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TOPOLOGICAL SOLITONS AND CONSERVATION LAWS OF THE COUPLED BURGERS EQUATIONS

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Abstract. In this paper, the Lie symmetry analysis is performed for (2+1)-dimensional coupled Burgers equations. Then we derive the symmetry group theorem. At the same time, a great many of new exact solutions are obtained by applying the group theorem and given solutions. Meanwhile, 1-soliton solutions of the coupled Burgers equations are obtained by use of the Ansatz method. In addition, we also obtain conservation laws of the equations with the corresponding Lie symmetry. At last, suitable Painlevé truncated expansions are used to derive new exact solutions of bell-type.

Key words: Burgers equations, Lie point symmetry groups, Ansatz method, conservation laws.

1. INTRODUCTION

Nonlinear evolution equations play a very important role in various fields of nonlinear science. Many powerful methods have been constructed and a great deal of works to carry out the investigation of nonlinear evolution equations have been reported in recent years [1-20]. It is generally known that *Lie's theory* provides a standard method for finding the Lie point symmetry group of a nonlinear system. And above all, Lie's method of infinitesimal transformation groups which essentially reduces the number of independent variables in partial differential equations (PDEs) and reduces the order of ordinary differential equations (ODEs) has been widely used in equations of mathematical physics. Lie's method is an effective tool and the simplest method among group theoretic techniques and a large number of equations are solved with the aid of this method. In addition, the *Ansatz method* is successfully used to get the soliton solutions of the PDEs [21-23].

On the other hand, Weiss, Tabor and Carnevale (WTC) [24] presented the Painlevé test for nonlinear evolution equations (NLEEs) directly in 1983. Up to the present time, the developments about this method include Kruskal's [25], Conte's invariant method [26], Pickering's approach [27], and Lou's extended method [28]. But more than anything, even if the model is non-Painlevé integrable, the method can be used to find some special important exact and explicit solutions for a given NLEE by means of the truncated WTC Painlevé expansion [29].

In this paper, by using the *symmetry group method*, we will consider the following (2+1)-dimensional coupled Burgers equations (BEs)

$$\begin{aligned} &u_t - 2uu_x - 2u_y v - u_{xx} - u_{yy} = 0, \\ &v_t - 2vv_y - 2uv_x - v_{xx} - v_{yy} = 0. \end{aligned} \tag{1}$$

Burgers first introduced the equation $u_t = uu_x + u_{xx}$ to capture some of the features of one-dimensional turbulent fluid in a channel caused by the interaction of the opposite effects of convection and diffusion [30]. It is one of the fundamental model equations in fluid mechanics. The Burgers equation demonstrates the coupling between diffusion u_{xx} and the convection process uu_x. The Burger's types of equations have been derived to model many physical phenomena, such as under the effect of gravity sedimentation or evolution of scaled volume concentrations of two kinds of particles in fluid suspensions or colloids, and so on. A great deal of research work has been invested in recent years for the study of these types of equations [31-34].

Our aim in the present work is to analyze the (2+1)-dimensional coupled Burgers equations with the help of symmetry method. Then we get many new and more general complicated solutions. At the same time, we obtain conservation laws of (2+1)-dimensional Burgers equation. The plan of the paper is as follows. Section 2 discusses the Lie symmetry analysis of the Burgers equation. Then in Section 3, the Ansatz method is applied to get the soliton solutions of the coupled Burgers equations. In Section 4, conservation laws are obtained. Section 5 investigates the Painlevé properties and some new exact soliton solutions are given. Finally, we present conclusions in the last section.

