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ANALYTICAL APPROXIMATE SOLUTIONS OF THE ZAKHAROV-KUZNETSOV EQUATIONS

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Abstract. In this paper, analytical approximate solutions for the Zakharov-Kuznetsov equations by homotopy analysis method (HAM) and the He's polynomials iterative method (HPIM) are presented. Our results indicate the remarkable efficiency of HAM as compared to HPIM. The convergence of these two methods is also analyzed.

Key words: Homotopy analysis method, Zakharov-Kuznetsov equation, He's polynomials iterative method, Analytical approximate solution.

1. INTRODUCTION

The Zakharov-Kuznetsov (ZK) equation was first derived for describing weakly nonlinear ion-acoustic waves in strongly magnetized lossless plasma in two dimensions. The nonlinear evolution equations find applications in many areas, such as hydrodynamics, plasma physics, nonlinear optics, etc. A large number of evolution equations in many areas of applied mathematics, physics and engineering appear as nonlinear wave equations [1–11]. For example, one of the most important one-dimensional nonlinear wave equation is the Korteweg-de Vries (KdV) equation

$$v_t + avv_x + v_{xxx} = 0.$$

One of the best known two-dimensional generalizations of the KdV equation is the ZK equation in the form

$$v_t + avv_x + (v_{xx} + v_{yy})_x = 0,$$

which governs the behavior of weakly nonlinear ion-acoustic waves in a plasma comprising cold-ions and hot isothermal electrons in the presence of a uniform magnetic field [12]. Most nonlinear equations are difficult to solve analytically, especially the ZK equation. In Ref. [12], the ZK equation was solved by the sine-cosine and the tanh-function methods and Hesameddini [13] applied the differential transform method (DTM) to solve the ZK equation.

In this paper a new analytic method is used to solve the Zakharov-Kuznetsov (ZK(m,n,k)) equations of the form

$$v_t + a(v^m)_x + b(v^n)_{xxx} + c(v^k)_{yyx} = 0, \;\; m,n,k \neq 0$$

where a, b, and c are arbitrary constants and m,n,k are integers.

Recently, the homotopy analysis method (HAM) has been successfully employed to solve many types of nonlinear problems in science and engineering [14–18]. In order to adjust and control the convergence region of HAM, we can use a proper value for the auxiliary parameter h. In this method the solution is considered as the sum of an infinite series which converges rapidly to the accurate solution. Moreover, the iterative methods have applied to find approximate analytical solution of fractional order differential equations [19–23].

The paper is organized as follows. In section 2, the basic ideas of the present approaches are described. In section 3, two special cases of the ZK equations are employed to illustrate the convergence, accuracy, and computational efficiency of these approaches. Finally, our conclusions are given.

2. METHODOLOGY

In this section, we give a brief summary of the Homotopy Analysis Method and He's polynomials iterative method [24].

2.1. HOMOTOPY ANALYSIS METHOD

We consider the following differential equation

$$N[v(x,t)] = 0, \tag{1}$$

where N is a nonlinear operator, t is independent variable, and $v(x,t)$ is an unknown function. For simplicity, we ignore all boundary or initial conditions, which can be treated in a similar way. By means of generalizing the traditional homotopy method, Liao [24] constructs the so-called zero-order deformation equation

$$(1-q)L[\phi(x,t;q)-v_0(x,t)] = qhH(x,t)N[\phi(x,t;q)], \tag{2}$$

where $q \in [0,1]$ is the embedding parameter, $h \neq 0$ is a non-zero auxiliary parameter, $H(x,t) \neq 0$ is a nonzero auxiliary function, L is an auxiliary linear operator, $v_0(x,t)$ is an initial guess of $v(x,t)$, and $\phi(x,t;q)$ is a unknown function. It is important that one has great freedom to choose auxiliary parameters in HAM. Obviously, when $q=0$ and $q=1$, it holds

