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SOLITONS AND OTHER SOLUTIONS TO THE GENERALIZED MACCARI SYSTEM

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Abstract. The generalized Maccari system is studied in this paper. The governing equation is solved both analytically and numerically. The travelling wave solution reveals 1-soliton solution. Subsequently, by using the mapping method and Lie symmetry analysis we also obtain several other solutions that include cnoidal waves and other doubly periodic functions. The parametric domain restrictions are also identified in this paper. Finally, some numerical simulations are given.

Key words: Solitons, travelling waves, integrability.

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1. INTRODUCTION

The nonlinear wave equations is one of the most important as well as exciting areas in the fields of applied mathematics and theoretical physics. There are several forms of nonlinear wave equations that arise in various areas of scientific and engineering research. There are various issues of these wave equations that are addressed in several papers. They are its integrability aspects, conservation laws, numerical simulations, wave interactions, Painlevé analysis and various others. In order to address these features, there are several tools that are necessary. Many such tools have started appearing in the past couple of decades [1] – [35]. This lead to a flourishing amount of results of various nonlinear evolution equations (NLEEs).

This paper will address one such nonlinear wave equation. It is called the Maccari system (MS) which will be considered from a generalized setting. The MS arises in the study of the motion of isolated waves that is localized in a tiny zone in space. Consequently, MS is studied in fluid dynamics and plasma physics [20]. This paper will address the integrability aspects and the numerical simulation of the MS with power law nonlinearity and that too in a generalized setting. There are three types of integration tools that will be applied to carry out the integration of this equation. They are travelling waves (TWSs) which will reveal 1-soliton solution to the system with a couple of parameter restrictions that are also known as the constraints in order for the solitons to exist. The mapping methods [7] will obtain cnoidal and snoidal wave solutions which in the limiting cases will lead to solitons or linear waves. Then, the Lie symmetry approach will give additional solutions to the MS. Finally, the numerical simulations will illustrate these solutions.

2. GOVERNING EQUATION

The generalized form of the MS that is going to be studied in this paper is given by

$$iq_t + a\left(q_{xx} + q_{yy}\right) + bqr = 0 \tag{1}$$

$$r_t + k_1 r_x + k_2 r_y + c\left(|q|^{2n}\right)_x = 0 \tag{2}$$

This is a coupled system of NLEEs. The dependent variables are $q(x,y,t)$ and $r(x,y,t)$ where q is a complex valued function while r is real-valued. The independent variables are x, y and t where t represents the temporal variable while the remaining are the independent spatial variables. The parameter n dictates the power law nonlinearity parameter. The special case when $n = 1$ leads to the case of cubic nonlinearity.

In the past, this MS system with $k_2 = 0$, $n = 1$ and $q_{yy} = 0$ was extensively studied using additional methods of integrability such as the G'/G-expansion approach, semi-inverse variational principle and others [1]. The first integral method was recently applied to solve MS system [2, 10]. Several elliptic function solutions were also reported [12]. A list of abundant solutions were listed by Wazwaz [31]. Additionally the G'/G-expansion method was applied to integrate the MS to extract a few other solutions [6–8]. It also needs to be noted that the generalized case of power law nonlinearity was however never studied in the past.

3. TRAVELLING WAVE HYPOTHESIS

The travelling wave hypothesis for solving (1) and (2) is given by

$$q(x,y,t) = g\left(B_1 x + B_2 y - vt\right) e^{i(-\kappa_1 x - \kappa_2 y + \omega t + \theta)} \tag{3}$$

and

$$r(x,y,t) = h\left(B_1 x + B_2 y - vt\right) \tag{4}$$

In (3) and (4), the wave profiles for the two dependent variables are given by $g(s)$ and $h(s)$ respectively where

$$s = B_1 x + B_2 y - vt. \tag{5}$$

The parameters B_j for $j = 1,2$ are proportions to the width of the wave in the x- and y-directions respectively while v is the velocity of the soliton. Also, from the phase component in (3), ω represents the wave number while κ_j for $j = 1,2$ gives the frequency in the x- and y-directions respectively and finally θ represents the phase constant. Substituting these hypothesis in (1) and (2) and decomposing into real and imaginary parts yield

$$a\left(B_1^2 + B_2^2\right) g'' = \left\{\omega + a\left(\kappa_1^2 + \kappa_2^2\right)\right\} g - bgh \tag{6}$$

$$v = 2a\left(\kappa_1 B_1 + \kappa_2 B_2\right) \tag{7}$$

and

$$\left(v - k_1 B_1 - k_2 B_2\right) h' = c\left(g^{2n}\right)' \tag{8}$$

where the first two relations are respectively the real and imaginary components of (1) while (8) is from (2). The notations $g' = dg/ds$ and $g'' = d^2g/ds^2$ are used. The relation (7) gives the velocity of the soliton. Equation (8) yields after integration

$$h(s) = \frac{cg^{2n}}{\left(v - k_1 B_1 - k_2 B_2\right)} \tag{9}$$

where the integration constant is taken to be zero, since the search is for a soliton solution. Now, multiplying both sides of (6) by g' and integrating, after taking the integration constant again to be zero for the same reason, yields, on using (9),

$$\begin{aligned} a\left(B_1^2 + B_2^2\right)\left(g'\right)^2 = {} & \left\{\omega + a\left(\kappa_1^2 + \kappa_2^2\right)\right\} g^2 \\ & - \frac{bc}{(n+1)\left(v - k_1 B_1 - k_2 B_2\right)} g^{2n+2} \end{aligned} \tag{10}$$