2. SYMMETRY ANALYSIS FOR THE (2+1)-DIMENSIONAL COUPLED BURGERS EQUATIONS

In this section, we will perform group method for Eq. (1). If Eq. (1) is invariant under a one-parameter Lie group of point transformations

$$t^* = t + \epsilon\tau(x,y,t,u,v) + O(\epsilon^2), x^* = x + \epsilon\xi(x,y,t,u,v) + O(\epsilon^2),$$

$$\begin{gathered} y^* = y + \epsilon\eta(x,y,t,u,v) + O(\epsilon^2), u^* = u + \epsilon\phi(x,y,t,u,v) + O(\epsilon^2), \\ v^* = v + \epsilon\psi(x,y,t,u,v) + O(\epsilon^2), \end{gathered} \tag{2}$$

the vector field of an evolution type of equation is as follows:

$$\begin{aligned} V =&\tau(x,y,t,u,v)\frac{\partial}{\partial t}+\xi(x,y,t,u,v)\frac{\partial}{\partial x}+\eta(x,y,t,u,v)\frac{\partial}{\partial y}\\ &+\phi(x,y,t,u,v)\frac{\partial}{\partial u}+\psi(x,y,t,u,v)\frac{\partial}{\partial v}, \end{aligned} \tag{3}$$

where the coefficient functions $\tau(x,y,t,u,v)$, $\xi(x,y,t,u,v)$, $\eta(x,y,t,u,v)$, $\phi(x,y,t,u,v)$, and $\psi(x,y,t,u,v)$ of the vector field are to be determined.

If the vector field (3) generates a symmetry of the coupled Burgers equations, then V must satisfy Lie's symmetry condition

$$pr^{(2)}V(\Delta_1,\Delta_2)|_{\Delta_1=0,\Delta_2=0}=0, \tag{4}$$

where $\Delta_1=u_t-2uu_x+2u_yv-u_{xx}-u_{yy}$ and $\Delta_2=v_t-2vv_y-2uv_x-v_{xx}-v_{yy}$. Applying the second prolongation $pr^{(2)}V$ to Eq. (1), we find the following system of symmetry equations and the invariant condition reads as

$$\begin{aligned} &\phi^t-2u_x\phi-2u\phi^x-2\phi^yv-2u_y\psi-\phi^{xx}-\phi^{yy}=0,\\ &\psi^t-2v_y\psi-2v\psi^y-2v_x\phi-2u\psi^x-\psi^{xx}-\psi^{yy}=0, \end{aligned} \tag{5}$$

where

$$\begin{aligned} &\phi^t=D_t(\phi)-u_xD_t(\xi)-u_yD_t(\eta)-u_tD_t(\tau),\\ &\phi^x=D_x(\phi)-u_xD_x(\xi)-u_yD_x(\eta)-u_tD_x(\tau),\\ &\phi^y=D_y(\phi)-u_xD_y(\xi)-u_yD_y(\eta)-u_tD_y(\tau),\\ &\psi^t=D_t(\psi)-v_xD_t(\xi)-v_yD_t(\eta)-v_tD_t(\tau),\\ &\psi^x=D_x(\psi)-v_xD_x(\xi)-v_yD_x(\eta)-v_tD_x(\tau),\\ &\psi^y=D_y(\psi)-v_xD_y(\xi)-v_yD_y(\eta)-v_tD_y(\tau),\\ &\phi^{xx}=D_x(\phi^x)-u_{xt}D_x(\tau)-u_{xx}D_x(\xi)-u_{xy}D_x(\eta),\\ &\phi^{yy}=D_y(\phi^y)-u_{yt}D_y(\tau)-u_{xy}D_y(\xi)-u_{yy}D_y(\eta),\\ &\psi^{xx}=D_x(\psi^x)-v_{xt}D_x(\tau)-v_{xx}D_x(\xi)-v_{xy}D_x(\eta),\\ &\psi^{yy}=D_y(\psi^y)-v_{yt}D_y(\tau)-v_{xy}D_y(\xi)-v_{yy}D_y(\eta). \end{aligned} \tag{6}$$

Here, D_i denotes the total derivative operator and is defined by

$$D_i=\frac{\partial}{\partial x^i}+u_i^p\frac{\partial}{\partial u}+u_{ij}^p\frac{\partial}{\partial u_j}+\cdots \qquad i=1,2,3,p=1,2, \tag{7}$$

and $(x^1,x^2,x^3)=(t,x,y),(u^1,u^2)=(u,v)$.

