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DYNAMICS OF TWO-LAYERED SHALLOW WATER WAVES WITH COUPLED KdV EQUATIONS

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Abstract. The dynamics of two-layered shallow water waves is studied in this paper by the aid of coupled potential-Korteweg-de Vries (pKdV) equation. There are two types of models that are considered. The first model yields solitary waves as well as shock wave solutions. The second model yields shock wave solutions only. The ansatz method is applied to retrieve the solitary wave as well as the shock wave solutions of both models. In both cases the time-dependent coefficients of dispersion and nonlinearity are considered, in order to keep the model closer to reality. Numerical simulations demonstrate the presence of soliton behaviour that is both rich and diverse.

Key words: KdV equation, soliton, travelling wave, ansatz method.

1. INTRODUCTION

The study of solitons appears in various areas of physical sciences including complex molecular systems [1], nonlinear lattices [2], fluid dynamics [3], plasma physics [4], nonlinear optics [5–12], and Bose-Einstein condensation [13–19]. There are several mathematical models that examine the dynamics of shallow water waves. A few of these are the Korteweg-de Vries (KdV) equation [20–22], Boussinesq equation [23, 24], Peregrine equation [25], Benjamin-Bona-Mahoey equation [26], and Sharma-Tasso-Olver equation [27]. In the case of the two-layered shallow water waves there are several associted models that describe their dynamics accurately. These are the coupled KdV [28], Boussinesq [23], Gear-Grimshaw [29], and Bona-Chen [30] equations. Two-layered shallow water waves arise in real-life scenarios across the globe. For example, the Alaska-Valdez oil spill formed a two-layered shallow water wave as it moved on to Alaskan beaches and shores. The top layer is the less dense gasoline while the lower layer is the water. In such a situation the dynamics of shallow water waves may be governed by the coupled potential-KdV (pKdV) equation [31]. The first model (Model-I) of the integrable coupled pKdV equation reads

$$\begin{aligned} u_t &= u_{xxx} + 3uu_{xx} - 3vu_{xx} + 3u_x^2 + 3u_x v_x + 3u^2 u_x - 6uvu_x + 3v^2 u_x, \\ v_t &= v_{xxx} - 3uv_{xx} + 3vv_{xx} + 3u_x v_x + 3v_x^2 + 3u^2 v_x - 6uvu_x + 3v^2 v_x, \end{aligned} \tag{1}$$

whereas the second model (Model-II) reads

$$\begin{aligned} u_t &= \frac{1}{2}u_{xxx} + \frac{1}{2}v_{xxx} + 2u_x^2 + v_x^2, \\ v_t &= \frac{1}{2}u_{xxx} + \frac{1}{2}v_{xxx} + u_x^2 + 2v_x^2. \end{aligned}$$

Exact travelling wave solutions for these two pKdV model equations were obtained using an extended tanh-function method by Yang [32].

This paper undertakes a study of a generalization of these models by including time-dependent coefficients. This makes the models much closer to practical situations where external environmental factors can alter the parameters of the model in time (and most likely in space as well). In the sections that follow we introduce the time-dependent pKdV equations and apply the so-called *ansatz method* to find exact travelling wave solutions. The ansatz method is possibly the simplest of all the algebraic methods. At the heart of it is the assumption that the exact solutions are prescribed functions depending on several yet to be determined parameters. These important parameters include quantities such as the amplitude, width, and speed of the solitons. The parameters are determined by direct substitution of the ansatz into the partial differential equations. This results in obtaining a set of compatibility conditions which relate the parameters of the solitons to the coefficients in the pKdV equations. These conditions must hold in order for solitons to exist.

