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EXACT SOLUTIONS OF BOUSSINESQ AND KdV-mKdV EQUATIONS BY FRACTIONAL SUB-EQUATION METHOD

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Abstract. A fractional sub-equation method is introduced to solve fractional differential equations. By the aid of the solutions of the fractional Riccati equation, we construct solutions of the Boussinesq and KdV-mKdV equations of fractional order. The obtained results show that this method is very efficient and easy to apply for solving fractional partial differential equations.

Key words: Sub-equation method, Modified Riemann-Liouville derivative, Fractional differential equation, Boussinesq equation, KdV-mKdV equation.

1. INTRODUCTION

During the last few decades it was reported that many areas of basic sciences and engineering such as fluid flow, control problem, signal processing, viscoelastic materials, polymers and others, are governed by models involving fractional partial differential equations (FPDEs) (see for example Refs. [1–7] and the references therein). Therefore, several different and powerful methods for solving nonlinear partial differential equations have been proposed in order to obtain the exact solutions *e.g.*, Sine-Cosine method [8], Bäcklund transformation [9], Adomian decomposition method [10, 11], Homotopy perturbation method [12, 13], Variational iteration method [14–16], Fractional sub-equation method [17–19] and so on.

The sub-equation method for the exact solutions of nonlinear FPDEs was in-

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troduced recently in [19]. This approach is based on the homogeneous balance principle [20] and the modified Riemann-Liouville-Jumarie derivative that construct the exact analytical solutions of nonlinear fractional partial differential equations. We notice that this modified Riemann-Liouville derivatives of order α is given as following [21]:

$$D_x^{\alpha} f(x) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-\xi)^{-\alpha-1}[f(\xi)-f(0)], & \alpha < 0, \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-\xi)^{-\alpha}[f(\xi)-f(0)], & 0 < \alpha < 1, \\ [f^{(\alpha-n)}(x)]^{(n)}, & n \leq \alpha < n+1, \ n \geq 1. \end{cases} \tag{1}$$

Below we specify some properties of this derivative [21], namely

$$D_x^{\alpha} x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} x^{\gamma-\alpha}, \quad \gamma > 0, \tag{2}$$

$$D_x^{\alpha}[f(x)g(x)] = g(x) D_x^{\alpha} f(x) + f(x) D_x^{\alpha} g(x), \tag{3}$$

$$D_x^{\alpha} f[g(x)] = f'_g[g(x)] D_x^{\alpha} g(x) = D_g^{\alpha} f[g(x)](g'_x)^{\alpha}. \tag{4}$$

In this manuscript, by introducing a new general ansatz, we applied the fractional sub-equation method for solving two FPDEs.
The first equation is called the space-time fractional Boussinesq equation which has the form

$$D_t^{2\alpha} u + b D_x^{2\alpha} u + \beta D_x^{2\alpha}(u^2) + \gamma D_x^{4\alpha} u = 0. \tag{5}$$

Here $\gamma = const > 0$ is the dispersion parameter depending on the compression and rigidity characteristics of the material, $\beta = const \in R$ is the coefficient controlling nonlinearity, $u(x,t)$ is the vertical deflection, and the quadratic nonlinearity $(u^2)_{xx}$ accounts for the curvature of the bending beam.
The second equation is the space-time fractional KdV-mKdV equation, namely

$$D_t^{\alpha} u + \mu D_x^{\alpha} u u + \beta u^2 D_x^{\alpha} u + D_x^{3\alpha} u = 0. \tag{6}$$

We mention that this equation is widely recognized as a paradigm for the description of weakly nonlinear long waves in many branches of physics and engineering. Here $u(x,t)$ denotes an appropriate field variable, t is the time, and x is the space coordinate in the relevant direction. mKdV equation arises in many different fields, such as shallow water model, plasma science, biophysics and so on. Also the nonlinear terms of KdV and mKdV equations appear simultaneously in physics and quantum theory that is called combined KdV-mKdV equation [22].

The structure of the manuscript is given below.

In Section 2, we describe the fractional sub-equation method for solving fractional partial differential equations. In Section 3, we give two applications of the proposed method to nonlinear equations. Conclusions are presented in Section 4.

1.1. BASICS OF THE FRACTIONAL SUB-EQUATION METHOD

In this section, we present a brief description of the fractional partial differential equation method. For a given fractional differential equation with two variables x and t, let us consider

$$p(u, u_x, u_t, D_t^\alpha u, D_x^\alpha u, \ldots) = 0, \quad 0 < \alpha < 1, \tag{7}$$

where $D_t^\alpha u$ and $D_x^\alpha u$ are Jumarie's modified Riemann-Liouville derivatives of u. By using the travelling wave transformation

$$u(x,t) = u(\xi), \qquad \xi = kx + ct, \tag{8}$$

where c and k are constants to be determined later, the FPDE (7) yields

$$p\left(u, ku', cu', k^\alpha D_\xi^\alpha u, c^\alpha D_\xi^\alpha u, \ldots\right) = 0, \quad 0 < \alpha < 1, \tag{9}$$

which is a nonlinear fractional ordinary differential equation for $u = u(\xi)$. We suppose that equation (9) has a solution in the form

$$u(\xi) = \sum_{i=0}^{n} a_i \varphi^i, \tag{10}$$

where a_i $(i = 0, 1, \cdots, n-1, n)$ are constants to be found later and the positive integer n is determined with the balancing of the highest order derivatives and nonlinear terms in Eq. (7) or Eq. (9), and $\varphi = \varphi(\xi)$ is a solution of the fractional Riccati equation that satisfies

$$D_\xi^\alpha \varphi = \sigma + \varphi^2, \qquad 0 < \alpha \leq 1. \tag{11}$$