Substituting (6) into (5), we obtain the determining equations for the symmetry group. One can get

$$\begin{aligned}
&\xi=(c_1t+c_2)x+2c_6t-c_3y+c_7,\quad \tau=c_1t^2+2c_2t+c_4,\\
&\eta=(c_1t+c_2)y+2c_5t+c_3x+c_8,\\
&\phi=-\Big[(c_1t+c_2)u+c_3v+\frac{c_1}{2}x+c_6\Big],\psi=-\Big[(c_1t+c_2)v-c_3u+\frac{c_1}{2}y+c_5\Big],
\end{aligned}\tag{8}$$

where $c_i(i=1,2\cdots 8)$ are arbitrary constants. Therefore the symmetry algebra of the coupled Burgers equations is spanned by the eight vector fields

$$\begin{aligned}
&V_1=\frac{\partial}{\partial x},V_2=\frac{\partial}{\partial y},V_3=\frac{\partial}{\partial t},V_4=2t\frac{\partial}{\partial y}-\frac{\partial}{\partial v},V_5=2t\frac{\partial}{\partial x}-\frac{\partial}{\partial u},\\
&V_6=x\frac{\partial}{\partial x}+2t\frac{\partial}{\partial t}+y\frac{\partial}{\partial y}-u\frac{\partial}{\partial u}-v\frac{\partial}{\partial v},\\
&V_7=xt\frac{\partial}{\partial x}+t^2\frac{\partial}{\partial t}+ty\frac{\partial}{\partial y}-(tu+\frac{1}{2}x)\frac{\partial}{\partial u}-(tv+\frac{1}{2}y)\frac{\partial}{\partial v},\\
&V_8=-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}-v\frac{\partial}{\partial u}+u\frac{\partial}{\partial v}.
\end{aligned}\tag{9}$$

It is easy to check that the symmetry generators found in (9) form a closed and an eight-dimensional Lie algebra. Here we do not list all of them. In order to get some exact solutions from the known ones, we should find the Lie symmetry groups from the related symmetries. To get the Lie symmetry group, we should solve the following initial problems

$$\frac{d}{d\varepsilon}(\bar{x},\bar{y},\bar{t},\bar{u},\bar{v})=\sigma(\bar{x},\bar{y},\bar{t},\bar{u},\bar{v}),(\bar{x},\bar{y},\bar{t},\bar{u},\bar{v})|_{\varepsilon=0}=(x,y,t,u,v),\tag{10}$$

where ε is a parameter and

$$\sigma=\xi u_x+\tau u_t+\eta u_y+\phi u+\psi v.\tag{11}$$

So we can obtain the Lie symmetry group

$$g:(x,y,t,u,v)\to(\bar{x},\bar{y},\bar{t},\bar{u},\bar{v}).\tag{12}$$

According to different ξ,τ,η,ϕ, and ψ, we have the following group by solving (10)

$$
\begin{aligned}
&g_1:(x+\varepsilon,y,t,u,v),\\
&g_2:(x,y+\varepsilon,t,u,v),\\
&g_3:(x,y,t+\varepsilon,u,v),\\
&g_4:(x,y-2t\varepsilon,t,u,v-\varepsilon),\\
&g_5:(x-2t\varepsilon,y,t,u-\varepsilon,v),\\
&g_6:(e^{\varepsilon}x,e^{\varepsilon}y,e^{2\varepsilon}t,e^{-\varepsilon}u,e^{-\varepsilon}v),\\
&g_7:\Big(\frac{x}{1-t\varepsilon},\frac{y}{1-t\varepsilon},\frac{t}{1-t\varepsilon},u(1-t\varepsilon)-\frac{\varepsilon}{2}x,v(1-t\varepsilon)-\frac{\varepsilon}{2}y\Big),\\
&g_8:(x\cos\varepsilon-y\sin\varepsilon,x\sin\varepsilon+y\cos\varepsilon,t,v\cos\varepsilon-u\sin\varepsilon,v\sin\varepsilon+u\cos\varepsilon).
\end{aligned}
\tag{13}
$$