Separating variables in (10) and integrating leads to

$$g(s) = A_1 \operatorname{sech}^{\frac{1}{n}}\left[B(B_1 x + B_2 y - vt)\right] \tag{11}$$

Thus (9) yields

$$h(s) = A_2 \operatorname{sech}^2\left[B(B_1 x + B_2 y - vt)\right] \tag{12}$$

where

$$A_1 = \left[\frac{(n+1)\left(v - k_1 B_1 - k_2 B_2\right)\left\{\omega + a\left(\kappa_1^2 + \kappa_2^2\right)\right\}}{bc}\right]^{\frac{1}{2n}} \tag{13}$$

$$A_2 = c(n+1)\left\{\omega + a\left(\kappa_1^2 + \kappa_2^2\right)\right\} \tag{14}$$

and

$$B = n\left[\frac{\omega + a\left(\kappa_1^2 + \kappa_2^2\right)}{a\left(B_1^2 + B_2^2\right)}\right]^{\frac{1}{2}} \tag{15}$$

Thus, finally, the 1-soliton solution is given by

$$q(x,y,t) = A_1 \operatorname{sech}^{\frac{1}{n}}\left[B(B_1 x + B_2 y - vt)\right] e^{i(-\kappa_1 x - \kappa_2 y + \omega t + \theta)} \tag{16}$$

and

$$r(x,y,t) = A_2 \operatorname{sech}^2\left[B(B_1 x + B_2 y - vt)\right] \tag{17}$$

where the amplitudes A_j for $j = 1,2$ are given by (13) and (14) respectively while the parameter B is given by (15). These relations immediately prompt the restrictions

$$a\left\{\omega + a\left(\kappa_1^2 + \kappa_2^2\right)\right\} > 0 \tag{18}$$

and

$$bc\left(v - k_1 B_1 - k_2 B_2\right)\left\{\omega + a\left(\kappa_1^2 + \kappa_2^2\right)\right\} > 0 \tag{19}$$

that must hold in order for the soliton solutions to exist.

4. MAPPING METHODS

4.1. BRIEF DESCRIPTION OF THE GENERAL METHOD

For a given nonlinear evolution equation of the form

$$N(u, u_t, u_x, ...) = 0, \tag{20}$$

we search for its TWS in the form

$$u(x,t) = u(\xi), \xi = k(x - ct), \tag{21}$$

where k, c are constants to be determined. Substituting Eq. (21) into Eq. (20) yields an ordinary differential equation (ODE) whose solution is searched in the form [7]

$$u(\xi) = \sum_{i=0}^{n} A_i f^i. \tag{22}$$

Here, n is a positive integer that can be determined by balancing the linear term of the highest order with the nonlinear term in Eq. (20), A_i are constants to be determined, and f satisfies the elliptic equation of first kind

$$f'' = \alpha f + \beta f^3, f'^2 = \alpha f^2 + \frac{1}{2}\beta f^4 + \gamma. \tag{23}$$

Prime denotes the derivative with respect to ξ and p, q and r are three parameters to be determined. After substituting Eq. (22) into the ODE and using Eq. (23), the constants A_i, k, c, p, q and r may be determined. The mapping relation is thus established through Eq. (22) between the solution to Eq. (23) and that of Eq. (20).

In the modified mapping method, we assume a solution in the form

$$u(\xi) = \sum_{i=0}^{n} A_i f^i + \sum_{i=1}^{n} B_i f^{-i}, \tag{24}$$

where f satisfies Eq. (23), where A_i and B_i are constants to be determined.

4.2. APPLICATION TO MS

Eq. (10) with $n = 1$ can be written as

$$a(B_1^2 + B_2^2)g'^2 = \{\omega + a(\kappa_1^2 + \kappa_2^2)\}g^2 - \frac{bcg^4}{2(v - k_1B_1 - k_2B_2)}. \tag{25}$$

Differentiating Eq. (25) with respect to s, we obtain the equation

$$Eg'' + Fg + Gg^3 = 0, \tag{26}$$

where $E = a(B_1^2 + B_2^2), F = -\{\omega + a(\kappa_1^2 + \kappa_2^2)\}, G = \dfrac{2bc}{v - k_1B_1 - k_2B_2}$.
We assume solution of Eq. (26) in the form

$$g(s) = A_0 + A_1 f \tag{27}$$

where f satisfies Eq. (23). Substitution of Eq. (27) in Eq. (26) and use of Eq. (23) yields the following set of algebraic equations which arises from different powers of f so that we have

$$\beta EA_1 + GA_1^3 = 0, \tag{28}$$

$$3GA_0A_1^2 = 0, \tag{29}$$

$$(\alpha E + F)A_1 + 3GA_0^2A_1 = 0, \tag{30}$$

$$FA_0 + GA_0^3 = 0, \tag{31}$$

Eqs. (28) to (31) will lead us to

$$A_0 = 0, A_1 = \pm\sqrt{\frac{\beta F}{\alpha G}}, \alpha E + F = 0. \tag{32}$$

