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FINE STRUCTURE OF RADIATION SPECTRUM OF SYSTEM OF ELECTRONS MOVING IN SPIRAL IN MEDIUM*

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Abstract. Using the improved Lorentz self-interaction method complemented by Dirac hypothesis, the fine structure of spectral distribution of the radiation power for the system of electrons moving in spiral in transparent isotropic medium is investigated. The fine structure of radiation spectrum of two and three electrons moving one by one along the spiral in transparent medium in non-relativistic case is studied. The conditions at which the radiation power for these systems tends to zero are analyzed. The oscillations in spectral distribution of synchrotron-Cherenkov radiation power for one electron, with relativistic transversal and smaller longitudinal velocity components, moving in spiral in transparent medium are obtained and investigated.

Key words: improved Lorentz self-interaction method, radiationless orbits, oscillation in synchrotron-Cherenkov radiation.

1. INTRODUCTION

The problem of radiation of charges particles moving in vacuum [1–2] and in medium [3–5] with acceleration presents a great interest. The formula obtained in 1897 by Larmor [6] permits to calculate the total power radiated by a non-relativistic point charge when it accelerates. In 1898 Lienard [7] gave a relativistic formula for the rate of radiation of a centripetally accelerated charge. In 1908 Schott [8] developed further the classical radiation theory of charge moving in a circle. The radiation spectrum of a system of electrons moving in vacuum in a circle was for the first time definitely investigated by G.A. Schott [8] in the

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framework of classical electrodynamics. The investigations of Schott [9] contain the surprising discovery that charged sphere in accelerated motion can have radiationless orbits. In ultra relativistic case for the bigger number of harmonics in Schott relationship for radiation power of electron Ivanenko, Sokolov [10] and Schwinger [11] replaced the Bessel functions by Mcdonald ones (ultra relativistic asymptotic approximation). Goedecke [12] investigated the general condition of non-radiation for an extended charge-current distribution. The particularities of properties of synchrotron radiation of charged particles moving in magnetic field in vacuum were examined by Ternov in report [1]. Radiation power as function of time shift between two electrons moving along the spiral in vacuum is investigated in [13–15].

The classical theory of radiation emitted by charged particles moving in non-dispersive medium with superluminal velocities was developed by Heaviside [16]. The classical theory of the Cherenkov phenomenon in a dispersive medium was at first formulated by Frank and Tamm in [17]. The different properties of the radiation of charged particles moving in a medium were analyzed in monographs [18–19]. The generalized Cherenkov-like effects based on four fundamental interactions was investigated and classified in [20].

The behaviour of electromagnetic radiation spectrum of the electron moving with superluminal velocities in medium in magnetic field was investigated in the papers [3–5, 21]. The oscillations in synchrotron-Cherenkov radiation spectrum of one electron were obtained for its motion in a circle [21] and in a spiral [22]. The oscillations in synchrotron-Cherenkov radiation spectrum of two, three, and four electrons moving one by one along a spiral in transparent medium were studied in [23, 24]. The hopping variation in synchrotron-Cherenkov radiation spectrum of one electron was studied in [14, 25–27].

The radiation spectrum of electrons in dispersive transparent medium was studied using the improved Lorentz self-interaction method complement by Dirac hypothesis [22, 28].

In this paper we investigate the fine structure of radiation spectrum of two and three electrons moving one by one along the spiral in transparent medium in non-relativistic case. We analyze the conditions at which the radiation power of these systems tends to zero. Also, we study the oscillations in synchrotron-Cherenkov radiation spectrum of one electron moving in the spiral in magnetic field in transparent medium with relativistic transversal velocity component (the component perpendicular to the magnetic induction vector) and smaller longitudinal velocity component (component parallel to the magnetic induction vector).