The symmetry groups g_1,g_2, and g_3 demonstrate the space- and time-invariance of the equation, while g_4,g_5 represent a kind of Galilean boost to a moving coordinate frame. The well-known scaling symmetry turns up in g_6, g_7 is a genuinely local group of transformations, and g_8 denotes a rotations group. We can obtain the corresponding new solutions by applying above groups $g_i(i=1,\dots 8)$

$$
\begin{aligned}
&u_1=f_1(x-\varepsilon,y,t),v_1=h_1(x-\varepsilon,y,t),\\
&u_2=f_2(x,y-\varepsilon,t),v_2=h_2(x,y-\varepsilon,t),\\
&u_3=f_3(x,y,t-\varepsilon),v_3=h_3(x,y,t-\varepsilon),\\
&u_4=f_4(x,y+2t\varepsilon,t),v_4=h_4(x,y+2t\varepsilon,t)-\varepsilon,\\
&u_5=f_5(x+2t\varepsilon,y,t)-\varepsilon,v_5=h_5(x+2t\varepsilon,y,t),\\
&u_6=e^{-\varepsilon}f_6(e^{-\varepsilon}x,e^{-\varepsilon}y,e^{-2\varepsilon}t),v_6=e^{-\varepsilon}h_6(e^{-\varepsilon}x,e^{-\varepsilon}y,e^{-2\varepsilon}t),\\
&u_7=f_7\Big(\frac{x}{1-t\varepsilon},\frac{y}{1-t\varepsilon},\frac{t}{1-t\varepsilon},\frac{1}{(1+t\varepsilon)}u-\frac{x\varepsilon}{2(1+t\varepsilon)}\Big),\\
&v_7=h_7\Big(\frac{x}{1-t\varepsilon},\frac{y}{1-t\varepsilon},\frac{t}{1-t\varepsilon},\frac{1}{(1+t\varepsilon)}v-\frac{y\varepsilon}{2(1+t\varepsilon)}\Big),\\
&u_8=f_8(x\cos\varepsilon-y\sin\varepsilon,x\sin\varepsilon+y\cos\varepsilon,t,v\cos\varepsilon-u\sin\varepsilon),\\
&v_8=h_8(x\cos\varepsilon-y\sin\varepsilon,x\sin\varepsilon+y\cos\varepsilon,t,v\sin\varepsilon+u\cos\varepsilon),
\end{aligned}
\tag{14}
$$

where ε is an arbitrary real number.

If taking the following two kink solutions [34] of Eq. (1)

$$
\begin{aligned}
u&=\frac{k_1\mathrm{e}^{k_1(x+y)+2k_1^2t}+k_2\mathrm{e}^{k_2(x+y)+2k_2^2t}}{1+\mathrm{e}^{k_1(x+y)+2k_1^2t}+\mathrm{e}^{k_2(x+y)+2k_2^2t}},\\
v&=\frac{k_1\mathrm{e}^{k_1(x+y)+2k_1^2t}+k_2\mathrm{e}^{k_2(x+y)+2k_2^2t}}{1+\mathrm{e}^{k_1(x+y)+2k_1^2t}+\mathrm{e}^{k_2(x+y)+2k_2^2t}},
\end{aligned}
\tag{15}
$$