By substituting (10) along with Eq. (11) into Eq. (9), we obtain a polynomial in $\varphi(\xi)$ by using the properties of Jumarie's modified Riemann-Liouville derivative (2)-(4). Setting all the coefficients of φ^k $(k = 0, 1, 2, \ldots)$ to zero, we obtain a set of overdetermined nonlinear algebraic equations for $c, k, a_i(i = 0, 1, \ldots, n-1, n)$. Solving this algebraic equations we determine the constants $c, k, a_i(i = 0, 1, \ldots, n-1, n)$. Thus, we get the exact solutions of Eq. (7) obtained by substituting these constants and the solutions of Eq. (11) into (10).

2. ILLUSTRATIVE EXAMPLES

In this section, we employ the fractional sub-equation method to solve the FPDEs (5) and (6).

Example 1. *The starting point is to consider the space-time fractional Boussinesq equation as:*

$$D_t^{2\alpha} u + b D_x^{2\alpha} u + \beta D_x^{2\alpha}(u^2) + \gamma D_x^{4\alpha} u = 0. \tag{12}$$

At first we used the travelling wave transformation, namely

$$u = u(\xi), \quad \xi = ct + kx \tag{13}$$

then Eq. (12) is reduced into a nonlinear fractional ordinal differential equation easy to solve

$$c^{2\alpha} D_\xi^{2\alpha} u + bk^{2\alpha} D_\xi^{2\alpha} u + \beta k^{2\alpha} D_\xi^{2\alpha}(u^2) + \gamma k^{4\alpha} D_x^{4\alpha} u = 0. \tag{14}$$

By balancing the highest order derivative terms and nonlinear terms in Eq. (14) gets $n = 2$, and we have a solution of Eq. (14) in the following form

$$u = \sum_{i=0}^{2} a_i \varphi^i, \tag{15}$$

where φ satisfies the sub-equation (11).

The next step is to substitute (15) into (14) and to collect the coefficients of φ^j and set them to be zero. Thus, we obtain

$$\begin{cases} a_0 = \frac{k^{-2\alpha}(-c^{2\alpha} - bk^{2\alpha})}{2\beta}, & a_1 = 0, \quad a_2 = \frac{-6k^{2\alpha}\gamma}{\beta}, \quad \sigma = 0 \\ a_0 = \frac{k^{-2\alpha}(-c^{2\alpha} - bk^{2\alpha} - 8k^{4\alpha}\gamma\sigma)}{2\beta}, & a_1 = 0, \quad a_2 = \frac{-6k^{2\alpha}\gamma}{\beta}, \quad \sigma \neq 0. \end{cases} \tag{16}$$

As a result, we give the solutions of Eqs. (11) (see for instance [23]), (13)-(16), and exact solutions of Eq. (12) as follows

$$u_1 = \frac{k^{-2\alpha}(-c^{2\alpha} - bk^{2\alpha} - 8k^{4\alpha}\gamma\sigma)}{2\beta} + \frac{6k^{2\alpha}\gamma}{\beta}(\sigma \tanh_\alpha^2(\sqrt{-\sigma}(kx+ct))), \sigma < 0,$$

$$u_2 = \frac{k^{-2\alpha}(-c^{2\alpha} - bk^{2\alpha} - 8k^{4\alpha}\gamma\sigma)}{2\beta} + \frac{6k^{2\alpha}\gamma}{\beta}(\sigma \coth_\alpha^2(\sqrt{-\sigma}(kx+ct))), \ \sigma < 0,$$

$$u_3 = \frac{k^{-2\alpha}(-c^{2\alpha} - bk^{2\alpha} - 8k^{4\alpha}\gamma\sigma)}{2\beta} - \frac{6k^{2\alpha}\gamma}{\beta}(\sigma \tan_\alpha^2(\sqrt{\sigma}(kx+ct))), \qquad \sigma > 0,$$

$$u_4 = \frac{k^{-2\alpha}(-c^{2\alpha} - bk^{2\alpha} - 8k^{4\alpha}\gamma\sigma)}{2\beta} - \frac{6k^{2\alpha}\gamma}{\beta}(\sigma \cot_\alpha^2(\sqrt{\sigma}(kx+ct))), \qquad \sigma > 0,$$

$$u_5 = \frac{k^{-2\alpha}(-c^{2\alpha} - bk^{2\alpha})}{2\beta} - \frac{6k^{2\alpha}\gamma\Gamma^2(1+\alpha)}{\beta(\xi^\alpha + \omega)^2}, \qquad \sigma = 0.$$

Figure 1 plots $u_1(x,t)$ and shows the exact solutions of Eq. (12) with $\alpha = -b = -\beta = \gamma = -\sigma = 1$ and values of $c = k = 0.5$.

