Romanian Reports in Physics, Vol. 66, No. 2, P. 319–335, 2014

THE SPINOR FIELD THEORY OF THE PHOTON

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Received October 28, 2013

Abstract. I introduce a spinor field theory for the photon. The three-dimensional vector electromagnetic field and the four-dimensional vector potential are components of this spinor photon field. A spinor equation for the photon field is derived from Maxwell's equations, the relations between the electromagnetic field and the four-dimensional vector potential, and the Lorentz gauge condition. The covariant quantization of free photon field is done, and only transverse photons are obtained. The vacuum energy divergence does not occur in this theory. A covariant "positive frequency" condition is introduced for separating the photon field from its complex conjugate in the presence of the electric current and charge.

Key words: photon, spinor.

1. INTRODUCTION

The electromagnetic interaction is the best studied one among the four known fundamental interactions. In the frame of quantum theory, the photon is the quantum of the electromagnetic field. There are several ways to represent the electromagnetic field: by the three-dimensional electric field and magnetic field vectors, by the 4×4 electromagnetic tensor, or by the four-dimensional potential vector [1, 2]. The electric and magnetic fields are directly related to the energy density of the electromagnetic field. The four-vector potential is directly related to the Lagrangian density of interaction. But however, none of these fields can be regarded as the photon field, because the photon density can not be expressed as inner products of these fields with their adjoint fields. In this paper, I introduce the spinor photon field that satisfies a spinor equation similar to the Dirac equation for the electron. The three-dimensional vector electric field, the three-dimensional magnetic vector field and the four-dimensional vector potential are components of this spinor photon field. The spinor equation for the photon field is based on the Maxwell equations, the relations between the electromagnetic field and the four-dimensional vector potential, and the Lorentz gauge condition. The Lagrangian densities for the free photon field and for the photon field in interaction with the matter are established. Covariant quantization of photon field is carried out, and only transverse photons emerge from the quantization procedure. The vacuum state of the photon field is found to have null energy. The solution for the photon field in the presence of the electric current and charge is found, and a covariant "positive frequency" condition is introduced for separating the photon field from its complex conjugate.

The Maxwell equations, the relations between the electromagnetic field and the four-dimensional vector potential, and the Lorentz gauge condition are rewritten as two eight-component spinor equations in Sec. 2. The spinor photon field is introduced in Sec. 3. The quantization of the photon field is treated in Sec. 4, and the photon field in the presence of electric current and charge is analysed in Sec. 5.

2. THE SPINOR EQUATION FOR THE ELECTROMAGNETIC FIELD

Maxwell's equations for the electric and magnetic fields $\mathbf{E}$ and $\mathbf{H}$ in the presence of a charge density ρ and a current density $\mathbf{j}$ can be written in the following form:

$$\frac{\partial}{\partial x_0}\left(\sqrt{\varepsilon_0}\mathbf{E}\right)=\nabla\times\left(\sqrt{\mu_0}\mathbf{H}\right)-\sqrt{\mu_0}\mathbf{j}, \tag{1}$$

$$\frac{\partial}{\partial x_0}\left(\sqrt{\mu_0}\mathbf{H}\right)=-\nabla\times\left(\sqrt{\varepsilon_0}\mathbf{E}\right), \tag{2}$$

$$0=-\nabla\cdot\left(\sqrt{\mu_0}\mathbf{H}\right), \tag{3}$$

$$0=\nabla\cdot\left(\sqrt{\varepsilon_0}\mathbf{E}\right)-\sqrt{\varepsilon_0}\rho, \tag{4}$$

where ε_0 is the vacuum permittivity, μ_0 is the magnetic permeability of the vacuum, and x_0=ct. Because the electric field, the magnetic field, the current density and the charge density are real quantities, they are completely described by their positive frequency components. An alternative way for writing the above equations is to introduce an eight components spinor electromagnetic field and an eight components spinor electric current density defined by

$$\psi_{em}(x)=\left(\sqrt{\varepsilon_0}E_1(x)\ \sqrt{\varepsilon_0}E_2(x)\ \sqrt{\varepsilon_0}E_3(x)\ 0\ \sqrt{\mu_0}H_1(x)\ \sqrt{\mu_0}H_2(x)\ \sqrt{\mu_0}H_3(x)\ 0\right)^T \tag{5}$$

and

$$j_e = \sqrt{\mu_0}\left(j_1(x)\ j_2(x)\ j_3(x)\ 0\ 0\ 0\ 0\ j_0(x)\right)^T, \tag{6}$$

where $j(x) = \left(c\rho(x), \mathbf{j}(x)\right)$ is the four-vector current density.