Case 1. $\alpha = -(1+m^2), \beta = 2m^2, \gamma = 1$.
In this case, the solutions of Eq. (23) are $f(s) = \mathrm{sn}(s)$ or $f(s) = \mathrm{cd}(s)$. Then the periodic wave solutions (PWSs) of Eq. (26) are,

$$g(s) = \pm\sqrt{\frac{\{\omega + a(\kappa_1^2 + \kappa_2^2)\}(v - k_1B_1 - k_2B_2)}{(1+m^2)bc}} m\,\mathrm{sn}(s), \tag{33}$$

and

$$g(s) = \pm\sqrt{\frac{\{\omega + a(\kappa_1^2 + \kappa_2^2)\}(v - k_1 B_1 - k_2 B_2)}{(1+m^2)bc}}\, m\,\mathrm{cd}(s), \tag{34}$$

where $\mathrm{sn}(s)$ is the Jacobi sine elliptic function, $\mathrm{cn}(s)$ is the Jacobi cosine elliptic function, $\mathrm{dn}(s)$ is the Jacobi elliptic function of the third kind and $\mathrm{cd}(s) = \dfrac{\mathrm{cn}(s)}{\mathrm{dn}(s)}$.

When $m \to 1$, Eq. (33) leads us to the solution of Eqs. (3) and (4) as

$$q(x,y,t) = \pm A_{11} \tanh(B_1 x + B_2 y - vt) e^{i(-\kappa_1 x - \kappa_2 y + \omega t + \theta)}, \tag{35}$$

and

$$r(x,y,t) = \frac{\{\omega + a(\kappa_1^2 + \kappa_2^2)\}}{2b} \tanh^2(B_x + B_2 y - vt) \times e^{2i(-\kappa_1 x - \kappa_2 y + \omega t + \theta)}, \tag{36}$$

where

$$A_{11} = \sqrt{\frac{\{\omega + a(\kappa_1^2 + \kappa_2^2)\}(v - k_1 B_1 - k_2 B_2)}{2bc}}. \tag{37}$$

Case 2. $\alpha = -(1+m^2), \beta = 2m^2, \gamma = 1$.

Here, the solution of Eq. (23) is $f(s) = \mathrm{cn}(s)$. Thus the PWS of Eq. (26) is,

$$g(s) = \pm\sqrt{\frac{\{\omega + a(\kappa_1^2 + \kappa_2^2)\}(v - k_1 B_1 - k_2 B_2)}{(2m^2-1)bc}}\, m\,\mathrm{cn}(s). \tag{38}$$

When $m \to 1$, Eq. (38) leads us to the solution of Eqs. (3) and (4) as

$$q(x,y,t) = \pm\sqrt{2} A_{11}\, \mathrm{sech}(B_1 x + B_2 y - vt) e^{i(-\kappa_1 x - \kappa_2 y + \omega t + \theta)}, \tag{39}$$

and

$$r(x,y,t) = \frac{\{\omega + a(\kappa_1^2 + \kappa_2^2)\}}{b} \mathrm{sech}^2(B_x + B_2 y - vt) e^{2i(-\kappa_1 x - \kappa_2 y + \omega t + \theta)}. \tag{40}$$

Case 3. $\alpha = 2 - m^2, \beta = -2\gamma = m^2 - 1$.

Here, the solution of Eq. (23) is $f(s) = \mathrm{dn}(s)$. Thus the PWS of Eq. (26) is,

$$g(s) = \pm\sqrt{\frac{\{\omega + a(\kappa_1^2 + \kappa_2^2)\}(v - k_1 B_1 - k_2 B_2)}{(2-m^2)bc}}\, \mathrm{dn}(s). \tag{41}$$

When $m \to 1$, Eq. (41) will lead us to the same solutions (39) and (40) for Eqs. (3) and (4).

Case 4. $\alpha = -(1+m^2), \beta = 2, \gamma = 1 - m^2$.

In this case, the solutions of Eq. (23) are $f(s) = \mathrm{ns}(s)$ or $f(s) = \mathrm{dc}(s)$. Then the periodic wave solutions (PWSs) of Eq. (26) are,

$$g(s)=\pm\sqrt{\frac{\{\omega+a(\kappa_1^2+\kappa_2^2)\}(v-k_1B_1-k_2B_2)}{(1+m^2)bc}}\,\mathrm{ns}(s), \tag{42}$$

and

$$g(s)=\pm\sqrt{\frac{\{\omega+a(\kappa_1^2+\kappa_2^2)\}(v-k_1B_1-k_2B_2)}{(1+m^2)bc}}\,\mathrm{cd}(s), \tag{43}$$

where $\mathrm{ns}(s)=\dfrac{1}{\mathrm{sn}(s)}$ and $\mathrm{dc}(s)=\dfrac{\mathrm{dn}(s)}{\mathrm{cn}(s)}$.