2. SOLITON SOLUTIONS TO MODEL-I

The problem of interest consists in solving the following class of nonlinear coupled pKdV equations with time-dependent coefficients:

$$\begin{aligned} u_t &= a_1(t)u_{xxx} + a_2(t)uu_{xx} - a_3(t)vu_{xx} + a_4(t)u_x^2 + a_5(t)u_x v_x \\ &\quad + a_6(t)u^2 u_x - a_7(t)uvu_x + a_8(t)v^2 u_x, \\ v_t &= b_1(t)v_{xxx} - b_2(t)uv_{xx} + b_3(t)vv_{xx} + b_4(t)u_x v_x + b_5(t)v_x^2 \\ &\quad + b_6(t)u^2 v_x - b_7(t)uvu_x + b_8(t)v^2 v_x, \end{aligned} \tag{2}$$

where $a_i(t)$ and $b_i(t)$ $(i = 1,...,8)$ are Riemann integrable but otherwise arbitrary time-dependent functions.

2.1. SOLITARY WAVES

To find solitary wave solutions of (2) we adopt a solitary wave ansatz of the form [29, 33]

$$u(x,t) = A_1(t)\,\mathrm{sech}^p\,\tau, \qquad v(x,t) = A_2(t)\,\mathrm{sech}^q\,\tau, \tag{3}$$

where $\tau = B(t)[x - c(t)t]$.

Here, $A_1(t)$ and $A_2(t)$ are the amplitudes of the u-soliton and v-soliton, respectively, while $c(t)$ is the velocity and $B(t)$ is the inverse width of the solitons, assumed to be the same for both u and v. The exponents p and q are unknown constants and will be determined later. Our aim is to determine the unknown functions $A_1(t)$, $A_2(t)$, $c(t)$, and $B(t)$ in terms of the model coefficients $a_i(t)$ and $b_i(t)$.

Taking into account the differential properties of the hyperbolic functions and defining the degree of u and v as $D(u) = p$ and $D(u) = q$, we have

$$\begin{aligned} D\left(u^n(\tau)v^m(\tau)\left(\frac{d^s u(\tau)}{d\tau^s}\right)^l\right) &= np + mq + l(p+s),\\ D\left(u^n(\tau)v^m(\tau)\left(\frac{d^s v(\tau)}{d\tau^s}\right)^l\right) &= np + mq + l(q+s). \end{aligned} \tag{4}$$

Balancing the highest derivative term u_{xxx} (or v_{xxx}) with nonlinear terms such as u_x^2, u^2u_x or u_xv_x yields $p = 1$, $q = 1$. Then substituting the *travelling wave ansatz* (3) into (2) yields the system

$$\begin{aligned} & A_1'\,\mathrm{sech}\,\tau - A_1\frac{B'}{B}\tau\,\mathrm{sech}\,\tau\,\tanh\tau + A_1B((c+tc') + a_1B^2)\,\mathrm{sech}\,\tau\,\tanh\tau \\ & \quad - A_1B^2(a_2A_1 - a_3A_2 + a_4A_1 + a_5A_2)\,\mathrm{sech}^2\,\tau \\ & \quad - A_1B(6a_1B^2 - a_6A_1^2 + a_7A_1A_2 - a_8A_2^2)\,\mathrm{sech}^3\,\tau\,\tanh\tau \\ & \quad + A_1B^2(2a_2A_1 - 2a_3A_2 + a_4A_1 + a_5A_2)\,\mathrm{sech}^4\,\tau = 0 \end{aligned} \tag{5}$$

$$\begin{aligned} & A_2'\,\mathrm{sech}\,\tau - A_2\frac{B'}{B}\tau\,\mathrm{sech}\,\tau\,\tanh\tau + A_2B((c+tc') + b_1B^2)\,\mathrm{sech}\,\tau\,\tanh\tau \\ & \quad + A_2B^2(b_2A_1 - b_3A_2 - b_4A_1 - b_5A_2)\,\mathrm{sech}^2\,\tau \\ & \quad - A_2B(6b_1B^2 - b_6A_1^2 + b_7A_1A_2 + b_8A_2^2)\,\mathrm{sech}^3\,\tau\,\tanh\tau \\ & \quad - A_2B^2(2b_2A_1 - 2b_3A_2 - b_4A_1 - b_5A_2)\,\mathrm{sech}^4\,\tau = 0. \end{aligned} \tag{6}$$

Equating the coefficients of the linearly independent functions to zero and noting that

