is the atomic density. The evolution of the density matrix ρ is determined by

$$i\hbar \frac{\partial \rho}{\partial t} = [H, \rho], \tag{4}$$

so that the total Hamiltonian is

$$H = H_0 - \vec{\mu} \cdot \vec{E}. \tag{5}$$

We consider a cubic (Kerr) nonlinear optical medium, *i.e.*, we assume that the material is centrosymmetric, so that the second order susceptibility $\chi^{(2)}$ vanishes. For the sake of simplicity, we will assume a linearly polarized wave. Thus $\vec{E}$ and $\vec{\mu}$ in Eq. (5) are replaced by scalar quantities E and μ, in which the matrix $\mu = (\mu_{nm})_{(n,m)\in[1,N]}$ is Hermitian, *i.e.*, $\mu_{mn} = \mu_{nm}^*$, where the star denotes the complex conjugate. Due to centrosymmetry of the medium, since μ_{mn} are the matrix elements of an odd operator, the matrix μ is off-diagonal. Next we assume that the characteristic pulse frequency ω_w has the same order of magnitude as the inverse of the pulse duration $1/t_w$, and is very small with respect to any resonance frequency $\Omega_{nm} = \omega_n - \omega_m$ in the atomic spectrum, *i.e.*, $1/t_w \sim \omega_w \ll \Omega_{nm}$, for all n and m. This motivates the use of the *long-wave approximation* [56] and the introduction of the slow variables $\tau = \varepsilon(t - z/V)$, and $\zeta = \varepsilon^3 z$, where $\varepsilon \ll 1$ is the small perturbation parameter used in the multiscale analysis [130]. By performing a multiscale analysis up to the third-order in the small parameter ε we derived in Ref. [48] a modified Korteweg-de Vries equation as a rigorous formal asymptotics of the Maxwell-Bloch equations for the most general Hamiltonian. This generic evolution equation describes the propagation of ultrashort (only a few femtosecond long) optical solitons in the so-called long-wave regime. It is well known that the generic mKdV equation is completely integrable by means of the inverse scattering transform [131]. The most general N-soliton solution has been obtained by Hirota [132]. It is worth noting that the *breather solution* of the mKdV equation, which is a particular case of the general two-soliton solution, is a prototype exact solution of a few-cycle soliton.

The mKdV equation can be written in its dimensionless form as [48]

$$\partial_Z u + 2\partial_T u^3 + \sigma \partial_T^3 u = 0, \tag{6}$$

where $\sigma = \pm 1$, u is a dimensionless electric field, and Z and T are dimensionless space and time variables

$$u = \frac{E}{E_0} \quad , \quad Z = \frac{z}{L} \quad , \quad T = \frac{t - z/V}{t_w}. \tag{7}$$

The reference time is thus chosen to be the pulse length t_w (in physical units). The atomic resonance frequencies Ω_{nm} have been chosen above as zero order quantities in the perturbative scheme, while t_w is assumed to be large, of order $1/\varepsilon$, with respect to the zero order times $1/\Omega_{nm}$, see Ref. [48] for more details concerning the characteristic electric field E_0, the characteristic propagation distance L and their dependence on the third-order susceptibility $\chi^{(3)}$ and on $n_0'' = \left.\frac{d^2 n_0}{d\omega^2}\right|_{\omega=0}$.

A typical example of ultrashort-soliton propagation is shown in Fig. 1. It is worth noting that the numerical simulation was run until $Z = 80$. We see that the input two-cycle pulse evolves with very few changes in shape and width. We chosed the propagation distance $Z = 79.72$ at which the carrier-envelope phase of the final FCP is the same as the initial one, moved to the initial position and we plotted it in Fig. 1 (green thick solid line) for comparison. A fit with the analytic mKdV breather, *i.e.*, with a special two-soliton solution of the mKdV equation [48] is also shown (dashed red line); note that the fit is very close to the numerical result.

3. COUPLED SYSTEM OF KORTEWEG-DE VRIES EQUATIONS FOR DESCRIBING ULTRASHORT-SOLITON PROPAGATION BY USING A GENERAL HAMILTONIAN: QUADRATIC NONLINEAR OPTICAL MEDIA

In Ref. [49] we considered the propagation of ultrashort solitons in quadratically nonlinear optical media described by a general Hamiltonian. We derived a coupled system of KdV equations describing ultrashort soliton evolution in such materials, by using a long-wave approximation and a rigorous application of the reductive perturbation formalism. We studied linear eigenpolarizations in the degenerate case and the formation of ultrashort (half-cycle) solitons from few-cycle inputs. We assumed in Ref. [49] that the absorption spectrum of the medium does not extend below some cutoff frequency, and that the typical frequency of the FCP is much less than that cutoff frequency. Thus we consider that the transparency range of the medium is very large, and we take into account only the frequencies located in the ultraviolet domain and further. It is worth noting that the effect of the infrared transitions, which yield a sine-Gordon model in the simplest case of two-level atoms and cubic nonlinearity, was considered in Ref. [50], see the next Section. Thus in Ref.