When $m\to 1$, Eq. (42) leads us to the solution of Eqs. (3) and (4) as

$$q(x,y,t)=\pm A_{11}\coth(B_1x+B_2y-vt)e^{i(-\kappa_1x-\kappa_2y+\omega t+\theta)}, \tag{44}$$

and

$$r(x,y,t)=\frac{\{\omega+a(\kappa_1^2+\kappa_2^2)\}}{2b}\coth^2(B_x+B_2y-vt)e^{2i(-\kappa_1x-\kappa_2y+\omega t+\theta)}. \tag{45}$$

We assume solution of Eq. (26) in the form

$$g(s)=A_0+A_1f+B_1f^{-1}, \tag{46}$$

where f satisfies Eq. (23). Substitution of Eq. (46) in Eq. (26) and use of Eq. (23) yields the following set of algebraic equations which arises from different powers of f so that we have

$$2\gamma EB_1+GB_1^3=0, 3GB_1^2A_0=0, \tag{47}$$

$$\alpha EB_1+FB_1+3GB_1A_0^2+3GB_1^2A_1=0, \tag{48}$$

$$FA_0+GA_0^3+6GB_1A_0A_1=0, \tag{49}$$

$$\alpha EA_1+FA_1+3GA_0^2A_1+3GB_1A_1^2=0, \tag{50}$$

$$3GA_0A_1^2=0, \beta EA_1+GA_1^3=0. \tag{51}$$

Eqs. (47) to (51) will give rise to

$$A_0=0, A_1=\pm\sqrt{-\frac{\beta E}{G}}, B_1=\pm\sqrt{-\frac{2\gamma E}{G}}, \alpha E+F+3GA_1B_1=0. \tag{52}$$

Case 5. $\alpha=-(1+m^2)$, $\beta=2m^2$, $\gamma=1$.

Here, the solution of Eq. (23) is $f(s)=\mathrm{sn}(s)$. Then the PWS of Eq. (26) is,

$$g(s)=\pm\sqrt{\frac{-a(B_1^2+B_2^2)(v-k_1B_1-k_2B_2)}{bc}}\,(m\,\mathrm{sn}(s)+\mathrm{ns}(s)). \tag{53}$$

When $m\to 1$, using Eq. (53) we obtain the solutions of Eqs. (3) and (4) as

$$q(x,y,t)=\pm B_{11}\left(\tanh(B_1x+B_2y-vt)+\coth(B_1x+B_2y-vt)\right)\times e^{i(-\kappa_1x-\kappa_2y+\omega t+\theta)} \tag{54}$$

and

$$r(x,y,t) = B_{22}\left(\tanh(B_1x+B_2y-vt)+\coth(B_1x+B_2y-vt)\right)^2 \times e^{2i(-\kappa_1 x-\kappa_2 y+\omega t+\theta)}, \quad (55)$$

where

$$B_{11} = \sqrt{-\frac{a(B_1^2+B_2^2)(v-k_1B_1-k_2B_2)}{bc}}, B_{22} = -\frac{a(B_1^2+B_2^2)}{b}. \quad (56)$$

Case 6. $\alpha = 2-m^2$, $\beta = 2$, $\gamma = 1-m^2$.

In this case, the solution of eq. (23) is $f(s) = \mathrm{cs}(s)$. Then the PWS of eq. (26) is,

$$g(s) = \pm\sqrt{\frac{-a(B_1^2+B_2^2)(v-k_1B_1-k_2B_2)}{bc}}\left(\mathrm{cs}(s)+\sqrt{1-m^2}\,\mathrm{sc}(s)\right), \quad (57)$$

where $\mathrm{cs}(s) = \dfrac{\mathrm{cn}(s)}{\mathrm{sn}(s)}$ and $\mathrm{sc}(s) = \dfrac{\mathrm{sn}(s)}{\mathrm{cn}(s)}$.

When $m \to 1$, eq. (57) gives rise to the solutions of eqs. (3) and (4) as

$$q(x,y,t) = \pm B_{11}\,\mathrm{csch}(B_1x+B_2y-vt)e^{i(-\kappa_1 x-\kappa_2 y+\omega t+\theta)} \quad (58)$$

and

$$r(x,y,t) = B_{22}\,\mathrm{csch}^2(B_1x+B_2y-vt)e^{2i(-\kappa_1 x-\kappa_2 y+\omega t+\theta)}. \quad (59)$$

5. LIE SYMMETRY APPROACH

In this section, we will apply the Lie group method [3,21,22], sometimes also called symmetry analysis, on Eqs. (1) and (2). Roughly speaking, a symmetry group of a system of differential equations is a group which transforms solutions of the system to other solutions. To start with, one can directly use the defining property of such a group and construct new solutions to the system from known ones. Some of the recent contributions in this field are [18,26].

As q is a complex variable, so to separate the real and imaginary parts of u, we will consider

$$q(x,t) = u(x,y,t)+iv(x,y,t). \quad (60)$$

Using (60), system of Eqs. (1) and (2) decomposes into following system of equations

$$\begin{aligned} -v_t+a(u_{xx}+u_{yy})+bur &= 0\\ u_t+a(v_{xx}+v_{yy})+bvr &= 0\\ r_t+k_1r_x+k_2r_y+c((u^2+v^2)^n)_x &= 0 \end{aligned} \quad (61)$$