$$\phi(x,t;0) = v_0(x,t), \quad \phi(x,t;1) = v(x,t), \tag{3}$$

Thus, as q increases from 0 to 1, The solution $\phi(x,t;q)$ varies from the initial guess $v_0(x,t)$ to the solution $v(x,t)$. Expanding by Taylor series with respect to q, we have

$$\phi(x,t;q) = v_0(x,t) + \sum_{m=1}^{\infty} v_m(x,t)\; q^m, \tag{4}$$

where

$$v_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;q)}{\partial q^m}\;|_{q=0}. \tag{5}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h, and the auxiliary function are so properly chosen, the series (4) converges at $q=1$, then we have

$$v(x,t) = v_0(x,t) + \sum_{m=1}^{\infty} v_m(x,t), \tag{6}$$

which is one of the solutions for the original nonlinear equation, as proved by Liao [24]. As $h=-1$ and $H(x,t)=1$, equation (2) becomes

$$(1-q)L[\phi(x,t;q)-v_0(x,t)] + qN[\phi(x,t;q)] = 0, \tag{7}$$

which is well known to the homotopy perturbation method (HPM). According to the definition (5), the governing equation can be deduced from the zeroth-order deformation equation (2). Next we define the vector

$$\vec{v}_n = \{v_0(x,t), v_1(x,t), \ldots, v_n(x,t)\}.$$

Differentiating equation (2) m times with respect to the embedding parameter q, then setting $q=0$ and finally dividing them by $m!$, one has the so-called mth-order deformation equation

$$L[v_m(x,t) - \chi_m v_{m-1}(x,t)] = hH(x,t)R_m(\vec{v}_{m-1}), \tag{8}$$

where

$$R_m(\vec{v}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(x,t;q)]}{\partial q^{m-1}}\;|_{q=0}, \tag{9}$$

and

$$\chi_m = \begin{cases} 0 & m \leq 1, \\ \\ 1 & m > 1. \end{cases} \tag{10}$$

It should be emphasized that $v_m(x,t)$ for $m \geq 1$ is governed by the linear equation (8) under the linear boundary condition that comes from the original problem. Then one can use the above algorithm and the Matlab computer software to obtain an approximate analytical solution.

2.2. HE'S POLYNOMIALS ITERATIVE METHOD

Consider the following general functional equation [25]

$$v = \mathcal{N}(v) + f, \tag{11}$$

where $\mathcal{N}$ is a nonlinear operator from a Banach space $B \to B$ and f is a known function. We are looking for a solution of the Eq. (11) having the series form

$$v = \sum_{i=1}^{\infty} v_i(t). \tag{12}$$

The nonlinear operator $\mathcal{N}$ can be decomposed as

$$\mathcal{N}\left(\sum_{i=1}^{\infty} v_i\right) = \mathcal{N}(v_0) + \sum_{i=1}^{\infty}\left[\mathcal{N}\left(\sum_{j=0}^{i} v_j\right) - \mathcal{N}\left(\sum_{j=0}^{i-1} v_j\right)\right]. \tag{13}$$

From equation (12) and (13), equation (11) is equivalent to

$$\sum_{i=1}^{\infty} v_i = f + \mathcal{N}(v_0) + \sum_{i=1}^{\infty}\left[\mathcal{N}\left(\sum_{j=0}^{i} v_j\right) - \mathcal{N}\left(\sum_{j=0}^{i-1} v_j\right)\right]. \tag{14}$$

We define the recurrence relation

$$\begin{cases} v_0 = f, \\ v_1 = \mathcal{N}(v_0), \\ v_{m+1} = \mathcal{N}(v_0 + ... + v_m) - \mathcal{N}(v_0 + ... + v_{m-1}), \;\; m = 1, 2, \dots \end{cases} \tag{15}$$

Then

$$(v_0 + \ldots + v_{m+1}) = \mathcal{N}(v_0 + \ldots + v_m), \;\; m = 1, 2, \ldots.$$

and

$$\sum_{i=1}^{\infty} v_i = f + \mathcal{N}\left(\sum_{j=0}^{\infty} v_j\right).$$

The k-term approximate solution of (11) and (12) is given by $v = \sum_{i=0}^{k} v_i$.