The Maxwell equations (1-4) now can be written as

$$\frac{\partial}{\partial x_0}\psi_{em}(x) = -\boldsymbol{\alpha}_e \cdot \nabla\psi_{em}(x) - j_e(x), \tag{7}$$

where

$$\alpha_{e1} = \begin{pmatrix} 0 & 0 & 0 & -i\sigma_2 \\ 0 & 0 & -i\sigma_2 & 0 \\ 0 & i\sigma_2 & 0 & 0 \\ i\sigma_2 & 0 & 0 & 0 \end{pmatrix}, \quad \alpha_{e2} = \begin{pmatrix} 0 & 0 & 0 & -I_2 \\ 0 & 0 & I_2 & 0 \\ 0 & I_2 & 0 & 0 \\ -I_2 & 0 & 0 & 0 \end{pmatrix},$$
$$\alpha_{e3} = \begin{pmatrix} 0 & 0 & i\sigma_2 & 0 \\ 0 & 0 & 0 & -i\sigma_2 \\ -i\sigma_2 & 0 & 0 & 0 \\ 0 & i\sigma_2 & 0 & 0 \end{pmatrix} \tag{8}$$

with

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{9}$$

We have

$$\alpha_{em} \cdot \alpha_{en} + \alpha_{en} \cdot \alpha_{em} = 2\delta_{mn}, \quad m, n = 1, 2, 3. \tag{10}$$

The electromagnetic field can be described by the four-vector potential $A(x) = \left(\phi(x)/c, \mathbf{A}(x)\right)$. The relation between the electric and magnetic fields and the four-vector potential, and the Lorentz gauge condition can be written as:

$$\frac{\partial}{\partial x_0}\left(\sqrt{\varepsilon_0}\mathbf{A}\right) = -\nabla\left(\sqrt{\varepsilon_0}A_0\right) - \frac{1}{c}\sqrt{\varepsilon_0}\mathbf{E}, \tag{11}$$

$$0 = \nabla \times \left(\sqrt{\varepsilon_0}\mathbf{A}\right) - \frac{1}{c}\sqrt{\mu_0}\mathbf{H}, \tag{12}$$

$$\frac{\partial}{\partial x_0}\left(\sqrt{\varepsilon_0}A_0\right) = -\nabla \cdot \left(\sqrt{\varepsilon_0}\mathbf{A}\right). \tag{13}$$

The relations (11–13) can be rewritten as

$$\frac{\partial}{\partial x_0}\psi_a(x) = \boldsymbol{\alpha}_e \cdot \nabla\psi_a(x) - \frac{1}{\hbar c}\psi_{em}(x), \tag{14}$$

where the spinor potential field $\psi_a(x)$ is defined by

$$\psi_a(x) = \frac{\sqrt{\varepsilon_0}}{\hbar}\left(A_1(x)\ A_2(x)\ A_3(x)\ 0\ 0\ 0\ 0\ A_0(x)\right)^T. \tag{15}$$

One may observe that the equation (14) also holds if we replace the Lorentz gauge condition with the following one:

$$\frac{\partial}{\partial x_0}\left(\sqrt{\varepsilon_0}A_0\right) = -\nabla\cdot\left(\sqrt{\varepsilon_0}\mathbf{A}\right) - \frac{1}{c}\psi_{em8}, \tag{16}$$

where ψ_{em8} is an arbitrary scalar constant. In this case, the spinor electromagnetic field takes the following form

$$\psi_{em}(x) = \left(\sqrt{\varepsilon_0}E_1(x)\ \sqrt{\varepsilon_0}E_2(x)\ \sqrt{\varepsilon_0}E_3(x)\ 0\ \sqrt{\mu_0}H_1(x)\ \sqrt{\mu_0}H_2(x)\ \sqrt{\mu_0}H_3(x)\ \psi_{em8}(x)\right)^T. \tag{17}$$