one can obtain new exact solutions of Eq. (1) by applying g_5 as follows

$$
\begin{aligned}
u &= \frac{k_1 \mathrm{e}^{k_1(x+2t\varepsilon+y)+2k_1^2 t} + k_2 \mathrm{e}^{k_2(x+2t\varepsilon+y)+2k_2^2 t}}{1+\mathrm{e}^{k_1(x+2t\varepsilon+y)+2k_1^2 t} + \mathrm{e}^{k_2(x+2t\varepsilon+y)+2k_2^2 t}} - \varepsilon, \\
v &= \frac{k_1 \mathrm{e}^{k_1(x+2t\varepsilon+y)+2k_1^2 t} + k_2 \mathrm{e}^{k_2(x+2t\varepsilon+y)+2k_2^2 t}}{1+\mathrm{e}^{k_1(x+2t\varepsilon+y)+2k_1^2 t} + \mathrm{e}^{k_2(x+2t\varepsilon+y)+2k_2^2 t}},
\end{aligned}
\tag{16}
$$

where k_1, k_2, and ε are arbitrary constants.

By selecting the arbitrary constants, one can obtain many other solutions.

Remark-1. A lot of new invariant solutions can be found through given solutions [34] for the coupled Burgers equations. Thus we generalize the corresponding results in [34].

Remark-2. Detailed similarity reductions of the coupled Burgers equations can be found in [31].

3. ANSATZ METHOD

In this section, we will study the coupled BEs with the Ansatz approach. The target will be to retrieve the *shock wave solutions* that are also known as topological solitons. In order to proceed, the starting hypothesis is taken to be

$$u(x,y,t) = A_1 \tanh^{p_1} \tau, \tag{17}$$

and

$$v(x,y,t) = A_2 \tanh^{p_2} \tau, \tag{18}$$

where

$$\tau = B_1 x + B_2 y - ct, \tag{19}$$

and A, B_j for $j = 1,2$ are free parameters while c is the speed of the shock wave. Substituting these hypothesis into Eq. (1) leads to the following coupled set of algebraic equations:

$$
\begin{aligned}
& c\left(\tanh^{p_1-1}\tau - \tanh^{p_1+1}\tau\right) + 2A_1B_1\left(\tanh^{2p_1-1}\tau - \tanh^{2p_1+1}\tau\right) \\
& + 2A_2B_2\left(\tanh^{p_1+p_2-1}\tau - \tanh^{p_1+p_2+1}\tau\right) \\
& + \left(B_1^2+B_2^2\right)\left\{(p_1-1)\tanh^{p_1-2}\tau - 2p_1\tanh^{p_1}\tau + (p_1+1)\tanh^{p_1+2}\tau\right\} = 0,
\end{aligned}
\tag{20}
$$

and

$$
\begin{aligned}
& c\left(\tanh^{p_2-1}\tau - \tanh^{p_2+1}\tau\right) + 2A_2B_2\left(\tanh^{2p_2-1}\tau - \tanh^{2p_2+1}\tau\right) \\
& + 2A_1B_1\left(\tanh^{p_1+p_2-1}\tau - \tanh^{p_1+p_2+1}\tau\right) \\
& + \left(B_1^2+B_2^2\right)\left\{(p_2-1)\tanh^{p_2-2}\tau - 2p_2\tanh^{p_2}\tau + (p_2+1)\tanh^{p_2+2}\tau\right\} = 0.
\end{aligned}
\tag{21}
$$

From Eq. (20), by the aid of balancing principle,

$$2p_1 - 1 = p_1 + p_2 - 1 = p_1, \tag{22}$$

which implies

$$p_1 = p_2 = 1. \tag{23}$$

Similarly from equation (21), the same conclusion as in Eq. (23) is obtained. Now from Eqs. (20) and (21), setting the coefficients of the linearly independent functions p_j and p_{j+2}, for $j = 1,2$, to zero leads to

$$c = 0, \tag{24}$$

and

$$A_1B_1 + A_2B_2 = B_1^2 + B_2^2. \tag{25}$$

This shows that the shock wave solution of the coupled BEs that can be obtained by the ansatz approach is a stationary shock wave given by

$$u(x,y,t) = A_1 \tanh(B_1x + B_2y), \tag{26}$$

and

$$v(x,y,t) = A_2 \tanh(B_1x + B_2y), \tag{27}$$

where the four free parameters are connected as given by Eq. (25).