3. APPLICATION

In order to assess the accuracy of HAM, and to illustrate the method in more details, we consider the following two examples.

Example 3.1. According to [26], consider the ZK(2,2,2), as the following

$$v_t-(v^2)_x+(\frac{1}{8})(v^2)_{xxx}+(\frac{1}{8})(v^2)_{yyx}=0, \tag{16}$$

subject to initial condition

$$v(x,y,0)=\frac{4}{3}\lambda\sinh^2(\frac{1}{2}(x+y)), \tag{17}$$

where λ is an arbitrary constant.

► Solution by HAM

To solve the equation (16) by means of HAM, according to the initial conditions denoted in equation (17), it is natural to choose

$$v_0(x,y,t)=\frac{4}{3}\lambda\sinh^2(\frac{1}{2}(x+y)). \tag{18}$$

We choose the linear operators

$$L[\phi(x,y,t;q)]=\frac{\partial\phi(x,y,t;q)}{\partial t},$$

with the property $L[c]=0$, where c is constant. From (16), we define nonlinear operators

$$N[\phi]=\phi_t-(\phi^2)_x+(\frac{1}{8})(\phi^2)_{xxx}+(\frac{1}{8})(\phi^2)_{yyx}. \tag{19}$$

We construct the zero-order deformation equation

$$(1-q)L[\phi(x,y,t;q)-v_0(x,y,t)]=qhH(t)N[\phi]. \tag{20}$$

Differentiating (20), m times with respect to q, then setting $q=0$ and finally dividing them by $m!$, we obtain the mth-order deformation equation

$$L[v_m(x,y,t)-\chi_m v_{m-1}(x,y,t)]=hR_m(\vec{v}_{m-1}), \tag{21}$$

where

$$R_m(\vec{v}_{m-1})=\frac{\partial v_{m-1}}{\partial t}-\frac{\partial}{\partial x}[\sum_{i=0}^{m-1}v_iv_{m-1-i}]+\frac{1}{8}\frac{\partial^3}{\partial x^3}[\sum_{i=0}^{m-1}v_iv_{m-1-i}]$$

$$+\frac{1}{8}\frac{\partial^3}{\partial x\partial y^2}[\sum_{i=0}^{m-1}v_iv_{m-1-i}]$$

Then $v_m(x,y,t)$ is

$$v_m(x,y,t) = \chi_m v_{m-1}(x,y,t) + h\int_0^t R_m(\vec{v}_{m-1})dt.$$

One can obtain the solution of the mth-order deformation equations as follows
$v_0(x,y,t) = (4\lambda \sinh(x/2+y/2)^2)/3,$
$v_1(x,y,t) = (2h\lambda^2 t\sinh(x+y))/3,$
$v_2(x,y,t) = (2h\lambda^2 t\sinh(x+y))/3 + (h^2\lambda^2 t(2\sinh(x+y)+$
$\lambda t\cosh(x+y)))/3,$
$\vdots$

Therefore using equation $v(x,y,t) = v_0(x,y,t) + v_1(x,y,t) + v_2(x,y,t) + \cdots$, we have
$v(x,y,t) = (2\lambda\cosh(x+y))/3 - (2\lambda)/3 + (2h^2\lambda^2 t\sinh(x+y))/3+$
$(h^2\lambda^3 t^2\cosh(x+y))/3 + (4h\lambda^2 t\sinh(x+y))/3 + \cdots$.

It is straightforward to choose an appropriate range for h that ensure the convergence of the solution series. We plot the h-curve of $v(1,1,0.1)$ in Fig. 1 which shows that the solution series is convergent when $-1.2 < h < -0.7$. Thus, the auxiliary parameter h plays an important role within the frame of the HAM.