2. INSTANTANEOUS AND TIME AVERAGED RADIATION POWER OF SYSTEM OF ELECTRONS MOVING ALONG THE SPIRAL IN TRANSPARENT MEDIUM

Using the improved Lorentz self-interaction method complement by Dirac hypothesis, we investigate the fine structure of spectral distribution of the radiation power of the electrons moving in the spiral in transparent isotropic medium. According to [5, 28, 29], the time averaged radiation power $\bar{P}^{rad}$ of charged particles moving in medium is:

$$\begin{aligned}\bar{P}^{rad} &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\mathrm{d}tP^{rad}(t) = \\ &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\mathrm{d}t\left\{\int_{\tau}\left(\vec{j}\left(\vec{r},t\right)\frac{1}{c}\frac{\partial\vec{A}^{Dir}\left(\vec{r},t\right)}{\partial t}-\rho\left(\vec{r},t\right)\frac{\partial\Phi^{Dir}\left(\vec{r},t\right)}{\partial t}\right)\mathrm{d}\vec{r}\right\}.\end{aligned} \tag{1}$$

Here $P^{rad}(t)$ is the instantaneous radiation power, $\vec{j}\left(\vec{r},t\right)$ is the current density and $\rho(\vec{r},t)$ is the charge density. The integration is performed over some volume τ. According to the Dirac hypothesis [3, 5, 29, 30], the scalar $\Phi^{Dir}\left(\vec{r},t\right)$ and vector $\vec{A}^{Dir}(\vec{r},t)$ potentials are defined as a half-differences of the retarded and advanced potentials:

$$\Phi^{Dir}=\frac{1}{2}\left(\Phi^{ret}-\Phi^{adv}\right),\qquad \vec{A}^{Dir}=\frac{1}{2}\left(\vec{A}^{ret}-\vec{A}^{adv}\right). \tag{2}$$

Then, the sources functions for N charged point particles are defined as [30]

$$\vec{j}\left(\vec{r},\ t\right)=\sum_{l=1}^{N}\vec{V}_l\left(t\right)\rho_l\left(\vec{r},t\right),\ \rho(\vec{r},t)=\sum_{l=1}^{N}\rho_l(\vec{r},t),\quad \rho_l(\vec{r},\ t)=e\delta(\vec{r}-\vec{r}_l(t)), \tag{3}$$

where $\vec{r}_l\left(t\right)$ and $\vec{V}_l(t)$ are the motion law and the velocity of the l^{th} particle, respectively, fixed by the expressions:

$$\vec{r}_l(t)=r_0\cos\{\omega_0(t+\Delta t_l)\}\vec{i}+r_0\sin\{\omega_0(t+\Delta t_l)\}\vec{j}+V_{\|}(t+\Delta t_l)\vec{k},\quad \vec{V}_l\left(t\right)=\frac{\mathrm{d}\vec{r}_l\left(t\right)}{\mathrm{d}t}. \tag{4}$$

Here $r_0=V_{\perp}\omega_0^{-1}$, $\omega_0=ceB^{ext}/\widetilde{E}$, $\widetilde{E}=c\sqrt{p^2+m_0^2c^2}$, the magnetic induction vector $\vec{B}^{ext}\|0Z$, $V_{\perp}$ and $V_{\|}$ are the components of the velocity, $\vec{p}$ and $\widetilde{E}$ are the momentum and energy of the electron, e and m_0 are its charge and rest mass, respectively.

In this case, the time-averaged radiation power of the sequence of point electrons can be obtained after substitution of (2) to (4) into (1). Then, it is found [28] that

$$\bar{P}^{rad} = \bar{P}_{med}^{\text{int}} = \int_0^\infty W(\omega)\,\mathrm{d}\omega\,, \tag{5}$$

$$W(\omega) = \frac{2e^2}{\pi c^2}\int_0^\infty \mathrm{d}x\,\mu(\omega)\,S_N(\omega)\,\omega\;\frac{\sin\left\{\frac{n(\omega)}{c}\omega\eta(x)\right\}}{\eta(x)}\cos\omega x\;\left[V_\perp^2\cos(\omega_0 x) + V_\parallel^2 - \frac{c^2}{n^2(\omega)}\right], \tag{6}$$

where $W(\omega)$ is the function of spectral distribution of radiation power, $\eta(x) = \sqrt{V_\parallel^2 x^2 + 4\frac{V_\perp^2}{\omega_0^2}\sin^2\left(\frac{\omega_0}{2}x\right)}$, ω is cyclic frequency, c is velocity of light in vacuum.

For the sequence of electrons moving one by one along a spiral the coherence factor takes the form [28]:

$$S_N(\omega) = \sum_{l,j=1}^{N}\cos\{\omega(\Delta t_l - \Delta t_j)\}. \tag{7}$$

The coherence factor $S_N(\omega)$ determines the redistribution of radiation power of electrons in spectral distribution of radiation.