According to properties of the electric field $\mathbf{E}(x)$, magnetic field $\mathbf{H}(x)$, and four-vector potential $A(x)$ under continuous space-time transformations, we have the following relation for $\psi_{em}(x)$ and $\psi_a(x)$ under a Lorentz transformation

$$\psi'_{em}(x') = \exp\left(-\boldsymbol{\varphi}\cdot\mathbf{l}\right)\psi_{em}(x), \tag{18}$$

and

$$\psi'_a(x') = \exp\left(\boldsymbol{\varphi}\cdot\left(\boldsymbol{\alpha}_e - \mathbf{l}\right)\right)\psi_a(x), \tag{19}$$

where

$$\boldsymbol{\varphi} = \frac{\mathbf{v}}{v}\left(\ln\sqrt{1+\frac{v}{c}} - \ln\sqrt{1-\frac{v}{c}}\right). \tag{20}$$

Under a rotation characterized by the rotation angle $\boldsymbol{\phi}$, expressed as an axial vector, we have

$$\psi'_{em}(x') = \exp\left(i\boldsymbol{\phi}\cdot\mathbf{s}\right)\psi_{em}(x), \tag{21}$$

and

$$\psi'_a(x') = \exp\left(i\boldsymbol{\phi}\cdot\mathbf{s}\right)\psi_a(x), \tag{22}$$

with

$$\mathbf{s}=\begin{pmatrix}\boldsymbol{\Sigma} & 0\\ 0 & \boldsymbol{\Sigma}\end{pmatrix},\quad \mathbf{l}=\begin{pmatrix}0 & i\boldsymbol{\Sigma}\\ -i\boldsymbol{\Sigma} & 0\end{pmatrix}, \tag{23}$$

and

$$\Sigma_1=\begin{pmatrix}0&0&0&0\\0&0&-i&0\\0&i&0&0\\0&0&0&0\end{pmatrix},\quad \Sigma_2=\begin{pmatrix}0&0&i&0\\0&0&0&0\\-i&0&0&0\\0&0&0&0\end{pmatrix},\quad \Sigma_3=\begin{pmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}. \tag{24}$$

The following commutation relation holds for $\mathbf{s}$:

$$[s_n, s_m]=i\sum_{p=1}^{3}\varepsilon_{nmp}s_p. \tag{25}$$

Equations (7) and (14) are invariant under continuous space-time transformations (See the Appendices).

3. THE PHOTON FIELD

One can use $\psi_{em}(x)$ or $\psi_a(x)$ to represent the electromagnetic field, but it is not possible to express the photon density as an inner product of $\psi_{em}(x)$ or $\psi_a(x)$ with its adjoint field. Therefore neither $\psi_{em}(x)$, nor $\psi_a(x)$ can be regarded as the photon field. The concept of photon is closely related to monochromatic electromagnetic plane waves, so we consider a monochromatic plane wave

$$\psi^k_{em}(x), \psi^k_a(x) \propto \exp(-ikx) \tag{26}$$

in the absence of electric current and charge. Let $\psi^{+k}_{em}(x)$ and $\psi^{+k}_a(x)$ be the positive frequency parts of $\psi^k_{em}(x)$ and $\psi^k_a(x)$. We find that the inner product $\psi^{+k\dagger}_{em}(x)\psi^{+k}_{em}(x)$ is equal to the time average of energy density, which should equal to, in the case of a monochromatic wave, the product of the photon density and the photon energy $\hbar ck_0$. On other hand, we have

$$\varepsilon_0\mathbf{E}^{+k*}(x)\cdot\mathbf{E}^{+k}(x)=\mu_0\mathbf{H}^{+k*}(x)\cdot\mathbf{H}^{+k}(x)=\frac{1}{2}\psi^{+k\dagger}_{em}(x)\psi^{+k}_{em}(x) \tag{27}$$

and

$$i\mathbf{E}^{+k*}(x)\cdot\mathbf{A}^{+k}(x) = -i\mathbf{A}^{+k*}(x)\cdot\mathbf{E}^{+k}(x) = \frac{1}{ck_0}\mathbf{E}^{+k*}(x)\cdot\mathbf{E}^{+k}(x)\,, \tag{28}$$

thus the photon density is equal to

$$-i\psi_a^{+k\dagger}(x)\psi_{em}^{+k}(x) + i\psi_{em}^{+k\dagger}(x)\psi_a^{+k}(x)\,. \tag{29}$$