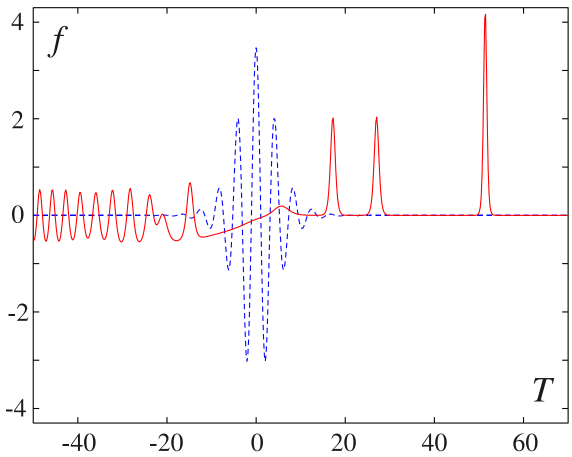

Fig. 2 – (Color online) Wave profile showing the formation of three unipolar half-cycle solitons from a FCP input. Dashed line is the FCP input ($Z=0$), solid line is output ($Z=3$). The dimensionless parameters are: amplitude $\mathcal{A}=3.5$, angular frequency $\omega=1.5$, carrier-envelope phase $\psi=0$, and duration $T_0=5.33$ (after [49]).

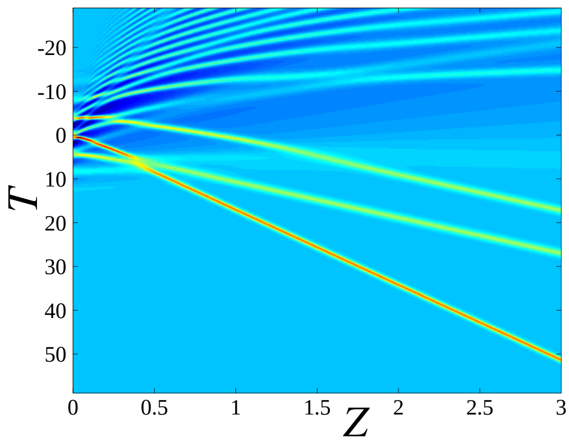

Fig. 3 – (Color online) Evolution of a FCP input showing the formation of three unipolar half-cycle solitons (after [49]).

[49] we considered in detail the case of quadratic nonlinear optical media, *i.e.*, we assumed a non-centrosymmetric optical medium with a nonvanishing second-order susceptibility $\chi^{(2)}$. We worked in the so-called *long-wave regime* and we performed the multiscale analysis [130] order by order. In this specific situation the natural slow variables are as follows: $\tau = \varepsilon(t - z/V)$, and $\zeta = \varepsilon^3 t$. Let us fix some propagation direction. If the two values of the refractive index corresponding to both possible polarization directions are distinct (we referred to this situation as the non-degenerate case), then the propagation of ultrashort solitons in quadratic media, assuming a general Hamiltonian with an arbitrary number of energy levels, is described by a KdV equation [49].

Rather than specifying some particular quadratic nonlinear optical medium, we considered in Ref. [49] the solution of the KdV equation in its standard dimensionless form:

$$f_Z + 6f\partial_T f + \partial_T^3 f = 0, \tag{8}$$

We recall that the well-known soliton of the KdV equation (8) is given by

$$f = 2p^2 \text{sech}^2\left(pT - 4p^3 Z\right), \tag{9}$$

where p is an arbitrary parameter.

The KdV equation can be solved by means of the *inverse scattering transform*; it is well known that any input evolves into a finite number of solitons, plus some dispersive wave, called ‘radiation’ in the terminology of the rigorous mathematical theory of solitons. We have solved numerically the KdV equation (8) by means of the modified Euler exponential time differencing scheme, starting from an input in the form of a FCP, as

$$f = \mathcal{A}\cos(\omega T + \psi)\,\text{sech}^2(T/T_0). \tag{10}$$

Here the reference time is $T_0 = 0.8\,\text{fs}$, so that a dimensionless angular frequency $\omega = 1.5$ corresponds to a wavelength very close to 1 μm, see Ref. [49] for more details of this numerical study. As expected from the general theory, the input FCP evolves into a few solitons, plus a dispersive wave. The number of solitons depends on the amplitude and duration of the initial pulse, and on the carrier envelope phase. A typical example is shown in Figs. 2 and 3, which show the formation of three solitons, a larger one and two smaller ones.