3. SPECTRAL-ANGULAR DISTRIBUTION OF THE RADIATION POWER OF SEQUENCE OF ELECTRONS MOVING IN THE SPIRAL IN TRANSPARENT MEDIUM

After some transformations of relationships (5) and (6), the contributions of separate harmonics to the averaged electrons radiation power can be expressed as:

$$\bar{P}^{rad}=\frac{e^2}{c^3}\sum_{m=-\infty}^{\infty}\int_0^{\infty}\mathrm{d}\omega\,\mu(\omega)\,n(\omega)\,\omega^2\int_0^{\pi}\sin\theta\,\mathrm{d}\theta S_N(\omega)\,\delta\left\{\omega\left(1-\frac{n(\omega)}{c}V_{\parallel}\cos\theta\right)-m\omega_0\right\}\times$$
$$\times\left\{V_{\perp}^2\left[\frac{m^2}{q^2}J_m^2(q)+J_m'^2(q)\right]+\left(V_{\parallel}^2-\frac{c^2}{n^2(\omega)}\right)J_m^2(q)\right\},\qquad(8)$$

where $q=\frac{n(\omega)}{c}\frac{\omega}{\omega_0}V_{\perp}\sin\theta$, $J_m(q)$ and $J_m'(q)$ are the Bessel function of integer index and its derivative, respectively. Each harmonic in relationship (8) is the set of the frequencies, which are the solutions of the equation

$$\omega\left(1-\frac{n(\omega)}{c}V_{\parallel}\cos\theta\right)-m\omega_0=0\,.\qquad(9)$$

The coherence factor $S_1(\omega)$ of a single electron is defined as

$$S_N(\omega)=S_1(\omega)=1\,.\qquad(10)$$

In the case of two electrons the coherence factor $S_2(\omega)$ is defined as [31]

$$S_2(\omega)=2+2\cos(\omega\Delta t_{12}).\qquad(11)$$

Here $\Delta t_{12}=\Delta t_2-\Delta t_1$ is the time shift between the first and second electrons moving along the spiral.

The coherence factor $S_3(\omega)$ of three electrons takes the form [31]

$$S_3(\omega)=3+2\cos(\omega\Delta t_{12})+2\cos(\omega\Delta t_{23})+2\cos\{\omega(\Delta t_{12}+\Delta t_{23})\}.\qquad(12)$$

Here Δt_{23} is the time shift between the second and third electrons.

4. FINE STRUCTURE OF RADIATION SPECTRUM OF ONE, TWO AND THREE NON-RELATIVISTIC ELECTRONS MOVING IN THE SPIRAL IN MEDIUM

Our high accuracy numerical method of calculations of the radiation spectra was carried out on the basis of relationships (5) to (7) and (10) to (12). The fine structure of radiation spectrum of one, two and three electrons, moving along the spiral in medium for the non-relativistic components of electrons velocity are presented at Figs. 1–3.

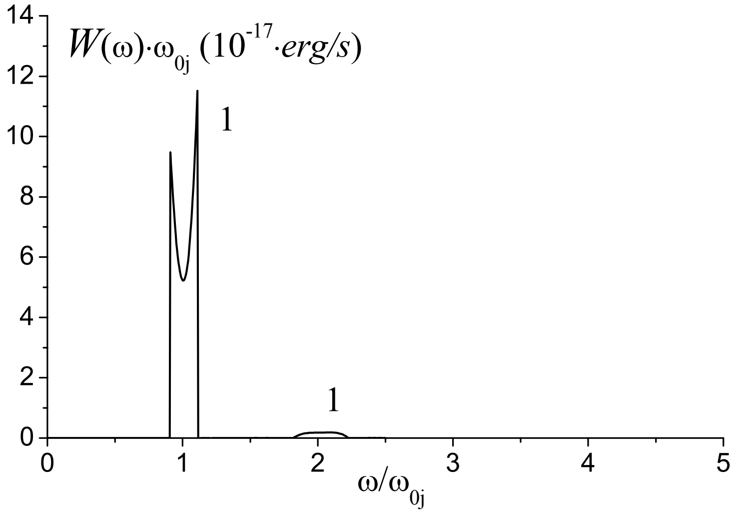

Fig. 1 – Fine structure of radiation spectrum at non-relativistic velocity components of one electron moving along the spiral at $B^{ext} = 1\,\text{Gs}$ $\mu = 1$, $n = 2.0$, $V_{\perp} = V_{\perp med} = 0.2\times10^{10}$ cm/s, $V_{\|} = V_{\|med} = 0.15\times10^{10}$ cm/s, $\omega_{0j} = 0.1753\times10^{8}$ rad/s, $r_{0j} = 114$ cm (j = 1, 2, 3).
Curve 1: one electron with radiation power $P^{\text{int}}_{med1} = 0.1512\times10^{-16}$ erg/s.