4. CONSERVATION LAWS OF COUPLED BURGERS EQUATIONS

In this section, we will study the conservation laws by using the adjoint equation and symmetries of the Burgers equations. For Eq. (1), the adjoint equation has the form

$$\begin{aligned} F_1 &= \theta_{1t} - 2u\theta_{1x} - 2v_y\theta_1 - 2v\theta_{1y} + \theta_{1xx} + \theta_{1yy} = 0, \\ F_2 &= \theta_{2t} - 2v\theta_{2y} - 2u\theta_{2x} - 2u_x\theta_2 + \theta_{2xx} + \theta_{2yy} = 0, \end{aligned} \tag{28}$$

and the Lagrangian in the symmetrized form is

$$L = \theta_1(u_t - 2uu_x - 2u_yv - u_{xx} - u_{yy}) + \theta_2(v_t - 2vv_y - 2uv_x - v_{xx} - v_{yy}). \tag{29}$$

Theorem. Every Lie point, Lie-Bäcklund and non-local symmetry of Eq. (1) provides a conservation law for this equation and the adjoint equation [35]. Then the elements of conservation vector (C^1, C^2, C^3) are defined by the following expression

$$\begin{aligned} C^i = \xi^i L + W^\alpha \Big[\frac{\partial L}{\partial u_i^\alpha} - D_j(\frac{\partial L}{\partial u_{ij}^\alpha}) + D_jD_k(\frac{\partial L}{\partial u_{ijk}^\alpha})\Big] + D_j(W^\alpha)\Big[(\frac{\partial L}{\partial u_{ij}^\alpha}) \\ - D_k((\frac{\partial L}{\partial u_{ijk}^\alpha})) + \dots\Big], \end{aligned} \tag{30}$$

where $W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$.

The conserved vector corresponds to an operator V

$$V=\xi^1(x,y,t,u,v)\frac{\partial}{\partial t}+\xi^2(x,y,t,u,v)\frac{\partial}{\partial x}+\xi^3(x,y,t,u,v)\frac{\partial}{\partial y}$$
$$+\eta^1(x,y,t,u,v)\frac{\partial}{\partial u}+\eta^2(x,y,t,u,v)\frac{\partial}{\partial v}. \tag{31}$$

The operator V yields the conservation law $D_t(C^1)+D_x(C^2)+D_y(C^3)=0$, where the conserved vector is given by Eq. (30) and has the components

$$C^1=\xi^1L+W^1\theta_1+W^2\theta_2, \tag{32}$$

$$C^2=\xi^2L+W^1[\theta_{1x}-2u\theta_1]+W^2[\theta_{2x}-2u\theta_2]-\theta_1D_x(W^1)-\theta_2D_x(W^2), \tag{33}$$

$$C^3=\xi^3L+W^1[\theta_{1y}-2v\theta_1]+W^2[\theta_{2y}-2v\theta_2]-\theta_1D_y(W^1)-\theta_2D_y(W^2). \tag{34}$$

Let us make more detailed calculations for the operator $v=\partial_t$. For this operator, we have

$$\xi^1=1,\xi^2=\xi^3=\eta^1=\eta^2=0,W^1=-u_t,W_2=-v_t. \tag{35}$$

In this case, Eq. (30) provides the conservation law $D_t(C^1)+D_x(C^2)+D_y(C^3)=0$ of Eq. (1) for the vector $C=(C^1,C^2,C^3)$ with the following components:

$$C^1=\xi^1\Big(\theta_1(u_t-2uu_x-2u_yv-u_{xx}-u_{yy})+\theta_2(v_t-2vv_y-2uv_x$$
$$-v_{xx}-v_{yy})\Big)-u_t\theta_1-v_t\theta_2, \tag{36}$$

$$C^2=-u_t[\theta_{1x}-2u\theta_1]-v_t[\theta_{2x}-2u\theta_2]-\theta_1D_x(-u_t)-\theta_2D_x(-v_t), \tag{37}$$

$$C^3=-u_t[\theta_{1y}-2v\theta_1]-v_t[\theta_{2y}-2v\theta_2]-\theta_1D_y(-u_t)-\theta_2D_y(-v_t). \tag{38}$$

This vector involves an arbitrary solution θ_1,θ_2 of the adjoint equation (28) and provides an infinite number of conservation laws.