In Ref. [49] we also analyzed in detail the degenerate case, *i.e.*, when we consider that the two possible linear polarizations propagating in the chosen direction have the same refractive index n. We derived a cumbersome coupled system of KdV equations, which describes ultrashort-soliton propagation in quadratic media, when both polarization components interact, still assuming a general N-level Hamiltonian [49]. In the particular case when the propagation axis z is one of the eigenaxes of the optical medium, the degeneracy conditions imply that the crystal is uniaxial and that

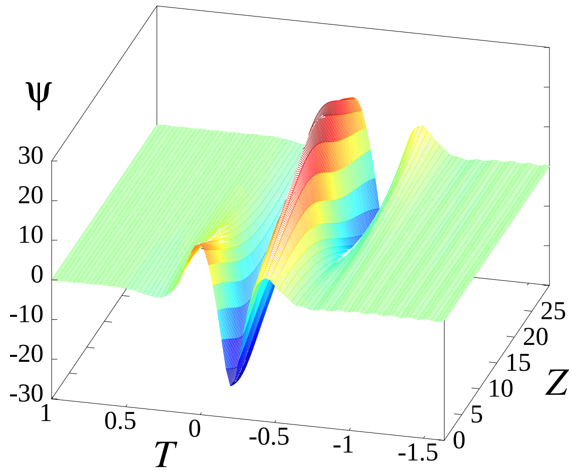

Fig. 4 – (Color online) One period of a breather of the generalized double-sine-Gordon equation, for $\lambda = \sqrt{3}$ and $q = 0.4$ (after [50]).

the optical axis is the propagation one. For the linear eigenpolarizations, the coupled KdV system reduces to a single KdV equation as in the non-degenerate case, and a set of half-cycle solitons will form from few-cycle inputs, too [49].

In conclusion, in Ref. [49] we have investigated two typical situations: (a) if the difference between the two refractive indices of the quadratic nonlinear optical medium in the considered propagation direction is large enough, so that the two eigenpolarizations cannot interact, a generic KdV model was derived, which describes the decay of a FCP input into either one or a few unipolar half-cycle solitons, and (b) if the difference between the two refractive indices can be neglected, a complicated coupled system of KdV equations describing ultrashort-soliton evolution in quadratic nonlinear optical media was derived by using a rigorous application of the reductive perturbation formalism.

4. GENERALIZED DOUBLE-SINE-GORDON EQUATION FOR DESCRIBING ULTRASHORT-SOLITON PROPAGATION IN MEDIA DESCRIBED BY A GENERIC HAMILTONIAN

In Ref. [50] we derived a generalized double-sG equation for describing ultrashort-soliton propagation in a medium decribed by a general Hamiltonian with an arbitrary number of energy levels, assuming that all transition frequencies of the medium are well below the typical wave frequency, *i.e.*, only the contribution of infrared transitions are taken into account. Thus we used a *short-wave approximation* and a rigorous application of the reductive perturbation formalism to derive a cumbersome coupled system of nonlinear partial differential equations describing soliton evolution in such atomic systems. In order to perform the multiscale analysis, order by order, we introduced the scaled variables τ (a retarded time) and ζ: $\tau = t - z/V$ and $\zeta = \varepsilon z$, where ε is the small parameter in the corresponding reductive perturbation formalism. It is worth noting that the variable τ is not a slow one, whereas the propagation variable ζ is. Thus the variable ζ gives account for long-distance propagation. Note that the electric field E is expanded as $E = E^{(0)} + \varepsilon E^{(1)} + \ldots$.

As we have shown in Ref. [50], this set of equations can be simplified when a few additional assumptions are considered. First, if no coupling between the various atomic transitions occurs, the FCP evolution is described by a sG-type equation involving one sine term for each transition, as

$$\frac{\partial \psi}{\partial Z} = \sum_j q_j \sin\left(\lambda_j \int_{-\infty}^{T} \psi dT'\right), \tag{11}$$

in dimensionless form, where j labels the atomic transitions. The dimensionless electric field is $\psi = E^{(0)}/E_r$, where E_r is some reference electric field, see Ref. [50].