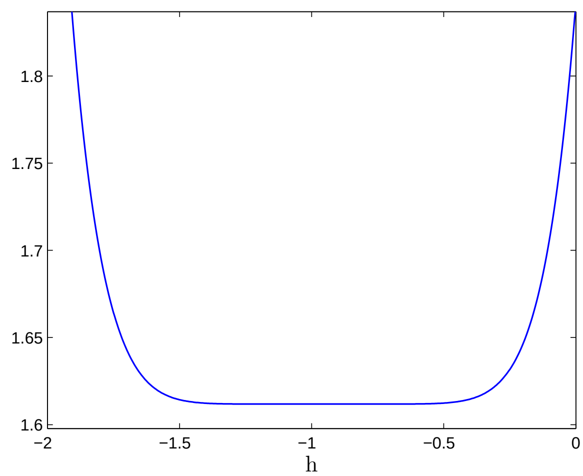

Fig. 1 – The h-curve of $v(1,1,0.1)$ given by the 9th-order approximate solution.

▶ **He's Polynomials Iterative Method**

Here, we solve ZK(2,2,2) explicitly by HPIM. Equation (16) is equivalent to the following integral equation

$$v = \frac{4}{3}\lambda \sinh^2(\frac{1}{2}(x+y)) + \int_0^t (v^2)_x - \frac{1}{8}(v^2)_{xxx} - \frac{1}{8}(v^2)_{yyx}\, dt,$$

where $f(x) = \frac{4}{3}\lambda \sinh^2(\frac{1}{2}(x+y))$ and $\mathcal{N} = \int_0^t (v^2)_x - \frac{1}{8}(v^2)_{xxx} - \frac{1}{8}(v^2)_{yyx}\, dt$.

Following the algorithm given in previous section, some first terms of the successive approximation series are as follows
$v_0(x,y,t) = (4\lambda \sinh(x/2+y/2)^2)/3,$
$v_1(x,y,t) = -(2\lambda^2 t \sinh(x+y))/3,$
$v_2(x,y,t) = (\lambda^3 t^2 \cosh(x+y))/3,$
$\vdots$

The approximate solution for equation (16) is given in the form
$v(x,y,t) = \sum_{i=1}^{\infty} v_i = -(\sinh(x+y)\lambda^4 t^3)/9 + (\cosh(x+y)\lambda^3 t^2)/3$
$-(2\sinh(x+y)\lambda^2 t)/3 + (4\lambda \sinh(x/2+y/2)^2)/3 + \cdots$.

► **Results**

In order to assess the accuracy of the two mentioned methods, a comparative study between HAM and HPIM is performed in order to investigate Zakharov-Kuznetsov equation with the following arbitrary constants $x = 0.1$, $y = 0.1$ and $\lambda = 1$.

Table 1 shows that the results of the HAM are in excellent agreement with those obtained by the HPIM.

Table 1

Absolute errors between the exact solution and the 10th-order approximate solution given when $x = 0.1$, $y = 0.1$, and $\lambda = 1$.

t	$\lvert Exact - HAM \rvert$	$\lvert Exact - HPIM \rvert$
0.1	1.214306433183765e-016	1.908195823574488e-017
0.2	1.898481372109018e-014	1.912706104612028e-014
0.3	1.101332566810775e-012	1.101391547408959e-012
0.4	1.953304878954398e-011	1.953318756742206e-011
0.5	1.817122043112907e-010	1.817123500280626e-010
0.6	1.124017288967050e-009	1.124017379172670e-009
0.7	5.246648351886485e-009	5.246648462908787e-009
0.8	1.993001279287920e-008	1.993001295941266e-008
0.9	6.468509036738901e-008	6.468509053392246e-008
1.0	1.854453278993251e-007	1.854453280103474e-007

Example 3.2.

Now we consider the ZK(3,3,3) equation

$$v_t-(v^3)_x+2(v^3)_{xxx}+2(v^3)_{yyx}=0, \tag{22}$$

subject to initial condition

$$v(x,y,0)=\sqrt{\frac{3\lambda}{2}}\sinh(\frac{1}{6}(x+y)), \tag{23}$$

where λ is an arbitrary constant.