Remark-3. With the aid of Maple 15, we have checked that the above vector (C^1,C^2,C^3) is the conservation vector of Eq. (1).

5. TRUNCATED PAINLEVÉ EXPANSION AND EXACT SOLITON SOLUTIONS

First, in [31] we can see that Eq. (1) fails to satisfy the generalized Painlevé test. Now, we assume the following Laurent expansion of the function $u\equiv u(x,y,t)$ and $v\equiv v(x,y,t)$ about a singular manifold $\phi\equiv\phi(x,y,t)$

$$u=\sum_{j=0}^{\infty}u_j\phi^{j+\alpha},\ \ v=\sum_{j=0}^{\infty}v_j\phi^{j+\beta}. \tag{39}$$

Substituting the leading term $(j=0)$ of (39) into Eq. (1) and by leading order analysis, we obtain $\alpha=\beta=-1$. Though the (2+1)-dimensional coupled Burgers equations is non-Painlevé integrable, we could still obtain some new exact solutions by means of some suitable Painlevé truncated expansion. Following [28], the Painlevé expansion (39) can be modified as follows

$$u=\sum_{j=0}^{\infty}U_j\xi^{j-1},\ \ v=\sum_{j=0}^{\infty}V_j\xi^{j-1}, \tag{40}$$

with ξ being decided by

$$\xi_x=\sum_{j=0}^{M}P_j\xi^j,\ \ \xi_y=\sum_{j=0}^{M}Q_j\xi^j,\ \ \xi_t=\sum_{j=0}^{M}R_j\xi^j. \tag{41}$$

To be more specific, we restrict to $M=2$ in Eq. (41). Thus Eq. (41) reads as follows

$$\xi_x=P_0+P_1\xi+P_2\xi^2,\ \ \xi_y=Q_0+Q_1\xi+Q_2\xi^2,\ \ \xi_t=R_0+R_1\xi+R_2\xi^2. \tag{42}$$

It is important to note, however, that the general expansion (40) with (41) is just the Pickering's modification [27]. Then, we will seek some special types of new explicit solutions of (2+1)-dimensional coupled Burgers equations by using the nonstandard truncation of (39). In particular, we provide the expansion function as in [29]

$$g\equiv\lambda-\chi,\chi=\Big(\frac{\phi_x}{\phi}-\frac{\phi_{xx}}{2\phi_x}\Big)^{-1}, \tag{43}$$

where λ is an arbitrary constant and ϕ is an arbitrary manifold. As we know, if we let $\lambda=0$, then the modified expansion will be reduced to the Conte's expansion. If in (43), we take some special selection, (41) can be writen [29]

$$\begin{aligned}&g_x=-1+\frac{P}{2}(\lambda-g)^2,g_y=Q-Q_x(\lambda-g)+\frac{1}{2}(Q_{xx}-PQ)(\lambda-g)^2,\\&g_t=R-R_x(\lambda-g)+\frac{1}{2}(R_{xx}-PR)(\lambda-g)^2.\end{aligned} \tag{44}$$

Here

$$P\equiv\frac{3}{2}\Big(\frac{\phi_{xx}}{\phi_x}\Big)^2-\frac{\phi_{xxx}}{\phi_x},Q\equiv-\frac{\phi_y}{\phi_x},R\equiv-\frac{\phi_t}{\phi_x}, \tag{45}$$

which are the Möbius transformation invariants. In view of Eq. (45), it is easy to prove that all the compatibility conditions $g_{xt}=g_{tx}$, $g_{xy}=g_{yx}$, $g_{yt}=g_{ty}$ are satisfied automatically.