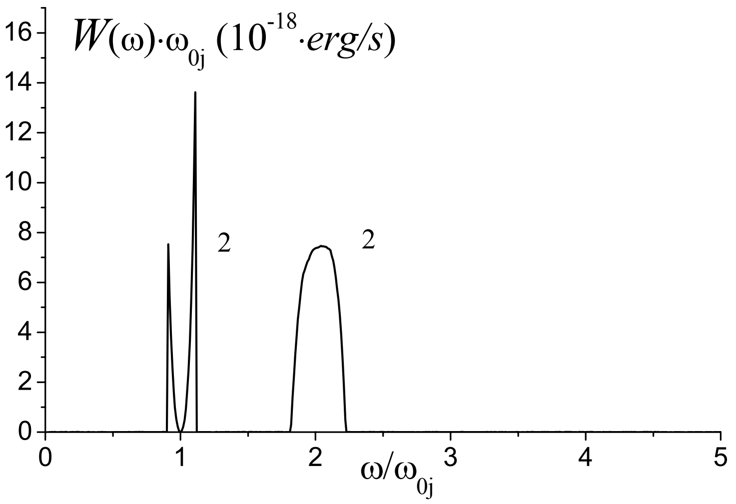

Fig. 2 – Fine structure of radiation spectrum at non-relativistic velocity components of sequence of two electrons moving along the spiral at $B^{ext} = 1\,\text{Gs}$ $\mu = 1$, $n = 2.0$, $V_{\perp med} = 0.2\times10^{10}$ cm/s, $V_{\|med} = 0.15\times10^{10}$ cm/s. Curve 2: two electrons with time shift $\Delta t^{(2)}_{12} = \pi/\omega_{02}$ and $P^{\text{int}}_{med2} = 0.3113\times10^{-17}$ erg/s, $P^{\text{int}}_{med2}/P^{\text{int}}_{med1} = 0.2059$.

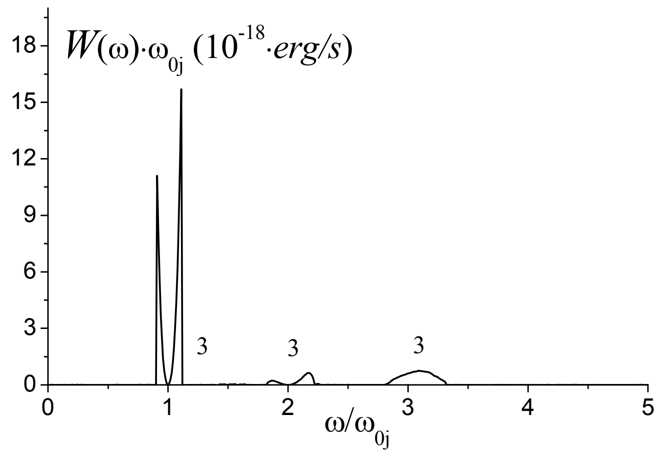

Fig. 3 – Fine structure of radiation spectrum at non-relativistic velocity components of three electrons moving along spiral for $B^{ext} = 1\,\text{Gs}$ $\mu = 1$, $n = 2.0$, $V_{\perp med} = 0.2\times10^{10}$ cm/s, $V_{\| med} = 0.15\times10^{10}$ cm/s. Curve 3: three electrons with time shifts $\Delta t_{12}^{(3)} = \Delta t_{23}^{(3)} = 2\pi/(3\omega_{03})$ and $P_{med3}^{\text{int}} = 0.1058\times10^{-17}$ erg/s, $P_{med3}^{\text{int}} / P_{med1}^{\text{int}} = 0.0699$.