► **Solution by HAM**

To solve the equation (22) by means of HAM, according to the initial conditions denoted in equation (23), it is natural to choose

$$v_0(x,y,t)=\sqrt{\frac{3\lambda}{2}}\sinh(\frac{1}{6}(x+y)). \tag{24}$$

We choose the linear operators

$$L[\phi(x,y,t;q)]=\frac{\partial\phi(x,y,t;q)}{\partial t},$$

with the property $L[c]=0$, where c is constant. From (22), we define nonlinear operators

$$N[\phi]=\phi_t-(\phi^3)_x+2(\phi^3)_{xxx}+2(\phi^3)_{yyx}. \tag{25}$$

We construct the zero-order deformation equation

$$(1-q)L[\phi(x,y,t;q)-v_0(x,y,t)]=qhH(t)N[\phi]. \tag{26}$$

Differentiating (26), m times with respect to q, then setting $q=0$ and finally dividing them by $m!$, the mth-order deformation equation is derived as

$$L[v_m(x,y,t)-\chi_m v_{m-1}(x,y,t)]=hR_m(\vec{v}_{m-1}), \tag{27}$$

where

$$R_m(\vec{v}_{m-1})=\frac{\partial v_{m-1}}{\partial t}-\frac{\partial}{\partial x}[\sum_{i=0}^{m-1}\sum_{j=0}^{i}v_{m-1-i}v_{i-j}v_j]+2\frac{\partial^3}{\partial x^3}[\sum_{i=0}^{m-1}\sum_{j=0}^{i}v_{m-1-i}v_{i-j}v_j]$$
$$+2\frac{\partial^3}{\partial x\partial^2 y}[\sum_{i=0}^{m-1}\sum_{j=0}^{i}v_{m-1-i}v_{i-j}v_j]$$

Therefore, the solution of the mth-order deformation equation will be

$$v_m(x,y,t)=\chi_m v_{m-1}(x,y,t)+h\int_0^t R_m(\vec{v}_{m-1})\,dt.$$

Subsequently, solving the mth-order deformation equations one has

$$v_0(x,y,t) = (2^{\frac{1}{2}}3^{\frac{1}{2}}\sinh(x/6+y/6))/2,$$

$$v_1(x,y,t) = -(6^{\frac{1}{2}}t\cosh(x/6+y/6))/12,$$

$$v_2(x,y,t) = (6^{\frac{1}{2}}t^2\sinh(x/6+y/6))/144,$$

$$\vdots$$

Finally, by view of equation $v(x,y,t) = v_0(x,y,t) + v_1(x,y,t) + v_2(x,y,t) + \cdots$, we have

$v(x,y,t) = (6^{\frac{1}{2}}\lambda^{\frac{1}{2}}(\sinh(x/6+y/6)h^2\lambda^2t^2 + 12\cosh(x/6+y/6)h^2\lambda t$
$+24\cosh(x/6+y/6)h\lambda t + 72\sinh(x/6+y/6)))/144 + \cdots$.

The influence of h on the convergence of the solution series is given in Fig. 2. It is easy to see that in order to have a good approximation, h has to be chosen in the interval $-1.25 < h < -0.8$.

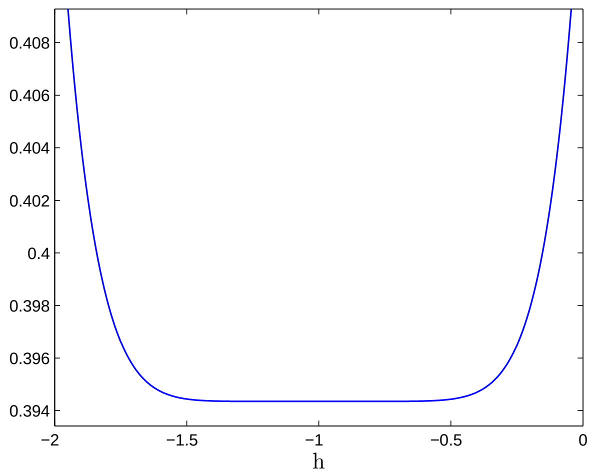

Fig. 2 – The h-curve of $v(1,1,0.1)$ given by the 9th-order approximate solution.