The derivation of the above sG-type equation was given in Ref. [50] in the simplified situation of a four-level system, which may interact with the light wave only through two independent transitions, *i.e.*, only two matrix elements of the dipolar momentum are nonzero, one connecting the energy level 1 with 2, the other one, the energy level 3 with 4. In this situation, the sG-type equation (11) reduces to a generalized double-sG equation of the form

$$\frac{\partial \psi}{\partial Z} = \sin\left(\int_{-\infty}^{T} \psi dT'\right) - q \sin\left(\lambda \int_{-\infty}^{T} \psi dT'\right). \tag{12}$$

A second rather simple special situation of physical interest is the case of two transitions coupled together in such a way that they have equal weights. Specifically, we considered a three-level atomic system [50], in which the two excited levels are not coupled together, and we assumed that both transition dipolar momenta are equal. Then, the complicated general system can be reduced to a standard double-sG equation, which can be written in its dimensionless form as

$$\frac{\partial \psi}{\partial Z} = \sin\left(\int_{-\infty}^{T} \psi dT'\right) + q \sin\left(2 \int_{-\infty}^{T} \psi dT'\right). \tag{13}$$

Here we only present the output of a typical numerical simulation showing the formation of robust breather-type solutions of the generalized double-sG equation from sinusoidal inputs with Gaussian envelopes. In Fig. 4 we show one period of a breather of the generalized double-sG equation (12), for the following choice of the parameters: $\lambda = \sqrt{3}$ and $q = 0.4$. In conclusion, the existence of robust breather-type solutions was evidenced in the most general case of a generalized double-sG equation, see Ref. [50].

5. CONCLUSIONS

We have overviewed a few generic models beyond the slowly varying envelope approximation of the nonlinear Schrödinger-type evolution equations, describing the propagation of ultrashort solitons in either quadratic or cubic nonlinear optical media. To this aim, we used the *density matrix formalism* for a general Hamiltonian, with an arbitrary number of energy levels. Assuming that all transition frequencies of the cubic nonlinear optical medium are well above the typical wave frequency, we used a long-wave approximation to derive an approximate evolution model of modified Korteweg-de Vries type. For non-centrosymmetric quadratically nonlinear optical media described by a general Hamiltonian we derived in the long-wave approximation regime a rather cumbersome coupled system of Korteweg-de Vries equations describing ultrashort soliton evolution in such media. Then the rigorous application of the multiscale analysis in the short-wave approximation regime, allowed us to obtain a cumbersome coupled system of nonlinear partial differential equations describing ultrashort-soliton evolution in nonlinear optical media. If any coupling or interaction between different atomic transitions is neglected, it simplifies to a rather cumbersome sine-Gordon-type equation. In the specific case of only two independent transitions, the general evolution system can be further reduced to a generalized double-sine-Gordon equation. By introducing these generic dynamical models, we performed an extensive study of soliton dynamics in nonlinear materials described by a general Hamiltonian of multilevel atoms. This allow us a better understanding of the physical phenomena and dynamical processes arising in nonlinear systems modeled by the obtained equations.

These studies might be extended in two main directions. First, a fully realistic model for ultrashort-soliton propagation might be obtained by putting together the results yielded by both long- and short-wave approaches, *i.e.*, by considering a medium containing two kinds of transitions, both in the infrared and in the ultraviolet spectral domains. In such a system, the prototype dynamical equation might be some generalized mKdV-sG equation which can adequately describe the propagation of few-cycle pulses beyond the SVEA. Second, some of the obtained results can be relatively easily extended to $(2+1)$ dimensions by incorporating into the generic model a transverse spatial coordinate; thus the formation of ultrashort spatiotemporal optical solitons (nonlinear light bullets) can be investigated in the more general physical setting. Especially, the lumps described by the Kadomtsev-Petviashvili (KP) equation in quadratic media [107], and the collapse of FCPs in Kerr media, described by a generalized KP equation [108], remain valid if an arbitrary number of energy levels are taken into account. However, the contribution of infrared transitions [109] does not generalize straightforwardly to (2 + 1) dimensions, and therefore, this issue deserves further investigation. Finally, generalizing the current models to the case where the vectorial nature of the electric field is taken into account would also constitute an interesting theme for future studies.