Thus Eq. (40) reads

$$u=\frac{U_0}{g}+U_1+U_2g,\ \ v=\frac{V_0}{g}+V_1+V_2g. \tag{46}$$

Substituting (46) with (44) into Eq. (1), one can get a set of complicated overdetermined equations to determine the functions $U_0, U_1, U_2, V_0, V_1, V_2$ and P, Q, R. It is difficult to find out all possible solutions of the overdetermined equations. However, in order to find the *single soliton solution*, for brevity, we consider only the constant solutions. After some calculations, the final result can be written as:

Case a.

$$\begin{aligned} V_0 = V_0, V_2 = 0, U_0 = \frac{P\lambda^2(1+Q^2)}{2} + V_0 Q - 1 - Q^2, \\ V_1 = V_1, U_1 = -\frac{P\lambda(1+Q^2)}{2} + V_1 Q - \frac{R}{2}, U_2 = 0. \end{aligned} \tag{47}$$

Case b.

$$\begin{aligned} V_2 = V_2, V_0 = 0, U_2 = -\frac{P(1+Q^2)}{2} + V_2 Q, \\ V_1 = V_1, U_1 = \frac{P\lambda(1+Q^2)}{2} + V_1 Q - \frac{R}{2}, U_0 = 0. \end{aligned} \tag{48}$$

Equation (44) has the *bell-type soliton solution* (which include five arbitrary constants (P, Q, R, λ, c_0))

$$g(x,y,t) = \lambda - \sqrt{\frac{P}{2}} \tanh\Big(\frac{\sqrt{2P}}{2}(x - Qy - Rt + c_0)\Big), \tag{49}$$

where c_0 is an arbitrary constant. Thus, one can get bell-type soliton solutions of Eq. (1) as follows

$$\begin{aligned} u_1(x,y,t) =& \frac{\frac{P\lambda^2(1+Q^2)}{2} + V_0 Q - 1 - Q^2}{\lambda - \sqrt{\frac{P}{2}} \tanh\Big(\frac{\sqrt{2P}}{2}(x - Qy - Rt + c_0)\Big)} \\ & - \frac{P\lambda(1+Q^2)}{2} + V_1 Q - \frac{R}{2}, \\ v_1(x,y,t) =& \frac{V_0}{\lambda - \sqrt{\frac{P}{2}} \tanh\Big(\frac{\sqrt{2P}}{2}(x - Qy - Rt + c_0)\Big)} + V_1, \\ u_2(x,y,t) =& \frac{P\lambda(1+Q^2)}{2} + V_1 Q - \frac{R}{2} - (\frac{P(1+Q^2)}{2} - V_2 Q) \\ & \Big[\lambda - \sqrt{\frac{P}{2}} \tanh\Big(\frac{\sqrt{2P}}{2}(x - Qy - Rt + c_0)\Big)\Big], \\ v_2(x,y,t) =& V_1 + V_2 \Big[\lambda - \sqrt{\frac{P}{2}} \tanh\Big(\frac{\sqrt{2P}}{2}(x - Qy - Rt + c_0)\Big)\Big]. \end{aligned} \tag{50}$$

Remark-4. Using Case b, one can get other exact solutions of Eqs. (1). The details are omitted here.

6. CONCLUSIONS

In this paper, according to the Lie symmetry analysis, we find some new exact solutions of the (2+1)-dimensional Burgers equations and generalize the corresponding results obtained in Ref. [34]. In the meantime, the topological 1-soliton solution is obtained for the coupled Burgers equations by the ansatz method. Meanwhile, we also give the conservation laws of Burgers equations which have not been reported until now, to the best of our knowledge. At last, though the model is non-Painlevé integrable, we could still obtain some new exact soliton solutions by means of some suitable Painlevé truncated expansion. The symmetry analysis based on the group method is a very powerful method and is worthy of studying further.

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