We have

$$-i\psi_a^{+k\dagger}(x)\psi_{em}^{+k}(x) + i\psi_{em}^{+k\dagger}(x)\psi_a^{+k}(x) = \left(\psi_{em}^{+k\dagger}(x)\;\;\psi_a^{+k\dagger}(x)\right)\tau_2\begin{pmatrix}\psi_{em}^{+k}(x)\\ \psi_a^{+k}(x)\end{pmatrix}, \tag{30}$$

where

$$\tau_2 = \begin{pmatrix}0 & i\beta_e\\ -i\beta_e & 0\end{pmatrix} \quad \text{with} \quad \beta_e = \begin{pmatrix}I_4 & 0\\ 0 & -I_4\end{pmatrix}, \tag{31}$$

and I_4 is the 4×4 unit matrix.

Based on the relation (30), we define the photon field $\psi_f(x)$ as

$$\psi_f(x) \equiv \frac{1}{(2\pi)^4}\int_{k_0>0} \mathrm{d}^4k\,\exp\left(ik\left(x'-x\right)\right)\begin{pmatrix}\psi_{em}(x')\\ \psi_a(x')\end{pmatrix}. \tag{32}$$

One may observe that the condition $k_0>0$ is covariant for the free photon field, because it does not contain Fourier components with $k_0<|\mathbf{k}|$. According to Eqs. (7) and (14), the free photon field satisfies the following equation:

$$i\hbar\frac{\partial}{\partial x_0}\psi_f(x) = -i\hbar\boldsymbol{\alpha}_w\cdot\nabla\psi_f(x) - \frac{i}{c}\beta_-\psi_f(x)\,, \tag{33}$$

with

$$\boldsymbol{\alpha}_w = \begin{pmatrix}\boldsymbol{\alpha}_e & 0\\ 0 & -\boldsymbol{\alpha}_e\end{pmatrix}, \quad \beta_- = \begin{pmatrix}0 & 0\\ I_8 & 0\end{pmatrix}, \tag{34}$$

where I_8 is the 8×8 unit matrix.

The invariance of Eq. (33) under continuous space-time transformations is assured by the invariance of Eqs. (7) and (14). We have

$$\psi'_f(x') = \exp\left(\boldsymbol{\varphi}\cdot\boldsymbol{\Lambda}\right)\psi_f(x), \tag{35}$$

with

$$\boldsymbol{\Lambda} = \begin{pmatrix}-\mathbf{l} & 0\\ 0 & \boldsymbol{\alpha}_e - \mathbf{l}\end{pmatrix} \tag{36}$$

for Lorentz transformations, and

$$\psi'_f(x') = \exp\left(i\boldsymbol{\phi}\cdot\mathbf{s}_f\right)\psi_f(x), \tag{37}$$

under space rotations, where

$$\mathbf{s}_f = \begin{pmatrix} \mathbf{s} & 0 \\ 0 & \mathbf{s} \end{pmatrix}. \tag{38}$$

Eq. (33) is invariant also under space inversion and time reversal. It is easy to verify that $\tau_0\psi_f(x_0,-\mathbf{x})$ and $\tau_3\psi_f(-x_0,\mathbf{x})$ satisfy the same spinor equation Eq. (33) as $\psi_f(x_0,\mathbf{x})$, where

$$\tau_0 = \begin{pmatrix} -\beta_e & 0 \\ 0 & -\beta_e \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} \beta_e & 0 \\ 0 & -\beta_e \end{pmatrix}. \tag{39}$$

The equation for the free photon field can be derived from the following Lagrangian density

$$\mathcal{L}_0 = i\hbar\overline{\psi}_f\left(\frac{\partial}{\partial t} + c\boldsymbol{\alpha}_w\cdot\nabla\right)\psi_f + i\overline{\psi}_f\beta_-\psi_f, \tag{40}$$

where $\overline{\psi}_f(x) = \psi_f^\dagger(x)\tau_2$ is the adjoint field.

One may observe that there are a total of 15 component equations for photon field. Among these 15 equations only 11 equations are independent, and the other 4 equations (corresponding to Eqs. (2) and (3)) are a direct conclusion of these 11 equations. By means of variational calculation, all these 11 independent equations can be obtained.