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REFERENCES

1. G.I. Stegeman, D.N. Christodoulides, M. Segev, IEEE J. Select. Top. Quant. Electron. **6**, 1419 (2000).
2. Z. Chen, M. Segev, D.N. Christodoulides, Rep. Prog. Phys. **75**, 086401 (2012).
3. Y.S. Kivshar, G.P. Agrawal, "*Optical solitons: From fibers to photonic crystals*" (Academic Press, San Diego, 2003).
4. H.G. Purwins, H.U. Bodeker, Sh. Amiranashvili, Adv. Physics **59**, 485 (2010); P. Grelu, N. Akhmediev, Nature Photon. **6**, 84 (2012).
5. Y.V. Kartashov, B.A. Malomed, L. Torner, Rev. Mod. Phys. **83**, 247 (2011).
6. B.A. Malomed *et al.*, J. Opt. B: Quantum Semiclassical Opt. **7**, R53 (2005).
7. E.L. Falcao-Filho *et al.*, Phys. Rev. Lett. **110**, 013901 (2013).
8. V. Skarka, N.B. Aleksić, Phys. Rev. Lett. **96**, 013903 (2006); V. Skarka, N.B. Aleksić, V.I. Berezhiani, Phys. Rev. A **81**, 045803 (2010); V. Skarka *et al.*, Phys. Rev. Lett. **105**, 213901 (2010); Yingji He *et al.*, Phys. Rev. E **85**, 066206 (2012); Y.J. He, D. Mihalache, J. Opt. Soc. Am. B **29**, 2554 (2012); C. Fernandez-Oto *et al.*, Phys. Rev. Lett. **110**, 174101 (2013).
9. A. Chong, W.H. Renninger, D.N. Christodoulides, F.W. Wise, Nat. Photonics **4**, 103 (2010).
10. D. Abdollahpour *et al.*, Phys. Rev. Lett. **105**, 253901 (2010).
11. F. Ye *et al.*, Phys. Rev. Lett. **104**, 106802 (2010); Fangwei Ye *et al.*, Opt. Lett. **36**, 1179 (2011); Yao Kou *et al.*, Phys. Rev. A **84**, 033855 (2011); Yao Kou *et al.*, Opt. Lett. **37**, 3822 (2012); Yao Kou *et al.*, Opt. Lett. **38**, 1271 (2013).
12. H. Leblond, D. Mihalache, Rom. Rep. Phys. **63**, 1254 (2011).
13. D. Mihalache, D. Mazilu, Rom. Rep. Phys. **60**, 749 (2008); L.-C. Crasovan, B.A. Malomed, D. Mihalache, Phys. Rev. E **63**, 016605 (2001); L.-C. Crasovan, B.A. Malomed, D. Mihalache, Phys. Lett. A **289**, 59 (2001).
14. D. Mihalache, Proc. Romanian Acad. A **11**, 142 (2010); D. Mihalache, Rom. Rep. Phys. **63**, 9 (2011).
15. Y.V. Kartashov *et al.*, Opt. Lett. **31**, 2329 (2006); D. Mihalache, D. Mazilu, M. Bertolotti, C. Sibilia, J. Opt. Soc. Am. B **5**, 565 (1988); D. Mihalache *et al.*, Opt. Lett. **12**, 187 (1987); D. Mihalache, D. Mazilu, F. Lederer, Opt. Commun. **59**, 391 (1986); D. Mihalache, R.G. Nazmitdinov, V.K. Fedyanin, Physica Scripta **29**, 269 (1984); V.K. Fedyanin, D. Mihalache, Z. Phys. B **47**, 167 (1982).
16. D. Mihalache, D. Mazilu, F. Lederer, Y.S. Kivshar, Opt. Lett. **32**, 3173 (2007).
17. D. Mihalache, D. Mazilu, L.-C. Crasovan, B.A. Malomed, F. Lederer, L. Torner, Phys. Rev. E **68**, 046612 (2003).
18. D. Mihalache, D. Mazilu, L.-C. Crasovan, L. Torner, B.A. Malomed, F. Lederer, Phys. Rev. E **62**, 7340 (2000).
19. D. Mihalache, D. Mazilu, L. Torner, Phys. Rev. Lett. **81**, 4353 (1998).