For the uniform distribution of two electrons along the spiral at time shift $\Delta t_{12}^{(2)} = \pi/\omega_{02}$, the radiation power of sequence of two electrons is equal to $P_{med2}^{\text{int}} = 0.3113\times10^{-17}$ erg/s (Fig. 2) and coherent factor is equal to $P_{med2}^{\text{int}} / P_{med1}^{\text{int}} = 0.2059$.

For the uniform distribution of three electrons along the spiral at time shifts: $\Delta t_{12}^{(3)} = \Delta t_{23}^{(3)} = 2\pi/(3\omega_{03})$, the radiation power of sequence of three electrons is equal to $P_{med3}^{\text{int}} = 0.1058\times10^{-17}$ epr/c (Fig. 3) and coherent factor is equal to $P_{med3}^{\text{int}} / P_{med1}^{\text{int}} = 0.0699$.

Decreasing magnitude of radiation power tending to zero is possible when the number of electrons moving along the spiral in sequence increases and the longitudinal component of the velocity (parallel to the magnetic induction vector) decreases. Our results are in good agreement with the data obtained in [9–10, 13–15] and present interest for the definition of radiationless orbits.

5. OSCILLATIONS IN FINE STRUCTURE OF SYNCHROTRON-CHERENKOV RADIATION OF ELECTRON MOVING IN MEDIUM

The spectral distribution of synchrotron-Cherenkov radiation power was calculated for $B^{ext} = 1\,\text{Gs}$, $\mu = 1$, $n = 1.3$, $V_{\perp med} = 0.27\times10^{11}$ cm/s, $V_{\| med} = 0.1\times10^{10}$ cm/s, $\omega_{0j} = 0.762\times10^{7}$ rad/s, $r_{0j} = 3543$ cm (j = 1, 2, 3) and

$V_{\perp med} = 0.27\times10^{11}$ m/s, $V_{\| med} = 0.5\times10^{9}$ cm/s, $\omega_{0j} = 0.764\times10^{7}$ rad/s, $r_{0j} = 3535$ cm (j = 4, 5, 6), $c = 0.2997925\times10^{11}$ cm/s.

The radiation power for one electron: $P^{\mathrm{int}}_{med1} = 0.8079\times10^{-13}$ erg/s in interval from 0 to $40\omega_{01}$ is determined after integration of relationship (5) taking into account (6), (7) when $S_N(\omega)$ is substituted by $S_1(\omega)=1$ (curve 1 in Fig. 4).

The radiation spectrum at the first thirteen harmonics (curve 1 in Fig. 4) and at the first thirty harmonics (curve 4 in Fig. 7) has a form of discrete bands. The broadening of the discrete harmonics in bands is caused by the influence of the Doppler effect (9).

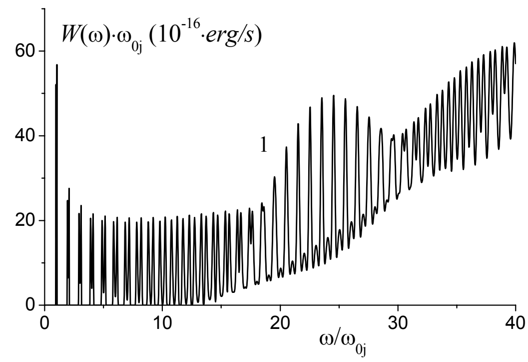

Fig. 4 – Oscillations in synchrotron-Cherenkov radiation spectrum at low harmonics for $B^{ext} = 1$ Gs, $\mu = 1$, $n = 1.3$, $V_{\perp med} = 0.27\times10^{11}$ cm/s, $V_{\| med} = 0.1\times10^{10}$ cm/s, $\omega_{0j} = 0.762\times10^{7}$ rad/s, $r_{0j} = 3543$ cm (j = 1,2, 3). Curve 1: one electron with radiation power $P^{\mathrm{int}}_{med1} = 0.808\times10^{-13}$ erg/s.