Example 2. *We start with the space-time fractional KdV-mKdV equation*

$$D_t^\alpha u + \mu u D_x^\alpha u + \beta u^2 D_x^\alpha u + D_x^{3\alpha} u = 0. \tag{17}$$

To solve Eq. (17), we consider the following travelling wave transformation

$$u = u(\xi), \quad \xi = kx + ct, \tag{18}$$

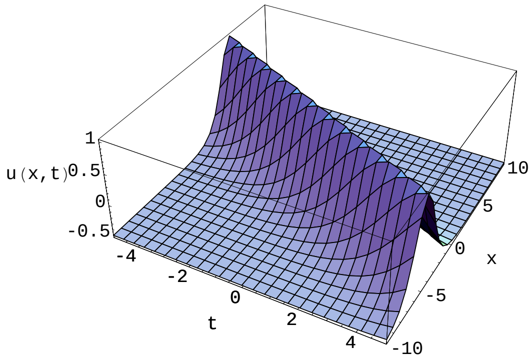

Fig. 1 – Exact solution for Eq. (12) with $\alpha = -b = -\beta = \gamma = -\sigma = 1$ and $c = k = 0.5$.

then Eq. (17) is reduced to a nonlinear fractional ordinary differential equation, namely

$$c^{\alpha} D_{\xi}^{\alpha} u + \mu k^{\alpha} u D_{\xi}^{\alpha} u + \beta k^{\alpha} u^2 D_{\xi}^{\alpha} u + k^{3\alpha} D_{\xi}^{3\alpha} u = 0. \tag{19}$$

By balancing the highest order derivative terms and nonlinear terms in Eq. (19), we get $n = 1$. Like in the previous case we obtain

$$\begin{cases} a_0 = -\frac{k^{\alpha}\mu}{2\beta}, \quad a_1 = \pm\frac{i\sqrt{6}\sqrt{k^{3\alpha}}}{\sqrt{\beta}}, \quad c^{\alpha} = \frac{k^{2\alpha}\mu^2 - 8k^{3\alpha}\beta\sigma}{4\beta}, \sigma \neq 0, \beta \neq 0 \\ a_0 = -\frac{k^{\alpha}\mu}{2\beta}, \quad a_1 = \pm\frac{i\sqrt{6}\sqrt{k^{3\alpha}}}{\sqrt{\beta}}, \quad c^{\alpha} = \frac{k^{2\alpha}\mu^2}{4\beta}, \quad \beta \neq 0, \sigma = 0. \end{cases} \tag{20}$$

Finally, from the solutions of Eqs. (11) [23], and (20) we obtain the following generalized hyperbolic function solutions, generalized trigonometric function solutions and rational solution of Eq. (17)

$$u_1 = -\frac{k^{\alpha}\mu}{2\beta} \pm \frac{i\sqrt{6}\sqrt{k^{3\alpha}}}{\sqrt{\beta}}(-\sqrt{-\sigma}\tanh_{\alpha}(\sqrt{-\sigma}(kx+ct))), \quad \sigma < 0,$$
$$u_2 = -\frac{k^{\alpha}\mu}{2\beta} \pm \frac{i\sqrt{6}\sqrt{k^{3\alpha}}}{\sqrt{\beta}}(-\sqrt{-\sigma}\coth_{\alpha}(\sqrt{-\sigma}(kx+ct))), \quad \sigma < 0,$$
$$u_3 = -\frac{k^{\alpha}\mu}{2\beta} \pm \frac{i\sqrt{6}\sqrt{k^{3\alpha}}}{\sqrt{\beta}}(\sqrt{\sigma}\tanh_{\alpha}(\sqrt{\sigma}(kx+ct))), \quad \sigma > 0,$$
$$u_4 = -\frac{k^{\alpha}\mu}{2\beta} \pm \frac{i\sqrt{6}\sqrt{k^{3\alpha}}}{\sqrt{\beta}}(-\sqrt{\sigma}\coth_{\alpha}(\sqrt{\sigma}(kx+ct))), \quad \sigma > 0,$$
$$u_5 = -\frac{k^{\alpha}\mu}{2\beta} \pm \frac{i\sqrt{6}\sqrt{k^{3\alpha}}\,\Gamma^2(1+\alpha)}{\sqrt{\beta}(\xi^{\alpha}+\omega)^2}, \quad \sigma = 0.$$

3. CONCLUSION

In this manuscript we have proposed a fractional sub-equation method to solve fractional differential equations. Exact analytical solutions of the space-time fractional Boussinesq equation (5) and the nonlinear KdV-mKdV equation (6) are successfully obtained. These solutions are useful for further understanding of the mechanisms of the complicated nonlinear physical phenomena and FPDEs. Moreover, the proposed method is simple but it represents a powerful algorithm for handling systems of FPDEs. `Mathematica` has been used for computations and programming in this paper.

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