$A_1(t) \neq 0$, $A_2(t) \neq 0$ and $B(t) \neq 0$ for all t, we obtain

$$\begin{aligned} A_1' = 0, \quad B' = 0, \\ c + tc' + a_1 B^2 = 0, \\ a_2 A_1 - a_3 A_2 + a_4 A_1 + a_5 A_2 = 0, \\ 2a_2 A_1 - 2a_3 A_2 + a_4 A_1 + a_5 A_2 = 0, \\ -6a_1 B^2 + a_6 A_1^2 - a_7 A_1 A_2 + a_8 A_2^2 = 0, \end{aligned} \tag{7}$$

and

$$\begin{aligned} A_2' = 0, \quad B' = 0, \\ c + tc' + b_1 B^2 = 0, \\ b_2 A_1 - b_3 A_2 - b_4 A_1 - b_5 A_2 = 0, \\ 2b_2 A_1 - 2b_3 A_2 - b_4 A_1 - b_5 A_2 = 0, \\ -6b_1 B^2 + b_6 A_1^2 - b_7 A_1 A_2 + b_8 A_2^2 = 0. \end{aligned} \tag{8}$$

From (7) and (8) we deduce that $A_1(t)$, $A_2(t)$ and $B(t)$ are all (positive) constants. We set $A_1(t) = A_1(0) = \lambda_1$, $A_2(t) = A_2(0) = \lambda_1$ and $B(t) = B(0) = B_0$, where λ_1 and λ_2 are integral constants related to the initial amplitudes of the two solitons, and B_0 is the initial inverse width. The remaining equations yield

$$\frac{a_2(t)}{a_3(t)} = \frac{b_2(t)}{b_3(t)} = \frac{\lambda_2}{\lambda_1}, \tag{9}$$

$$c(t) = -\frac{B_0^2}{t} \int_0^t a_1(t')\, dt' = -\frac{B_0^2}{t} \int_0^t b_1(t')\, dt', \tag{10}$$

$$B_0^2 = \frac{a_6 \lambda_1^2 - a_7 \lambda_1 \lambda_2 + a_8 \lambda_1^2}{6a_1} = \frac{b_6 \lambda_1^2 - b_7 \lambda_1 \lambda_2 + b_8 \lambda_1^2}{6b_1}. \tag{11}$$

Since $c(t)$ is single-valued we require that $a_1(t) = b_1(t)$ and from the expression for B_0 we get

$$(a_6 - b_6)\lambda_1^2 + (b_7 - a_7)\lambda_1 \lambda_2 = (b_8 - a_8)\lambda_2^2, \tag{12}$$

which serves as a constraint between the coefficients and soliton amplitudes.

Having obtained expressions for all of the parameters we construct a family of solitary wave solutions for Model-I in the form

$$\begin{aligned} u(x,t) = \lambda_1 \operatorname{sech}\{B_0[x - c(t)t]\}, \\ v(x,t) = \lambda_2 \operatorname{sech}\{B_0[x - c(t)t]\}, \end{aligned} \tag{13}$$

where λ_1, λ_2 and B_0 are arbitrary constants and $c(t)$ is the time-varying velocity given by (10). These solutions exist provided that the conditions (9) – (12) are satisfied. We note that the values of a_4, a_5, b_4, and b_5 do not appear to have any effect on the solitary wave parameters nor on its existence.