► **Solution by HPIM**

We solve ZK(3,3,3) explicitly by HPIM; thus equation (22) is equivalent to the following integral equation

$$v = \sqrt{\frac{3\lambda}{2}}\sinh(\frac{1}{6}(x+y)) + \int_0^t (v^3)_x - 2(v^3)_{xxx} - 2(v^3)_{yyx}\, dt,$$

where $f(x) = \sqrt{\frac{3\lambda}{2}}\sinh(\frac{1}{6}(x+y))$ and $\mathcal{N} = \int_0^t (v^3)_x - 2(v^3)_{xxx} - 2(v^3)_{yyx}\, dt$.

Following the algorithm given in previous section, some first terms of the successive approximation series are as follows
$v_0(x,y,t) = (\frac{3\lambda}{2})^{\frac{1}{2}} \sinh(\frac{1}{6}(x+y))$,
$v_1(x,y,t) = -(6^{\frac{1}{2}}\lambda^{\frac{3}{2}} t \cosh(x/6+y/6))/12$,
$v_2(x,y,t) = (6^{\frac{1}{2}}\lambda^{\frac{5}{2}} t^2(-\sinh(x/6+y/6)\lambda^2 t^2 + 8\cosh(x/6+y/6)\lambda t +$
$72\sinh(x/6+y/6)))/10368$,
$\vdots$

The series solutions with three terms are given as
$v(x,y,t) = \sinh(x/6+y/6)((3\lambda)/2)^{\frac{1}{2}} - (6^{\frac{1}{2}}\lambda^{\frac{3}{2}} t(\sinh(x/6+y/6)\lambda^3 t^3$
$-8\cosh(x/6+y/6)\lambda^2 t^2 - 72\sinh(x/6+y/6)\lambda t +$
$864\cosh(x/6+y/6)))/10368 + \cdots$.

► **Results**

In order to illustrate the methods in more details, a comparative study between HAM and HPIM is employed to investigate Zakharov-Kuznetsov equation with the following arbitrary constants $x = 0.1$, $y = 0.1$ and $\lambda = 1$.

Table 2 shows that the results of the HAM are in excellent agreement with those obtained by the HPIM.

Table 2

Absolute errors between the exact solution and the 5th-order approximate solution given when $x = 0.1$, $y = 0.1$ and $\lambda = 1$.

t	$\lvert Exact - HAM \rvert$	$\lvert Exact - HPIM \rvert$
0.1	1.313142303227544e-011	1.181661955784730e-010
0.2	4.201752071275600e-010	3.780843409184609e-009
0.3	3.190515643347602e-009	2.871008818183807e-008
0.4	1.344418436993733e-008	1.209934948273594e-007
0.5	4.102703823199994e-008	3.693094416459819e-007
0.6	1.020863746176159e-007	9.192207001823549e-007
0.7	2.206474220695087e-007	1.987568575875787e-006
0.8	4.301911441656481e-007	3.877001102423061e-006
0.9	7.752337946587140e-007	6.990670517625919e-006
1.0	1.312908663064771e-006	1.184713729135423e-005

4. CONCLUSIONS

In this paper two iterative methods have been successfully employed to obtain the approximate analytical solutions of Zakharov-Kuznetsov equations. A major difference between HAM and HPIM is that HAM can be used as a reasonable approach for controlling the convergence of approximation series. The basic ideas of this approach can be widely employed to efficiently solve other nonlinear dynamical problems. The obtained results demonstrate the accuracy and efficiency of the proposed method.

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