20. D. Mihalache, D. Mazilu, L.-C. Crasovan, L. Torner, Opt. Commun. **137**, 113 (1997).
21. D. Mihalache, F. Lederer, D. Mazilu, L.-C. Crasovan, Opt. Eng. **35**, 1616 (1996).
22. D. Mihalache, L. Torner, F. Moldoveanu, N.-C. Panoiu, N. Truta, Phys. Rev. E **48**, 4699 (1993).
23. E. Fazio, F. Renzi, R. Rinaldi, M. Bertolotti, M. Chauvet, W. Ramadan, A. Petris, V.I. Vlad, Appl. Phys. Lett. **85**, 2193 (2004); M. Alonzo *et al.*, J. Optics **12**, 015206 (2010).
24. V.I. Vlad, E. Fazio, M. Bertolotti, A. Bosco, A. Petris, Appl. Surf. Sci. **248**, 484 (2005).
25. A. Petris, A. Bosco, V.I. Vlad, E. Fazio, M. Bertolotti, J. Optoelectron. Adv. Mat. **7**, 2133 (2005).
26. V.I. Vlad, A. Petris, A. Bosco, E. Fazio, M. Bertolotti, J. Opt. A: Pure Appl. Opt. **8**, S477 (2006).
27. F. Pettazzi *et al.*, Phys. Rev. A **76**, 063818 (2007).
28. A. Petris, S.T. Popescu, V.I. Vlad, E. Fazio, Rom. Rep. Phys. **64**, 492 (2012).
29. S.T. Popescu, A. Petris, V.I. Vlad, Appl. Phys. B-Lasers and Optics **108**, 799 (2012).
30. A.T. Grecu *et al.*, Rom. J. Phys. **56**, 339 (2011); D. Grecu *et al.*, Rom. J. Phys. **57**, 180 (2012); D. Grecu, A.T. Grecu, Rom. Rep. Phys. **63**, 1302 (2011).
31. Z.J. Ren, C.F. Ying, C.J. Fan, Q. Wu, Chin. Phys. Lett. **29**, 124209 (2012).
32. Zhijun Ren, Qiong Wu, Hefa Mao, Yile Shi, Changjiang Fan, Opt. Express **21**, 4481 (2013); Bin Liu *et al.*, Opt. Express **21**, 5561 (2013).
33. A. Biswas *et al.*, Proc. Romanian Acad. A **13**, 32 (2012); L. Girgis *et al.*, Rom. Rep. Phys. **64**, 663 (2012); H. Triki *et al.*, Rom. Rep. Phys. **64**, 357 (2012).
34. G. Ebadi *et al.*, Rom. Rep. Phys. **64**, 915 (2012); A. Salas *et al.*, Proc. Romanian Acad. A **14**, 28 (2013).
35. G. Ebadi *et al.*, Proc. Romanian Acad. A **13**, 215 (2012); A.G. Johnpillai, A. Yildirim, A. Biswas, Rom. J. Phys. **57**, 545 (2012); A. Jafarian *et al.*, Rom. Rep. Phys. **65**, 76 (2013).
36. H. Triki *et al.*, Proc. Romanian Acad. A **13**, 103 (2012); G. Ebadi *et al.*, Rom. Rep. Phys. **65**, 27 (2013).
37. H. Triki *et al.*, Rom. J. Phys. **57**, 1029 (2012); B.S. Ahmed, A. Biswas, Proc. Romanian Acad. A **14**, 111 (2013).
38. H. Triki, A. Yildirim, T. Hayat, O.M. Aldossary, A. Biswas, Rom. Rep. Phys. **64**, 672 (2012); G. Ebadi *et al.*, Rom. J. Phys. **58**, 3 (2013).
39. A.M. Wazwaz, Phys. Scripta, **85**, 065011 (2012); A.M. Wazwaz, Phys. Scripta **84**, 035010 (2011); A.M. Wazwaz, Rom. Rep. Phys. **65**(2), 383 (2013).
40. D. Mihalache, Rom. J. Phys. **57**, 352 (2012); Ying Wang, Feng-De Zong, Feng-Bo Li, Chin. Phys. B **22**, 030315 (2013); H. Sakaguchi, B.A. Malomed, Opt. Express, **21**, 9813 (2013).
41. Li Chen, Rujiang Li, Na Yang, Da Chen, Lu Li, Proc. Romanian Acad. A **13**, 46 (2012); A.S. Rodrigues *et al.*, Rom. Rep. Phys. **65**, 5 (2013); I.V. Barashenkov *et al.*, Phys. Rev. A **86**, 053809 (2012); N.V. Alexeeva *et al.*, Phys. Rev. A **85**, 063837 (2012); I.V. Barashenkov *et al.*, Phys. Rev. A **87**, 033819 (2013); Yunji Meng, Youwen Liu, J. Opt. Soc. Am. B **30**, 1148 (2013); Jianing Xie *et al.*, J. Opt. Soc. Am. B **30**, 1216 (2013); Rujiang Li, Pengfei Li, Lu Li, Proc. Romanian Acad. A **14**, 121 (2013).
42. Changming Huang, Chunyan Li, Haidong Liu, Liangwei Dong, Proc. Romanian Acad. A **13**, 329 (2012).
43. Guangye Yang, Lu Li, Suotang Jia, Dumitru Mihalache, Rom. Rep. Phys. **65**, 391 (2013).
44. Y. J. He, D. Mihalache, Rom. Rep. Phys. **64**, 1243 (2012).
45. H. Wang, W. He, X. Zhu, Y.J. He, Rom. Rep. Phys. **64**, 1391 (2012).
46. W. P. Zhong *et al.*, Rom. Rep. Phys. **64**, 1399 (2012).
47. H. Leblond, D. Mihalache, Phys. Rev. A **86**, 043832 (2012).
48. H. Triki, H. Leblond, D. Mihalache, Opt. Commun. **285**, 3179 (2012).