2.1.1. Numerical simulations

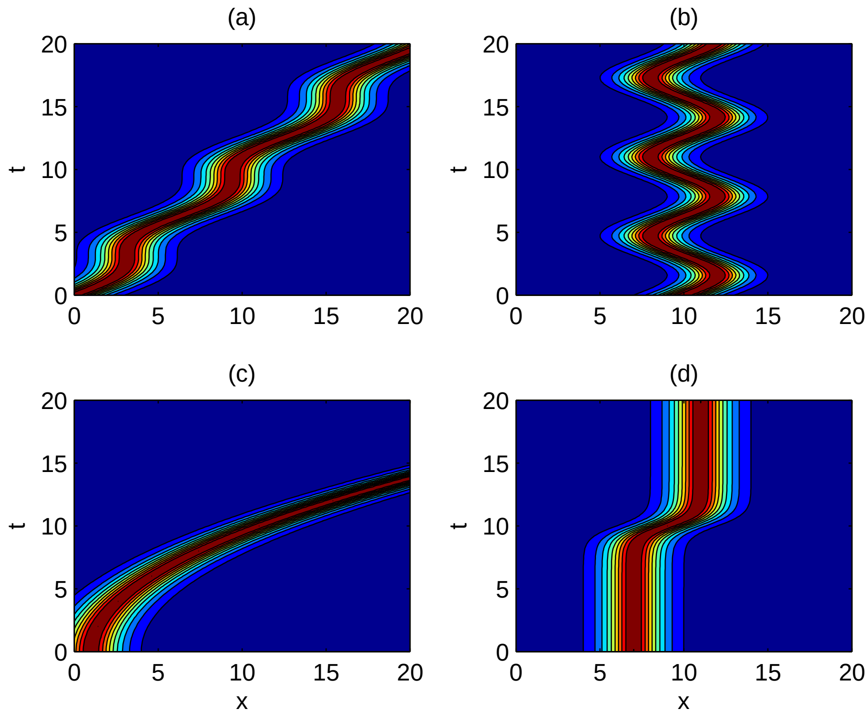

Fig. 1 – Contour plots of the u wave with $\lambda_1 = 1$ indicating a variety of wave behaviours: (a) wobbling, (b) swinging, (c) accelerating, and (d) laterally shifted solitary waves.

Comparing (1) with (2) we observe that $a_1 = b_1 = 1$, $a_2 = a_3 = a_4 = a_5 = a_6 = a_8 = 3$, $b_2 = b_3 = b_4 = b_5 = b_6 = b_8 = 3$, and $a_7 = b_7 = 6$. These choices satisfy the compatability conditions (9) and (12). According to (11) we must have $\lambda_1 \neq \lambda_2$, that is, it is not possible to have solitary waves of equal amplitudes with the same width. However, this is no longer true if the parameters are allowed to be time-dependent. In providing a few numerical examples we consider the degenerate case only ($\lambda_1 = \lambda_2$) with different functional forms for $a_1(t)$ ($b_1(t)$) and $a_8(t)$ ($b_8(t)$), whilst keeping all other parameters as for (1). We consider cases for which $a_1(t)$ and $a_8(t)$ are both periodic and when both are not periodic functions. In all of our examples the parameters are chosen so that $B_0 = 1$. For our first example we set $a_1(t) = -(1 + \cos(t))$ and $a_8(t) = 3 - 6(1 + \cos(t))$. Figure 1(a) illustrates the evolution of an initial profile $u(x,0) = \text{sech}(x)$ into a *'wobbling' soliton*. The frequency of $a_1(t)$ and $a_8(t)$ determine the degree of wobbling experienced by the soliton. If instead we choose $a_1(t) = -2\cos(t)$ and $a_8(t) = 3 - 12\cos(t)$, an initial wave profile $u(x,0) = \text{sech}(x-10)$ will give rise to a *'swinging' soliton*. The swings are centered about $x = 10$ with an amplitude of two units (see Fig. 1(b)).

Figures 1(c) and 1(d) depict solitons for non-periodic function of $a_1(t)$ and $a_8(t)$. Setting $a_1(t) = -0.2t$ and $a_8(t) = 3 - 1.2t$ results in a soliton whose speed increases linearly, $c(t) = 0.1t$, thus representing an *accelerating soliton* travelling along the parabolic trajectory $x = 0.1t^2$. Figure 1(c) shows a soliton that is narrowing as it evolves. A rather interesting behaviour emerges when $a_1(t) = -2\text{sech}^2(t)$ and $a_8(t) = 3 - 12\text{sech}^2(t)$. Figure 1(d) shows that an initial soliton with profile $u(x,0) = \text{sech}(x-7)$ remains stationary over $x = 7$ for most of the time when suddenly shifts laterally over a relatively short time frame before coming to a stop again at $x = 11$.