Let us consider the Lie group of point transformations

$$
\begin{aligned}
x^* &\to x+\epsilon\xi(x,t,y,u,v,r)\\
y^* &\to y+\epsilon\zeta(x,t,y,u,v,r)\\
t^* &\to t+\epsilon\tau(x,t,y,u,v,r)\\
u^* &\to u+\epsilon\eta(x,t,y,u,v,r)\\
v^* &\to v+\epsilon\phi(x,t,y,u,v,r)\\
r^* &\to r+\epsilon\psi(x,t,y,u,v,r)
\end{aligned}
\tag{62}
$$

which leaves the system (61) invariant. In other words, the transformations are such that if u, v, r is the solution of system (61) then u^*, v^*, r^* is also a solution. The method for determining the symmetry group of system (61), mainly consists in finding the infinitesimals ξ, ζ, τ, η, ϕ and ψ, which are functions of x, y, t, u, v and r. One can obtain reductions of system (61) to a system of PDEs in 2 dimensions after getting the similarity variable and similarity solution by solving the characteristic equations

$$
\frac{dx}{\xi}=\frac{dy}{\zeta}=\frac{dt}{\tau}=\frac{du}{\eta}=\frac{dv}{\phi}=\frac{dr}{\psi}. \tag{63}
$$

For system (61), the Lie classical method provides the following forms for the infinitesimal elements ξ, ζ, τ, η, ϕ and ψ:

$$
\begin{gathered}
\xi=a_2+\frac{x+k_1t}{2}a_6,\quad \zeta=a_3+\frac{y+k_2t}{2}a_6,\quad \tau=a_1+ta_6,\\
\eta=va_4+tva_5-\frac{nv(k_1x+k_2y)+3ua}{4na}a_6,\\
\phi=-ua_4-tua_5+\frac{nu(k_1x+k_2y)-3va}{4na}a_6,\\
\psi=-\frac{1}{b}a_5-ra_6,
\end{gathered}
\tag{64}
$$

where a_1, a_2, a_3, a_4, a_5 and a_6 are arbitrary constants.

Corresponding vector fields are

$$
\begin{aligned}
&V_1=\frac{\partial}{\partial t},\quad V_2=\frac{\partial}{\partial x},\quad V_3=\frac{\partial}{\partial y}\quad V_4=v\frac{\partial}{\partial u}-u\frac{\partial}{\partial v},\\
&V_5=tv\frac{\partial}{\partial u}-tu\frac{\partial}{\partial v}-\frac{1}{b}\frac{\partial}{\partial r},\\
&V_6=\frac{x+k_1t}{2}\frac{\partial}{\partial x}+\frac{y+k_2t}{2}\frac{\partial}{\partial y}+t\frac{\partial}{\partial t}-\frac{nv(k_1x+k_2y)+3ua}{4na}\frac{\partial}{\partial u}\\
&\qquad+\frac{nu(k_1x+k_2y)-3va}{4na}\frac{\partial}{\partial v}-r\frac{\partial}{\partial r}.
\end{aligned}
\tag{65}
$$

To obtain the reductions and exact solutions of Eqs. (1) and (2), we will consider following cases: $(i)\ V_5+\gamma V_1$, $(ii)\ V_4+\alpha V_1+\beta V_2+\lambda V_3$.

Vector Field $V_5+\gamma V_1$

Corresponding to this vector field, similarity variables are given as

$$\xi = x, \quad \tau = y, \quad q = F(\xi,\tau)e^{i\left(-\frac{t^2}{2\gamma}+G(\xi,\tau)\right)}, \quad r = -\frac{t}{b\gamma}+H(\xi,\tau), \tag{66}$$

where ξ, τ are new independent variables and F, G, H are new dependent variables.

Using (66) in Eq. (1) and (2), we have following reduced system of PDEs

$$\begin{aligned} a(F_{\xi\xi}+F_{\tau\tau}-FG_\xi^2-FG_\tau^2)+bFH=0,\\ FG_{\xi\xi}+FG_{\tau\tau}+2F_\xi G_\xi+2F_\tau G_\tau=0,\\ -\frac{1}{b\gamma}+k_1H_\xi+k_2H_\tau+c(F^{2n})_\xi=0. \end{aligned} \tag{67}$$

Again applying Lie symmetry method on system of Eqs. (67), we obtain following symmetries

$$\xi_{11}=C_1, \quad \xi_{12}=C_2, \quad \eta_{11}=0, \quad \eta_{12}=0, \quad \eta_{13}=0, \tag{68}$$

where C_1 and C_2 are arbitrary constants.

Corresponding similarity variables for system (67) are given as

$$\zeta=\xi-\alpha\tau, \quad F=K(\zeta), \quad G=L(\zeta), \quad H=M(\zeta), \tag{69}$$

where ζ is new independent variable and K, L, M are new dependent variables. Using (69) in Eqs. (67), we have following system of ODEs

$$\begin{aligned} a(1+\alpha^2)[K''-KL'^2]+bKM=0\\ 2K'L'+KL''=0\\ -\frac{1}{b\gamma}+(k_1-\alpha k_2)M'+c(K^{2n})'=0, \end{aligned} \tag{70}$$

where $(')$ denotes derivative with respect to ζ.