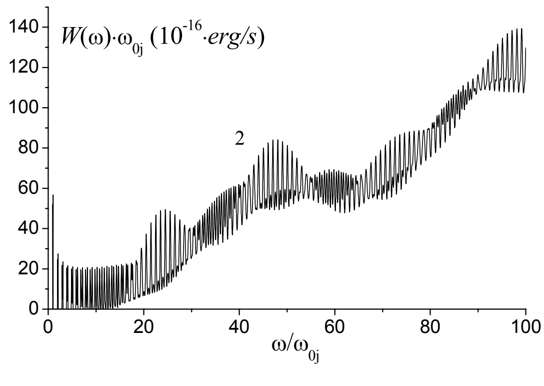

Fig. 5 – Oscillations in synchrotron-Cherenkov radiation spectrum at low ad middle harmonics for $B^{ext} = 1$ Gs, $\mu = 1$, $n = 1.3$, $V_{\perp med} = 0.27\times10^{11}$ cm/s, $V_{\| med} = 0.1\times10^{10}$ cm/s. Curve 2: one electron with radiation power $P^{\mathrm{int}}_{med2} = 0.5520\times10^{-12}$ erg/s.

The near-periodical variations (curves 1 to 6 in Figs. 4 to 9) of the spectral distribution of the electromagnetic radiation become more essential at decreasing longitudinal component of the velocity $V_{\|med}$.

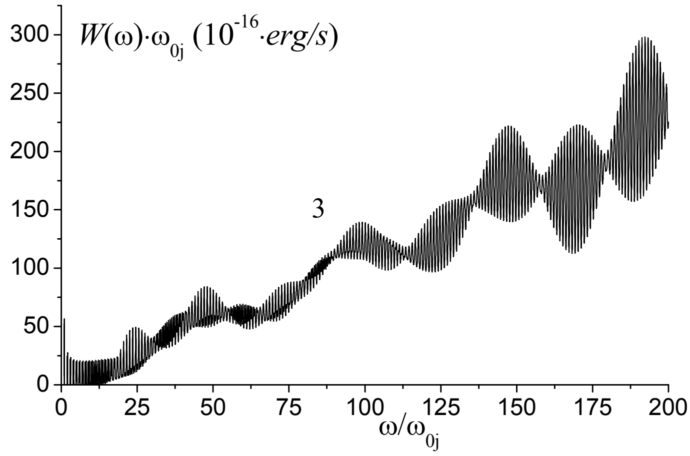

Fig. 6 – Oscillations in synchrotron-Cherenkov radiation spectrum at low, middle, and high harmonics for $B^{ext} = 1$ Gs, $\mu = 1$, $n = 1.3$, $V_{\perp med} = 0.27\times10^{11}$ cm/s, $V_{\|med} = 0.1\times10^{10}$ cm/s.
Curve 3: one electron with radiation power $P^{\text{int}}_{med3} = 0.2198\times10^{-11}$ erg/s.

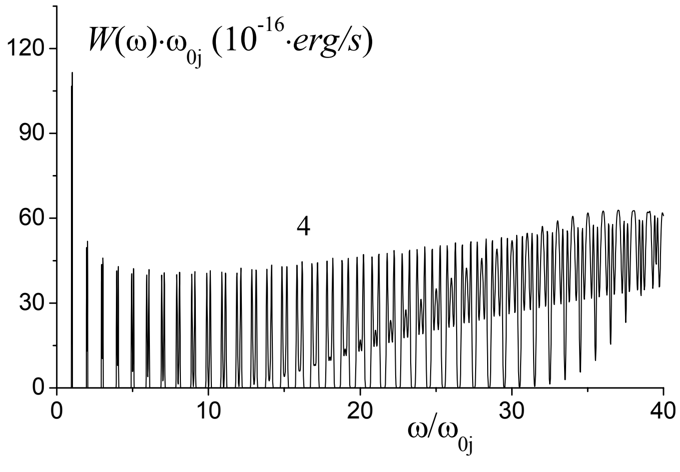

Fig. 7 – Oscillations in synchrotron-Cherenkov radiation spectrum at low harmonics for $B^{ext} = 1$ Gs, $\mu = 1$, $n = 1.3$, $V_{\perp med} = 0.27\times10^{11}$ cm/s, $V_{\|med} = 0.5\times10^{9}$ cm/s, $\omega_{0j} = 0.764\times10^{7}$ rad/s, $r_{0j} = 3535$ cm (j = 4, 5, 6). Curve 4: one electron with radiation power $P^{\text{int}}_{med4} = 0.808\times10^{-13}$ erg/s.