2.2. SHOCK WAVES

To obtain *shock wave solutions* of (2), we adopt the shock wave ansatz of the form [34,35]

$$\begin{aligned} u(x,t) &= A_1(t)\tanh^p \tau, \\ v(x,t) &= A_2(t)\tanh^q \tau, \end{aligned} \tag{14}$$

where $\tau = B(t)[x - c(t)t]$.

Using (4) and again balancing third-order dispersion with nonlinear terms we arrive at $p = 1$ and $q = 1$. Inserting the ansatz (14) into (2) and following a similar procedure as for the solitary waves of the previous section, we get for the shock wave parameters

$$\begin{gathered} A_1' = 0, \quad B' = 0, \\ (tc)' - 2a_1B^2 + a_4A_1B + a_5A_2B = 0, \\ (tc)' - 8a_1B^2 + B(2a_2A_1 - 2a_3A_2 + 2a_5A_2) - a_6A_1^2 \\ +a_7A_1A_2 - a_8A_2^2 = 0, \\ 6a_1B^2 - B(2a_2A_1 - 2a_3A_2 - a_4A_1 + a_5A_2) + a_6A_1^2 \\ -a_7A_1A_2 + a_8A_2^2 = 0, \end{gathered} \tag{15}$$

and

$$\begin{gathered} A_2' = 0, \quad B' = 0, \\ (tc)' - 2b_1B^2 + b_4A_1B + b_5A_2B = 0, \\ (tc)' - 8b_1B^2 - B(2b_2A_1 - 2b_3A_2 - 2b_4A_1 - 2b_5A_2) - b_6A_1^2 \\ +b_7A_1A_2 - b_8A_2^2 = 0, \\ 6b_1B^2 + 2b_2A_1B - b(2b_3A_2 + b_4A_1 + b_5A_2) + b_6A_1^2 \\ -b_7A_1A_2 + b_8A_2^2 = 0. \end{gathered} \tag{16}$$

Again $A_1(t) = \lambda_1$, $A_2(t) = \lambda_2$ and $B(t) = B_0$ are constants of the motion. From (15) an expression for B_0 reads

$$B_0 = \frac{1}{12a_1}\left[C_1 + \sqrt{C_1^2 - 24a_1(a_6\lambda_1^2 - a_7\lambda_1\lambda_2 + a_8\lambda_2^2)}\right], \tag{17}$$

where $C_1 = (2a_2 - a_4)\lambda_1 + (a_5 - 2a_3)\lambda_2$. From (16) another expression for B_0 is

$$B_0 = \frac{1}{12b_1}\left[D_1 + \sqrt{D_1^2 - 24b_1(b_6\lambda_1^2 - b_7\lambda_1\lambda_2 + b_8\lambda_2^2)}\right], \tag{18}$$

where $D_1 = (2b_2 - b_4)\lambda_1 + (b_5 - 2b_3)\lambda_2$. Equality between these gives rise to a complicated compatibility condition between the time-dependent coefficients and the amplitudes λ_1 and λ_2,

$$\begin{aligned} b_1C_1 - a_1D_1 = a_1&\sqrt{C_1^2 - 24a_1(a_6\lambda_1^2 - a_7\lambda_1\lambda_2 + a_8\lambda_2^2)} \\ &- b_1\sqrt{D_1^2 - 24b_1(b_6\lambda_1^2 - b_7\lambda_1\lambda_2 + b_8\lambda_2^2)}. \end{aligned} \tag{19}$$

Further, the expressions for the speed are

$$c(t) = \frac{B_0}{t}\int_0^t [2B_0a_1(t') - \lambda_1a_4(t') - \lambda_2a_5(t')]dt', \tag{20}$$

$$c(t) = \frac{B_0}{t}\int_0^t [2B_0b_1(t') - \lambda_1b_4(t') - \lambda_2b_5(t')]dt', \tag{21}$$

and requiring the speed be the same for both u and v leads to the constraint

$$2B_0(a_1 - b_1) = \lambda_1(a_4 - b_4) + \lambda_2(a_5 - b_5). \tag{22}$$

This latter equation can easily be satisfied if we choose $a_1(t) = b_1(t)$, $a_4(t) = b_4(t)$, and $a_5(t) = b_5(t)$.