Employing condition $\alpha=\frac{k_1}{k_2}$, we obtain following solution of system (70)

$$K=\left[\frac{\zeta}{bc\gamma}+C_3\right]^{\frac{1}{2n}}, \qquad L=\frac{nbc\gamma C_1}{n-1}\left[\frac{\zeta}{bc\gamma}+C_3\right]^{\frac{n-1}{n}}+C_2$$

$$M=\frac{a\left(k_2{}^2+k_1{}^2\right)}{4k_2{}^2bn^2\left(\zeta+C_3bc\gamma\right)^2}\left[-1+2n+4\left(\frac{\zeta+C_3bc\gamma}{bc\gamma}\right)^{-\frac{2}{n}}C_1{}^2\zeta^2n^2\right. \tag{71}$$

$$\left.+8\left(\frac{\zeta+C_3bc\gamma}{bc\gamma}\right)^{-\frac{2}{n}}C_1{}^2bc\gamma\zeta C_3n^2+4\left(\frac{\zeta+C_3bc\gamma}{bc\gamma}\right)^{-\frac{2}{n}}C_1{}^2b^2c^2\gamma^2C_3{}^2n^2\right]$$

Corresponding solution of main system of Eqs. (1) and (2) is given as

$$q=\left[\frac{\zeta}{bc\gamma}+C_3\right]^{\frac{1}{2n}}\exp\left\{i\left[-\frac{t^2}{2\gamma}+\frac{nbc\gamma C_1}{n-1}\left(\frac{\zeta}{bc\gamma}+C_3\right)^{\frac{n-1}{n}}+C_2\right]\right\}$$

$$r=-\frac{t}{b\gamma}+\frac{a\left(k_2{}^2+k_1{}^2\right)}{4k_2{}^2bn^2\left(\zeta+C_3bc\gamma\right)^2}\left[-1+2n+4\left(\frac{\zeta+C_3bc\gamma}{bc\gamma}\right)^{-\frac{2}{n}}C_1{}^2\zeta^2n^2\right. \tag{72}$$

$$\left.+8\left(\frac{\zeta+C_3bc\gamma}{bc\gamma}\right)^{-\frac{2}{n}}C_1{}^2bc\gamma\zeta C_3n^2+4\left(\frac{\zeta+C_3bc\gamma}{bc\gamma}\right)^{-\frac{2}{n}}C_1{}^2b^2c^2\gamma^2C_3{}^2n^2\right],$$

where $\zeta=x-\frac{k_1}{k_2}y$ and C_1, C_2, C_3 are arbitrary constants.

Vector Field $V_4+\alpha V_1+\beta V_2+\lambda V_3$

Corresponding to this vector field, similarity variables are given as

$$\xi=\alpha x-\beta t,\ \tau=\alpha y-\lambda t,\ q=F(\xi,\tau)e^{i\left(-\frac{t}{\alpha}+\frac{G(\xi,\tau)}{\alpha}\right)},r=H(\xi,\tau), \tag{73}$$

where ξ, τ are new independent variables and F, G, H are new dependent variables.

Using (73) in Eq. (1) and (2), we have following reduced system of PDEs

$$\begin{aligned}-\alpha\beta F_\xi-\alpha\lambda F_\tau+2a\alpha^2F_\xi G_\xi+a\alpha^2FG_{\xi\xi}+2a\alpha^2F_\tau G_\tau+a\alpha^2FG_{\tau\tau}=0\\ F+\beta FG_\xi+\lambda FG_\tau+a\alpha^3F_{\xi\xi}-a\alpha FG_\xi^2+a\alpha^3F_{\tau\tau}-a\alpha FG_\tau^2+b\alpha FH=0\\ -\beta H_\xi-\lambda H_\tau+k_1\alpha H_\xi+k_2\alpha H_\tau+c\alpha(F^{2n})_\xi=0.\end{aligned} \tag{74}$$

Again applying Lie classical method on system of Eqs. (74), we obtain only trivial symmetries. Thus corresponding similarity variables are

$$\zeta=\xi-\theta\tau,\quad F=K(\zeta),\quad G=L(\zeta),\quad H=M(\zeta), \tag{75}$$

where ζ is new independent variable and K, L, M are new dependent variables. Using similarity variables (75) in system of Eqs. (74), we obtain following system of ODEs

$$\begin{aligned}-\alpha\beta K'+\alpha\lambda\theta K'+2a\alpha^2K'L'+a\alpha^2KL''+2a\alpha^2\theta^2K'L'+a\alpha^2\theta^2KL''=0\\ K+\beta KL'-\lambda\theta KL'+a\alpha^3K''-a\alpha KL'^2+a\alpha^3\theta^2K''\\ -a\alpha\theta^2KL'^2+b\alpha KM=0\\ -\beta M'+\lambda\theta M'+k_1\alpha M'-k_2\alpha\theta M'+c\alpha(K^{2n})'=0,\end{aligned} \tag{76}$$

where prime ($'$) denotes derivative with respect to ζ.

We employ the condition $\theta=\frac{\beta}{\lambda}$ and obtain the following solutions of system

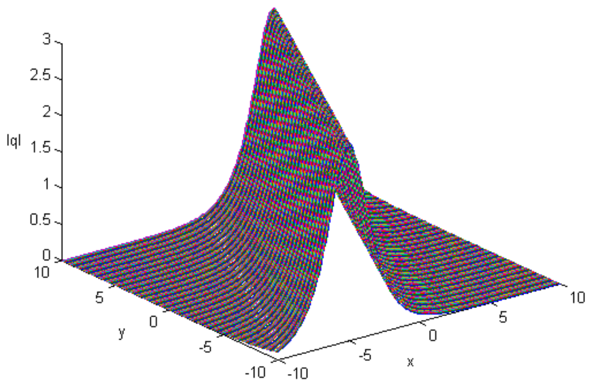

Fig. 1 – Single soliton of $Q^n_{l,m}$ with $t = 2.0$, $n = 1$.