The conjugate field of ψ_f is

$$\pi_f = \frac{\partial\mathcal{L}_0}{\partial\dot{\psi}_f} = i\hbar\overline{\psi}_f. \tag{41}$$

The Hamiltonian now can be calculated:

$$\begin{aligned} H_0 &= \int \mathrm{d}^3\mathbf{x}\left(\pi\dot{\psi}_f - \mathcal{L}_0\right) = \\ &= \int \mathrm{d}^3\mathbf{x}\overline{\psi}_f\left(-i\hbar c\boldsymbol{\alpha}_w\cdot\nabla - i\beta_-\right)\psi_f. \end{aligned} \tag{42}$$

The Lagrangian density (40) is invariant under a global phase change of the photon field $\psi_f(x)$. This implies the conservation of the photon number N for free photon field:

$$N = \int \rho_{ph} \mathrm{d}^3\mathbf{x} \tag{43}$$

and

$$\frac{\partial}{\partial t}\rho_{ph} + \nabla \cdot \mathbf{j}_{ph} = 0, \tag{44}$$

where the photon density ρ_{ph} is given by the inner product between the photon field $\psi_f(x)$ and its adjoint $\overline{\psi}_f(x)$:

$$\rho_{ph}(x) = \overline{\psi}_f(x)\psi_f(x) \tag{45}$$

and

$$\mathbf{j}_{ph} = c\overline{\psi}_f(x)\boldsymbol{\alpha}_w\psi_f(x) \tag{46}$$

is the photon current density. One may observe that the photon density defined by Eq. (45) may take negative values. But however, when we talk about photons we refer to electromagnetic fields with well-defined frequencies. By direct calculation, one can verify that $\rho_{ph} \geq 0$ if $\psi_f(x)$ has a well-defined frequency.

According to the relation between symmetries and conservation laws [3, 4], we may obtain the following expressions for the momentum $\mathbf{P}$ and the angular momentum $\mathbf{M}$ of the free photon field:

$$\mathbf{P} = -i\hbar\int \mathrm{d}^3\mathbf{x}\overline{\psi}_f\nabla\psi_f \tag{47}$$

and

$$\mathbf{M} = \int \mathrm{d}^3\mathbf{x}\overline{\psi}_f\left[\mathbf{x}\times(-i\hbar\nabla)\right]\psi_f + \int \mathrm{d}^3\mathbf{x}\overline{\psi}_f\left(\hbar\mathbf{s}_f\right)\psi_f. \tag{48}$$

It is clear that $\mathbf{s}_f$ can be interpreted as the spin operator of the photon field. According to expression (25), we have

$$[s_{fn}, s_{fm}] = i\sum_{p=1}^{3}\varepsilon_{nmp}s_{fp}, \quad n, m = 1, 2, 3. \tag{49}$$

4. QUANTIZATION OF THE PHOTON FIELD

It is convenient to quantize the photon field in the momentum space. To do this, we have to find firstly the plane wave solutions of the photon field. By substituting the following form of solution

$$\psi_f(x) \propto \exp\left(-ikx\right)w\left(\mathbf{k}\right) \tag{50}$$

into the spinor equation (33), we find

$$\left(\boldsymbol{\alpha}_w \cdot \mathbf{k} - k_0 - \frac{i\beta_-}{\hbar c}\right) w(\mathbf{k}) = 0. \tag{51}$$

Eq. (51) permits two independent nontrivial solutions with $k_0 = |\mathbf{k}|$. They can be chosen as

$$\begin{aligned} w_{\pm 1}(\mathbf{k}) = \frac{1}{2\sqrt{\hbar c}} \big(& \hbar c|\mathbf{k}|(q_1 \pm ir_1)\ \hbar c|\mathbf{k}|(q_2 \pm ir_2)\ \hbar c|\mathbf{k}|(q_3 \pm ir_3)\ 0 \\ & \hbar c|\mathbf{k}|(r_1 \mp iq_1)\ \hbar c|\mathbf{k}|(r_2 \mp iq_2)\ \hbar c|\mathbf{k}|(r_3 \mp iq_3)\ 0 \\ & -iq_1 \pm r_1\ -iq_2 \pm r_2\ -iq_3 \pm r_3\ 0 \quad 0\ 0\ 0\ 0 \big)^T, \end{aligned} \tag{52}$$

where $\mathbf{q}$ and $\mathbf{r}$ are two unity vectors satisfying the following conditions:

$$\hat{\mathbf{k}} \times \mathbf{q} = \mathbf{r}, \quad \hat{\mathbf{k}} \times \mathbf{r} = -\mathbf{q}, \quad \mathbf{q} \times \mathbf{r} = \hat{\mathbf{k}}, \quad \text{and} \quad \mathbf{r}(-\mathbf{k}) = -\mathbf{r}(\mathbf{k}), \tag{53}$$

with $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$.