If we insert the soliton parameters into (14) the shock wave solutions for the coupled pKdV system (2) are

$$\begin{aligned} u(x,t) &= \lambda_1 \tanh\{B_0[x - c(t)t]\}, \\ v(x,t) &= \lambda_2 \tanh\{B_0[x - c(t)t]\}, \end{aligned} \tag{23}$$

where λ_1, λ_2, B_0, and speed $c(t)$ are determined from the above set of compatibility conditions.

2.2.1. Numerical simulations

As with the solitary waves solutions, using the coefficients of (2) we deduce that no shock waves can exist whose amplitudes are equal. But, unlike the solitary waves case, the coefficients $a_4(t)$ ($b_4(t)$) and $a_5(t)$ ($b_5(t)$) now play an important role in determining the speed $c(t)$.

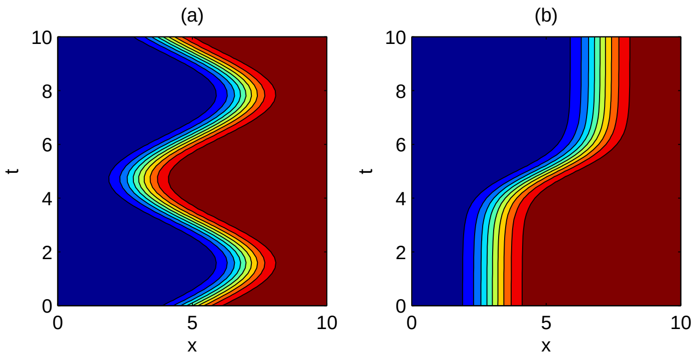

Fig. 2 – Contour plots of the u wave with $\lambda_1 = 1$ showing (a) an oscillating and (b) a laterally shifted shock wave.

For illustrative purposes we choose $a_1 = 1 + \cos(t)$, $a_8 = 3 - 6(1 + \cos(t))$ and $a_4 = a_5 = 1$. A plot of the contours of the u-soliton is given in Fig. 2(a) for the initial profile $u(x,0) = \tanh(x-5)$. The front of the shock wave varies periodically with a frequency fixed by $a_1(t)$ ($a_8(t)$). The time evolution of this shock wave is reminiscent of the 'swinging' solitary wave discussed earlier.

We can also obtain a laterally shifted shock wave, as depicted in Fig. 2(b), by setting $a_1 = \text{sech}^2(t-5) + 1$, $a_8 = 3 - 6(\text{sech}^2(t-5) + 1)$ and $a_4 = a_5 = 1$ with initial profile given by $u(x,0) = \tanh(x-3)$.

3. SOLITON SOLUTIONS TO MODEL-II

Let us now consider the coupled pair of pKdV equations given by

$$\begin{aligned} u_t &= a_1(t)u_{xxx} + a_2(t)v_{xxx} + a_3(t)u_x^2 + a_4(t)v_x^2, \\ v_t &= b_1(t)u_{xxx} + b_2(t)v_{xxx} + b_3(t)u_x^2 + b_4(t)v_x^2, \end{aligned} \tag{24}$$

where $a_i(t)$ and $b_i(t)$ $(i = 1, ..., 4)$ are Riemann integrable arbitrary real functions of the variable t. It should be noted that this system does not admit solitary wave solutions, thus we shall only seek to find exact shock wave solutions. The starting hypothesis is the same as (14). Balancing the dispersive terms with nonlinear terms we once again get $p = q = 1$. Setting the coefficients of independent functions to zero we obtain the following system of equations:

$$\begin{gathered}
A_1' = 0, \quad B' = 0, \\
A_1(c+tc') - B^2(2a_1A_1 + 2a_2A_2) + B(a_3A_1^2 + a_4A_2^2) = 0, \\
A_1(c+tc') - B^2(8a_1A_1 + 8a_2A_2) + B(2a_3A_1^2 + 2a_4A_2^2) = 0, \\
B(6a_1A_1 + 6a_2A_2) - a_3A_1^2 - a_4A_2^2 = 0,
\end{gathered} \tag{25}$$

and

$$\begin{gathered}
A_2' = 0, \quad B' = 0 \\
A_2(c+tc') - B^2(2b_1A_1 + 2b_2A_2) + B(b_3A_1^2 + b_4A_2^2) = 0, \\
A_2(c+tc') - B^2(8b_1A_1 + 8b_2A_2) + B(2b_3A_1^2 + 2b_4A_2^2) = 0, \\
B(6b_1A_1 + 6b_2A_2) - b_3A_1^2 - b_4A_2^2 = 0.
\end{gathered} \tag{26}$$

Solving these equations, we let as before $A_1(t) = \lambda_1$, $A_2(t) = \lambda_2$, and $B(t) = B_0$, where

$$B_0 = \frac{a_3\lambda_1^2 + a_4\lambda_2^2}{6(a_1\lambda_1 + a_2\lambda_2)} = \frac{b_3\lambda_1^2 + b_4\lambda_2^2}{6(b_1\lambda_1 + b_2\lambda_2)}. \tag{27}$$

The expressions for the speed are

$$\begin{aligned}
c(t) &= \frac{B_0}{\lambda_1 t}\int_0^t \left\{2B_0[\lambda_1 a_1(t') + \lambda_2 a_2(t')] - [\lambda_1^2 a_3(t') + \lambda_2^2 a_4(t')]\right\} dt', \\
c(t) &= \frac{B_0}{\lambda_2 t}\int_0^t \left\{2B_0[\lambda_1 b_1(t') + \lambda_2 b_2(t')] - [\lambda_1^2 b_3(t') + \lambda_2^2 b_4(t')]\right\} dt'.
\end{aligned}$$

Equating the two expressions of $c(t)$ gives the compatability condition:

$$2B_0\left[(a_1 - b_2)\lambda_1\lambda_2 + \lambda_2^2 a_2 - \lambda_1^2 b_1\right] = \lambda_2^3 a_4 - \lambda_1^3 b_3 + (\lambda_1 a_3 - \lambda_2 b_4)\lambda_1\lambda_2. \tag{28}$$

Also from equating the two values of B from (27) leads to the additional compatability condition

$$\left(a_3A_1^2 + a_4A_2^2\right)(b_1A_1 + b_2A_2) = (a_1A_1 + a_2A_2)\left(b_3A_1^2 + b_4A_2^2\right). \tag{29}$$

In closing, the shock wave solutions of (24) have a rather simple form

$$\begin{aligned}
u(x,t) &= \lambda_1 \tanh\{B_0[x - c(t)t]\}, \\
v(x,t) &= \lambda_2 \tanh\{B_0[x - c(t)t]\},
\end{aligned} \tag{30}$$

where λ_1, λ_2 and B_0 are arbitrary constants whose values, including that of the speed $c(t)$, must satisfy the compatibility conditions (28) and (29).

Letting $\lambda_1 = \lambda_2 = 1$, the compatibility conditions are satisfied if $a_1(t) = b_1(t)$, $a_2(t) = b_2(t)$, $a_3(t) = b_3(t)$, and $a_4(t) = b_4(t)$. By judicious choices of these parameters it is possible to obtain shock waves characteristics similar to those of Model-I, among many others.

4. CONCLUSION

In this paper, we have found exact algebraic soliton solutions to two coupled pKdV equations with arbitrary time-varying coefficients. For Model-I both solitary and shock wave solutions were demonstrated to exist, whereas, only shock wave solutions were found for Model-II. It is shown that certain constraints or compatibility conditions between the model coefficients and the soliton parameters must be satisfied to ensure the existence of these solitons. For different choices of the coefficients soliton solutions exhibiting a variety of evolutionary behaviours were realised. These include wobbling solitons, swinging solitons, and laterally shifted solitons.

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