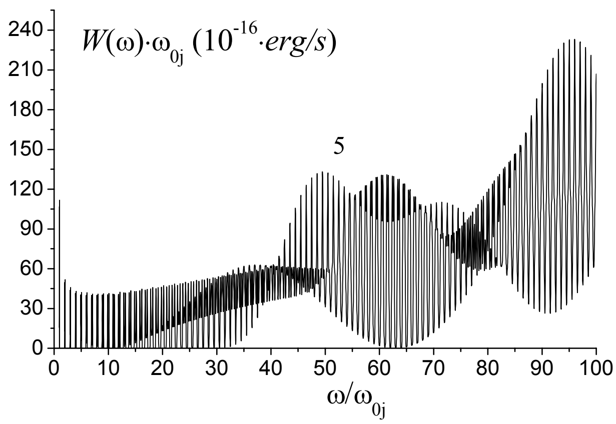

Fig. 8 – Oscillations in synchrotron-Cherenkov radiation spectrum at low and middle harmonics for $B^{ext} = 1\,\mathrm{Gs}$, $\mu = 1$, $n = 1.3$, $V_{\perp med} = 0.27 \times 10^{11}$ cm/s, $V_{\| med} = 0.5 \times 10^{9}$ cm/s. Curve 5: one electron with radiation power $P^{\text{int}}_{med5} = 0.5521 \times 10^{-12}$ erg/s.

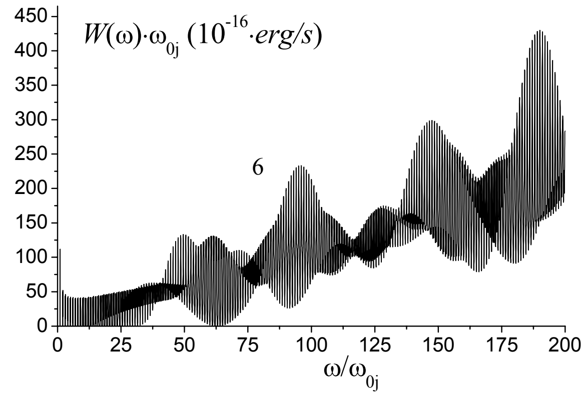

Fig. 9 – Oscillations and near-periodical variations in synchrotron-Cherenkov radiation spectrum at low, middle and high harmonics for $B^{ext} = 1\,\mathrm{Gs}$, $\mu = 1$, $n = 1.3$, $V_{\perp med} = 0.27 \times 10^{11}$ cm/s, $V_{\| med} = 0.5 \times 10^{9}$ cm/s. Curve 6: one electron with $P^{\text{int}}_{med6} = 0.2199 \times 10^{-11}$ erg/s.

For the velocities $c > V_{\perp med} > c/n$ ($V_{\perp med} = 0.2 \times 10^{11}$ cm/s, $V_{\| med} = 0.1 \times 10^{10}$ cm/s and $V_{\perp med} = 0.2 \times 10^{11}$ cm/s, $V_{\| med} = 0.5 \times 10^{9}$ cm/s) we have found the oscillations in the radiation spectrum of one electron (Curves 1 to 6 in Figs. 4 to 9). Our results are in good agreement to those obtained in [22] and complement these data.

The oscillating character of the spectral distribution of the synchrotron-Cherenkov radiation of electron moving in the spiral in magnetic field in the medium at $c > V_{\perp med} > c/n$ is determined by the properties of the Bessel functions

[32]. The spectral distribution of the synchrotron-Cherenkov radiation of electron moving in a circle at $V_{\perp med} > c/n$, has an oscillating character too [21].

6. CONCLUSIONS

Decreasing magnitude of radiation power tending to zero is possible when the number of electrons moving along the spiral in sequence increases and the longitudinal component of the velocity (parallel to the magnetic induction vector) decreases.

The oscillations in the synchrotron-Cherenkov radiation spectrum of one electron moving in the spiral are observed at the velocities $c > V_{\perp med} > c/n$ ($V_{\perp med} = 0.2\times10^{11}$ cm/s, $V_{\| med} = 0.1\times10^{10}$ cm/s and $V_{\perp med} = 0.2\times10^{11}$ cm/s, $V_{\| med} = 0.5\times10^{9}$ cm/s).

The near-periodical variations of the spectral distribution of the synchrotron-Cherenkov radiation become more essential at decreasing longitudinal component of the velocity ($V_{\| med}$).