49. H. Leblond, H. Triki, D. Mihalache, Phys. Rev. A **85**, 053826 (2012).
50. H. Leblond, H. Triki, D. Mihalache, Phys. Rev. A **86**, 063825 (2012).
51. Zhenya Yan, J. Phys. A: Math. Theor. **45**, 444035 (2012).
52. V. Besse, H. Leblond, D. Mihalache, B.A. Malomed, Phys. Rev. E **87**, 012916 (2013).
53. Yingji He, Dumitru Mihalache, Phys. Rev. A **87**, 013812 (2013).
54. Sh. Amiranashvili, U. Bandelow, N. Akhmediev, Phys. Rev. A **84**, 043834 (2011).
55. Sh. Amiranashvili, U. Bandelow, N. Akhmediev, Phys. Rev. A **87**, 013805 (2013).
56. H. Leblond, D. Mihalache, Phys. Reports **523**, 61 (2013).
57. Y. Silberberg, Opt. Lett. **15**, 1282 (1990).
58. A.B. Blagoeva *et al.*, IEEE J. Quantum Electron. **27**, 2060 (1991).
59. D.E. Edmundson, R.H. Enns, Opt. Lett. **17**, 586 (1992).
60. N. Akhmediev, J.M. Soto-Crespo, Phys. Rev. A **47**, 1358 (1993).
61. R. McLeod, K. Wagner, S. Blair, Phys. Rev. A **52**, 3254 (1995).
62. X. Liu, L.J. Qian, F.W. Wise, Phys. Rev. Lett. **82**, 4631 (1999).
63. S. Minardi *et al.*, Phys. Rev. Lett. **105**, 263901 (2010).
64. F. Eilenberger *et al.*, Phys. Rev. A **84**, 013836 (2011).
65. F. Eilenberger *et al.*, Opt. Express **19**, 23171 (2011).
66. M.O. Williams, C.W. McGrath, J. Nathan Kutz, Opt. Express **18**, 11671 (2010).
67. M.O. Williams, J. Nathan Kutz, Opt. Quantum Electron. **44**, 247 (2012).
68. N. Akhmediev, A. Ankiewicz (eds.), "*Dissipative solitons: From optics to biology and medicine*", Lect. Notes Phys. vol. **751** (Springer, Berlin, 2008).
69. P.G. Kevrekidis, D.J. Frantzeskakis, R. Carretero-Gonzalez (eds.), "*Emergent nonlinear phenomena in Bose-Einstein condensates: Theory and experiment*", Springer Series on Atomic, Optical, and Plasma Physics, vol. **45** (Springer, Berlin, 2008).
70. L. Gallmann *et al.*, Opt. Lett. **24**, 1314 (1999).
71. U. Morgner *et al.*, Opt. Lett. **24**, 411 (1999).
72. D.H. Sutter *et al.*, Opt. Lett. **24**, 631 (1999).
73. A. Shirakawa, I. Sakane, M. Takasaka, T. Kobayashi, Appl. Phys. Lett. **74**, 2268 (1999).
74. M. Wegener, "*Extreme Nonlinear Optics*" (Springer, Berlin, 2005).
75. E. Goulielmakis *et al.*, Science **320**, 1614 (2008).
76. J. Bonse *et al.*, Appl. Phys. A: Materials Science and Processing **74**, 19 (2002).
77. A. Scrinzi, M.Yu. Ivanov, R. Kienberger, D.M. Villeneuve, J. Phys. B **39**, R1 (2006).
78. F. Krausz, M. Ivanov, Rev. Mod. Phys. **81**, 163 (2009).
79. T. Brabek, F. Krausz, Rev. Mod. Phys. **72**, 545 (2000).
80. H.-C. Wu, J. Meyer-ter-Vehn, Nature Photonics **6**, 304 (2012).
81. K. Zhao *et al.*, Opt. Lett. **37**, 3891 (2012).
82. D. Strickland, G. Mourou, Opt. Commun. **56**, 219 (1985).
83. G.A. Mourou, T. Tajima, S.V. Bulanov, Rev. Mod. Phys. **78**, 309 (2006).
84. G.A. Mourou, T. Tajima, Science **331**, 41 (2011).
85. A. Dubietis, G. Jonusauskas, A. Piskarskas, Opt. Commun. **88**, 437 (1982).
86. V.M. Malkin, G. Shvets, N.J. Fisch, Phys. Rev. Lett. **82**, 4448 (1999).
87. G.A. Mourou *et al.*, Opt. Commun. **285**, 720 (2012).
88. A.E. Kaplan, Lasers in Engineering **24**, 3 (2013).
89. O. Tesileanu *et al.*, Rom. Rep. Phys. **64**, 1373 (2012).
90. X. Tan, X. Fan, Y. Yang, D. Tong, J. Mod. Opt. **55**, 2439 (2008).