$w_{+1}(\mathbf{k})$ and $w_{-1}(\mathbf{k})$ are orthogonal:

$$\bar{w}_h(\mathbf{k}) w_{h'}(\mathbf{k}) = w_h^\dagger(\mathbf{k}) \tau_2 w_{h'}(\mathbf{k}) = \delta_{hh'} |\mathbf{k}|. \tag{54}$$

We also have

$$\left(\hat{\mathbf{k}} \cdot \mathbf{s}_f\right) w_h(\mathbf{k}) = h w_h(\mathbf{k}), \quad h = \pm 1, \tag{55}$$

and

$$\mathbf{s}_f^2 w_h(\mathbf{k}) = s(s+1) w_h(\mathbf{k}) = 2 w_h(\mathbf{k}). \tag{56}$$

Therefore photons are particles of spin $s = 1$. One may also observe that the components in $w_{+1}(\mathbf{k})$ and $w_{-1}(\mathbf{k})$ corresponding to ψ_{em8} are zero, so the Lorentz gauge condition is re-obtained.

Having plane wave solutions of the photon field $\psi_f(x)$, we may now expand $\psi_f(x)$ in plane waves

$$\begin{aligned} \psi_f(x) &= \sum_{\mathbf{k}} \sum_h \frac{1}{\sqrt{V|\mathbf{k}|}} e^{-ikx} w_h(\mathbf{k}) b_h(\mathbf{k}), \\ \bar{\psi}_f(x) &= \sum_{\mathbf{k}} \sum_h \frac{1}{\sqrt{V|\mathbf{k}|}} e^{ikx} \bar{w}_h(\mathbf{k}) b_h^\dagger(\mathbf{k}), \end{aligned} \tag{57}$$

with $k_0 = |\mathbf{k}|$. According to relations (40) and (57), the Lagrangian of the photon field can be expressed as a function of the variables $q_{h\mathbf{k}}(t)$:

$$L_0(t,q) = \sum_{\mathbf{k}} \sum_h \hbar q^\dagger_{h\mathbf{k}}(t) \left(i\frac{\partial}{\partial t} - c|\mathbf{k}| \right) q_{h\mathbf{k}}(t), \tag{58}$$

with

$$q_{h\mathbf{k}}(t) = b_h(\mathbf{k}) \exp(-i\omega t), \quad \omega = ck_0. \tag{59}$$

The conjugate momentum of $q_{h\mathbf{k}}(t)$ can be calculated, and we have

$$p_{h\mathbf{k}}(t) = \frac{\partial L_0}{\partial \dot{q}_{h\mathbf{k}}(t)} = i\hbar b^\dagger_h(\mathbf{k}) \exp(i\omega t). \tag{60}$$

By applying the quantization condition $\left[q_{h\mathbf{k}}, p_{h\mathbf{k}}\right] = i\hbar \delta_{hh'} \delta_{\mathbf{kk}'}$, we find the following commutation relation for $b_{\pm 1}(\mathbf{k})$ and $b^\dagger_{\pm 1}(\mathbf{k})$

$$\left[b_h(\mathbf{k}), b^\dagger_{h'}(\mathbf{k}')\right] = \delta_{hh'} \delta_{\mathbf{kk}'}. \tag{61}$$

$b_{\pm 1}(\mathbf{k})$ and $b^\dagger_{\pm 1}(\mathbf{k})$ are just the photon annihilation operator and the photon creation operator. The Hamiltonian of the photon field can also be calculated. We obtain

$$H_0 = \sum_{\mathbf{k}} \sum_h p_{h\mathbf{k}} \dot{q}_{h\mathbf{k}} - L_0 = \sum_{\mathbf{k}} \sum_h \hbar\omega\, b^\dagger_h(\mathbf{k}) b_h(\mathbf{k}). \tag{62}$$

We observe that the vacuum energy of the photon field is zero.