REFERENCES

1. I.M. Ternov, Usp. Fiz. Nauk (in Russian), **165**, *4*, 429-456 (1995).
2. J.D. Jackson, *Classical Electrodynamics*, John Wiley & Sons, New York, London, Sydney, 1998.
3. A.V. Konstantinovich, V.M. Nitsovich, Izv. Vuzov. Fizika, *2*, 59–62 (1973).
4. A.B. Kukanov, A.V. Konstantinovich, Russian Phys. Journ., **18**, *8*, 1061–1065 (1976).
5. 15. J. Schwinger, Tsai Wu-yang, T. Erber, Ann. of Phys., **96**, *2*, 303-332, (1976).
6 1. J. Larmor, Philos. Trans.Royal Soc., **190**, 205–300 (1897).
7. A. Lienard, L'Eclairage Electrique, **16**, 5 (1898).
8. G.A. Schott, *Electromagnetic Radiation*, Cambridge University Press, Cambridge, 1912.
9. G.A. Schott, Philos. Mag., **7**, *15*, p. 752–761 (1933).
10. G.H. Goedecke, Physical Review, **135**, *8*, p. B281–B288 (1964)
11. D.I. Iwanenko, A.A. Sokolov. Dokl. Akad. Nauk SSSR (in Russian), **59**, *9*, 1551–1554 (1948).
12. J. Schwinger, Phys. Rev., **75**, *12*, 1912–1925 (1949).
13. A.V. Konstantinovich, I.A. Konstantinovich, Rom. Rep. Phys., **58**, *2*, 101–106 (2006).
14. A.V. Konstantinovich, I.A. Konstantinovich, Proc. Rom. Acad. A, **7**, *3*, 183–192 (2006).
15, A.V. Konstantinovich, S.V. Melnychuk, I.A. Konstantinovich, Rom. Journ. Phys., **50**, *3–4*, 347–356 (2005).
16. O. Heaviside, Philos. Mag., **27**, *5*, 324–339 (1889).
17. I.M. Frank, I.E. Tamm, Dokl. Akad. Nauk SSSR, **14**, *3*, 107–112 (1937).
18. I.M. Frank, *Vavilov-Cherenkov Radiation*, Nauka, Moscow, 1988.
19. G.N. Afanasiev, V*avilov-Cherenkov and Synchrotron Radiation: Foundations and Applications,* Kluwer Academic Publishers, Dordrecht, Boston, London, 2004.
20. D. B. Ion, W. Stocker, Phys. Rev. C., **52**, *6*, 3332–3343 (1995).
21. V.N. Tsytovich, Bulletin of Moscow State University (in Russian), *11*, 27–36 (1951).

22. A.V. Konstantinovich, I.A. Konstantinovich, Astroparticles Physics., **30**, *3*, 142–148 (2008).
23. A.V. Konstantinovich, I.A. Konstantinovich, Rom. Journ, Phys., **56**, *1–2*, 45-52 (2011).
24. A.V. Konstantinovich, I.A. Konstantinovich, Probl. At. Sci. Techn. Ser.: Nucl. Phys. Invest., *5*, 67–74 (2011).
25. A.V. Konstantinovich, S.V. Melnychuk, I.A. Konstantinovich, J. Opt. Adv. Mater., **5**, *5*, 1423–1431 (2003).
26. A.V. Konstantinovich, S.V. Melnychuk, I.A. Konstantinovich, Journal of Materials Science: Materials in Electronics, **17**, *4*, 315–320 (2006).
27. A.V. Konstantinovich, I.A. Konstantinovich, Bulletin of Karazin Kharkov National University, Radiophysics and Electronics. Ser. (in Russian), *19*, 38–47 (2011).
28. A.V. Konstantinovich, I.A Konstantinovich, Physics and Chemistry of Solid State. (in Ukrainian), **8**, *2*, 535–541 (2007).
29. A.V. Konstantinovich, S.V. Melnychuk, I.A. Konstantinovich, Bulletin of Chernivtsi National University, Physics and Electronics (in Ukrainian), **102**, 5–13 (2001).
30. P.A.M. Dirac, Proc Roy. Soc. A., **167**, *1*, 148-169 (1938).
31. 32. A.V. Konstantinovich, I.A. Konstantinovich, J. Opt. Adv. Mater., **8**, *5*, 2143–2147 (2006).
32. Janke-Emde-Lösch, *Tafeln Höherer Funktionen*, B. G. Teubner, Verlagsgesellshaft, Stuttgart, 1960.