(76)

$$
\begin{aligned}
K &= \left[-n\sqrt{\frac{bc}{a\alpha^2(k_1 - k_2(\frac{\beta}{\lambda}))(1+\frac{\beta^2}{\lambda^2})(n+1)}}\zeta - nC_3 \right]^{-\frac{1}{n}} \\
L &= C_1 \\
M &= \left(\frac{c}{(k_2(\frac{\beta}{\lambda}) - k_1)} \right) \left[-n\sqrt{\frac{bc}{a\alpha^2(k_1 - k_2(\frac{\beta}{\lambda}))(1+\frac{\beta^2}{\lambda^2})(n+1)}}\zeta - nC_3 \right]^{-2} \\
&\quad - b\alpha
\end{aligned}
\tag{77}
$$

Consequently, solution of main system of Eqs. (1) and (2) is given as

$$
\begin{aligned}
q &= \left[-n\sqrt{\frac{bc}{a\alpha^2(k_1 - k_2(\frac{\beta}{\lambda}))(1+\frac{\beta^2}{\lambda^2})(n+1)}}\zeta - nC_3 \right]^{-\frac{1}{n}} e^{i\left(-\frac{t}{\alpha}+\frac{C_1}{\alpha}\right)} \\
r &= \left(\frac{c}{(k_2(\frac{\beta}{\lambda}) - k_1)} \right) \left[-n\sqrt{\frac{bc}{a\alpha^2(k_1 - k_2(\frac{\beta}{\lambda}))(1+\frac{\beta^2}{\lambda^2})(n+1)}}\zeta - nC_3 \right]^{-2} \\
&\quad - b\alpha
\end{aligned}
\tag{78}
$$

where $\zeta = \alpha x - \beta y$ and C_1, C_2, C_3 are arbitrary constants.

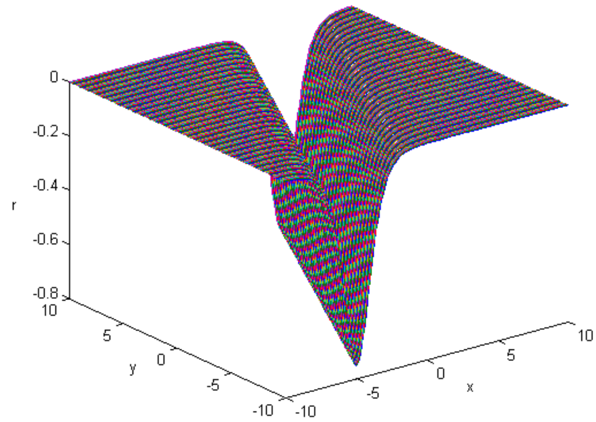

Fig. 2 – Single soliton of $R^n_{l,m}$ with $t = 2.0$, $n = 1$.

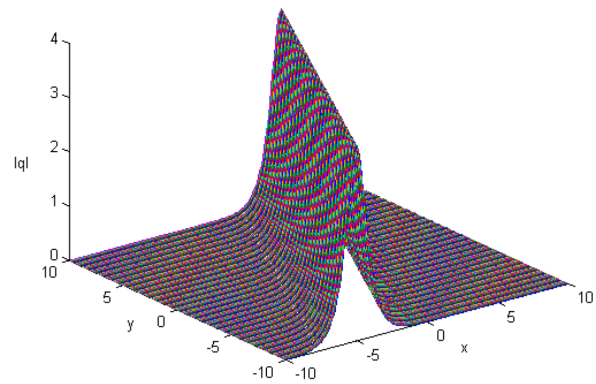

Fig. 3 – Single soliton of $Q^n_{l,m}$ with $t = 0$, $n = 2$.

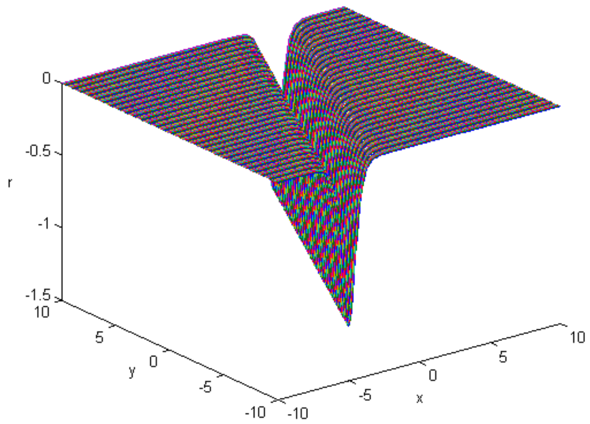

Fig. 4 – Single soliton of $R^n_{l,m}$ with $t=0$, $n=2$.