91. A.I. Maimistov, Quantum Electron. **40**, 756 (2010).
92. T. Brabec, F. Krausz, Phys. Rev. Lett. **78**, 3282 (1997).
93. A.V. Husakou, J. Herrmann, Phys. Rev. Lett. **87**, 203901 (2001).
94. J.K. Ranka, A.L. Gaeta, Opt. Lett. **23**, 534 (1998).
95. E.M. Belenov, A.V. Nazarkin, JETP Lett. **51**, 288 (1990).
96. Q. Lin, E. Wintner, Opt. Commun. **150**, 185 (1998).
97. I.V. Melnikov, D. Mihalache, F. Moldoveanu, N.-C. Panoiu, Phys. Rev. A **56**, 1569 (1997).
98. I.V. Mel'nikov, D. Mihalache, N.-C. Panoiu, Opt. Commun. **181**, 345 (2000).
99. H. Leblond, F. Sanchez, Phys. Rev. A **67**, 013804 (2003).
100. H. Leblond *et al.*, Mathematics and Computers in Simulation **69**, 378 (2005).
101. H. Leblond *et al.*, Phys. Rev. A **74**, 063815 (2006).
102. Sh. Amiranashvili, A.G. Vladimirov, U. Bandelow, Phys. Rev. A **77**, 063821 (2008).
103. A.I. Maimistov, J.-G. Caputo, Optics and Spectroscopy **94**, 245 (2003).
104. H. Leblond, Phys. Rev. A **78**, 013807 (2008).
105. L. Bergé, S. Skupin, Phys. Rev. Lett. **100**, 113902 (2008).
106. E.V. Kazantseva, A.I. Maimistov, B.A. Malomed, Opt. Commun. **188**, 195 (2001).
107. H. Leblond, D. Kremer, D. Mihalache, Phys. Rev. A **80**, 053812 (2009).
108. H. Leblond, D. Kremer, D. Mihalache, Phys. Rev. A **81**, 033824 (2010).
109. H. Leblond, D. Mihalache, Phys. Rev. A **81**, 063815 (2010).
110. Xiao-Tao Xie, M.A. Macovei, Phys. Rev. Lett. **104**, 073902 (2010).
111. N.N. Rosanov, V.V. Kozlov, S. Wabnitz, Phys. Rev. A **81**, 043815 (2010).
112. V.V. Kozlov, N.N. Rosanov, S. Wabnitz, Phys. Rev. A **84**, 053810 (2011).
113. H. Leblond, D. Mihalache, J. Phys. A: Math. Theor. **43**, 375205 (2010).
114. V.V. Kozlov, N.N. Rosanov, Phys. Rev. A **87**, 043836 (2013).
115. D.S. Steingrube *et al.*, Opt. Express **20**, 24049 (2012).
116. Faezeh Kimiaee Asadi, Babak Shokri, Hervé Leblond, Opt. Commun. **294**, 283 (2013).
117. Xianglong Zeng, Hairun Guo, Binbin Zhou, Morten Bache, Opt. Express **20**, 27071 (2012).
118. D.V. Novitsky, Phys. Rev. A **86**, 063835 (2012).
119. A. Pusch, I.V. Shadrivov, O. Hess, Y.S. Kivshar, Opt. Express **21**, 1121 (2013).
120. A.A. Drozdov, S.A. Kozlov, A.A. Sukhorukov, Y.S. Kivshar, Phys. Rev. A **86**, 053822 (2012).
121. Y. Shen, N. Whitaker, P.G. Kevrekidis, N.L. Tsitsas, D.J. Frantzeskakis, Phys. Rev. A **86**, 023841 (2012).
122. M. Kolesik, P. Jakobsen, J.V. Moloney, Phys. Rev. A **86**, 035801 (2012).
123. P. Whalen, J.V. Moloney, A.C. Newell, K. Newell, M. Kolesik, Phys. Rev. A **86**, 033806 (2012).
124. Y. Xiao, D.N. Maywar, G.P. Agrawal, Opt. Lett. **38**, 724 (2013).
125. N.N. Akhmediev, I.V. Mel'nikov, A.V. Nazarkin, Sov. Phys. Lebedev. Inst. Rep. **2**, 66 (1989) [Kratk. Soobshch. Fiz. FIAN **2**, 49 (1989)].
126. E.D. Farnum, J. Nathan Kutz, Opt. Lett. **35**, 3033 (2010).
127. A. Kujawski, Z. Phys. **66**, 271 (1987).
128. A.V. Andreev, Zh. Éksp. Teor. Phys. **108**, 796 (1995) [JETP **81**, 434 (1995)].
129. A.Yu. Parkhomenko, S.V. Sazonov, Zh. Éksp. Teor. Phys. **114**, 1595 (1998) [JETP **87**, 864 (1998)].
130. H. Leblond, J. Phys. B: At. Mol. Opt. Phys. **41**, 043001 (2008).
131. M. Wadati, J. Phys. Soc. Japan **34**, 1289 (1973).
132. R. Hirota, Lecture Notes in Mathematics, vol. **515**, pp. 40-68 (Springer, Berlin, 1976).