6. NUMERICAL SIMULATIONS

We first separate the function q in (1) of the form

$$q = u + iv.$$

Substituting in (1) and (2), we have

$$\begin{aligned} u_t + au_{xx} + au_{yy} + cuv = 0, &\quad x_L \le x \le x_R, \quad y_L \le y \le y_R \\ v_t - av_{xx} - av_{yy} - cuv = 0, &\quad x_L \le x \le x_R, \quad y_L \le y \le y_R \\ r_t + k_1 r_x + k_2 r_y + c(u^2+v^2)^n_x = 0, &\quad x_L \le x \le x_R, \quad y_L \le y \le y_R. \end{aligned} \tag{79}$$

We assume that $q^n_{l,m}$, $r^n_{l,m}$ be the exact solution and $Q^n_{l,m}$, $R^n_{l,m}$ be the approximate solution at the grid point (x_l, y_m, t_n). The proposed scheme can be displayed as

$$\begin{aligned} \frac{1}{k}\delta_t U^n_{\ell,m} + \frac{a}{h^2}(\delta_x^2+\delta_y^2)U^{n+\frac{1}{2}}_{\ell,m} + cU^{n+\frac{1}{2}}_{\ell,m}V^{n+\frac{1}{2}}_{\ell,m} = 0 \\ \frac{1}{k}\delta_t V^n_{\ell,m} - \frac{a}{h^2}(\delta_x^2+\delta_y^2)V^{n+\frac{1}{2}}_{\ell,m} - cU^{n+\frac{1}{2}}_{\ell,m}V^{n+\frac{1}{2}}_{\ell,m} = 0 \\ \frac{1}{k}\delta_t R^n_{\ell,m} + \frac{a}{h^2}(\kappa_1\delta_x+\kappa_2\delta_y)R^{n+\frac{1}{2}}_{\ell,m} + \frac{c}{h}\delta_x((U^{n+\frac{1}{2}}_{\ell,m})^2 + (V^{n+\frac{1}{2}}_{\ell,m})^2)^n = 0 \end{aligned}$$

where

$$U_{\ell,m}^{n+\frac{1}{2}} = \frac{U_{\ell,m}^{n+1} + U_{\ell,m}^{n}}{2}, \quad V_{\ell,m}^{n+\frac{1}{2}} = \frac{V_{\ell,m}^{n+1} + V_{\ell,m}^{n}}{2}, \quad R_{\ell,m}^{n+\frac{1}{2}} = \frac{R_{\ell,m}^{n+1} + R_{\ell,m}^{n}}{2}$$

$$\delta_t U_{\ell,m}^{n} = U_{\ell,m}^{n+1} - U_{\ell,m}^{n}, \quad \delta_x U_{\ell,m}^{n} = U_{\ell+1,m}^{n} - U_{\ell,m}^{n}, \quad \delta_y U_{\ell,m}^{n} = U_{\ell,m+1}^{n} - U_{\ell,m}^{n}$$

$$\delta_x^2 U_{\ell,m}^{n} = U_{\ell+1,m}^{n} - 2U_{\ell,m}^{n} + U_{\ell-1,m}^{n},$$

$$\delta_y^2 U_{\ell,m}^{n} = U_{\ell,m+1}^{n} - 2U_{\ell,m}^{n} + U_{\ell,m-1}^{n}$$

The similar notation for $\delta_t V_{l,m}^n$, $\delta_t R_{l,m}^n$, ... The proposed scheme is implicit and can be easily solved by the fixed point method. The scheme is second order in space and time directions. The initial conditions are extracted from the exact solutions

$$q(x,y,0) = A_1 \operatorname{sech}^{\frac{1}{n}}(B(B_1 x + B_2 y))e^{i(-k_1 x - k_2 y)}$$

$$r(x,y,0) = A_2 \operatorname{sech}^2(B(B_1 x + B_2 y))$$

We choose the parameter

$$h = 0.1, \qquad k = 0.01,$$

$$x_L = y_L = -10, \qquad x_R = y_R = 10,$$

$$B_1 = 1, \quad B_2 = -0.5, \quad k_1 = 1, \quad k_2 = -0.05,$$

$$\omega = 0.5, \quad a = -1, \quad b = 1, \quad c = 0.5, \quad v = 0.3.$$

Figures 1 and 2 display the numerical solutions of $|q|$ and r at $n = 1$ respectively, while Figures 3 and 4 display the numerical solutions of $|q|$ and r at $n = 2$, respectively.

7. CONCLUSIONS

This paper discussed the MS in details. The power law nonlinearity was taken into account rather than the usually studied cubic nonlinearity. This law sets the equation on a generalized flavour. Therefore, the results of this paper are on a richer taste than that of other papers that are seen in other publications on this topic. The travelling wave hypothesis yields the soliton solutions to the system when the integration constants are all taken to be zero. The mapping method is subsequently employed to reveal cnoidal waves, snoidal waves, dark solitons, solitary waves, singular solitons and other doubly periodic functions. The Lie symmetry approach was then subsequently applied to obtain several other types of solutions. Finally, the numerical simulations for the MS are displayed. In this context, the finite difference approach was employed.

These results have a profound impact on this equation, especially with power law nonlinearity. There is an enormous possibility of venturing into this equation in future. The time-dependent coefficients will be considered. Additionally, this equation will be further considered when the solutions will be extracted via the ansatz method which will reveal the solitary waves, dark solitons and others [27–30]. These studies will be compared with other forms of numerical analysis such as variational iteration method and several others. These just form the tip of the iceberg.

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