The commutation relations for the photon field can be written in a covariant form. According to the commutation relations (61) and the expression (57), we have

$$\left[\psi^\dagger_{fl}(x), \psi_{fm}(x')\right] = D_{lm}(x - x'), \quad \text{with} \quad l, m = 1, 2, \cdots, 8, \tag{63}$$

where the 8×8 matrix $D(x)$ is given by the following expression

$$D(x) = \frac{\hbar c}{2(2\pi)^3} \int_{k_0 > 0} \mathrm{d}^4 k \delta(k^2) \left[k_0 \mathbf{k} \cdot \mathbf{l} + (\mathbf{k} \cdot \mathbf{l})(\mathbf{k} \cdot \mathbf{l}) \right] e^{-ikx}. \tag{64}$$

The replacement

$$\frac{1}{V} \sum_{\mathbf{k}} \to \frac{1}{(2\pi)^3} \int \mathrm{d}^3 \mathbf{k} \tag{65}$$

was used in obtaining the relation (63). Under Lorentz transformations, $D(x)$ transforms to

$$D'(x') = \exp(-\boldsymbol{\varphi}\cdot\mathbf{l})D(x)\exp(-\boldsymbol{\varphi}\cdot\mathbf{l}). \tag{66}$$

One can verify with no difficulty that by using the expansions (57), the commutation relations (61) can be derived from commutation relation (63). Therefore, the commutation relations (61) and (63) are equivalent.

5. INTERACTION BETWEEN PHOTON FIELD AND MATTER

In the presence of electric current and charge, according to Eqs. (7) and (14), we have the following equation for the photon field:

$$i\hbar\frac{\partial}{\partial x_0}\psi_f(x) = \left(-i\hbar\boldsymbol{\alpha}_w\cdot\nabla - \frac{i}{c}\beta_-\right)\psi_f(x) - i\hbar J_f(x), \tag{67}$$

where the spinor current density $J_f(x)$ is given by

$$J_f(x) = \begin{pmatrix} j_e^+(x) \\ 0 \end{pmatrix}, \tag{68}$$

and $j_e^+(x)$ satisfies the relation

$$j_e^+(x) + j_e^{+*}(x) = j_e(x). \tag{69}$$

It would be natural to request that the spinor current density $J_f(x)$ to have only positive frequency Fourier components. But the spinor current density $j_e^+(x)$ may contain Fourier components with $|\mathbf{k}| > k_0$, and for these Fourier components this separation does hold for all frames of reference. In other words, the positive frequency condition is not covariant. A covariant form of this condition can be written as: $kp > 0$, where p is a well-defined 4-vector. For each physical system, there always is a well-defined 4-vector, namely the total energy-momentum 4-vector of the system. Therefore, we have

$$\int \mathrm{d}^4x\, e^{ikx} J_f(x) = \theta(kp)\int \mathrm{d}^4x\, e^{ikx}\begin{pmatrix} j_e^+(x) \\ 0 \end{pmatrix}, \tag{70}$$

with

$$\theta(z) = \begin{cases} 1 & \text{if } z > 0 \\ 1/2 & \text{if } z = 0 \\ 0 & \text{if } z < 0 \end{cases}, \tag{71}$$

and $p=(p_0,\mathbf{p})$ the total energy-momentum 4-vector of the photon field and the charged matter field under consideration. One may observe that, in the "centre of mass" frame in which the total momentum $\mathbf{p}$ of the whole system in interaction is zero, $J_f(x)$ have only positive frequency Fourier components.

It is easy to verify that the equation (67) can be derived from the following Lagrangian density

$$\mathcal{L}(x)=\mathcal{L}_0(x)+\mathcal{L}_{\text{int}}^f(x), \tag{72}$$

with the Lagrangian density of interaction given by

$$\mathcal{L}_{\text{int}}^f(x)=i\hbar c\left[\psi_f^\dagger(x)\tau_2 J_f(x)-J_f^\dagger(x)\tau_2\psi_f(x)\right]. \tag{73}$$

According to the definition of $J_f(x)$, we have

$$\int_{-\infty}^{\infty}\mathrm{d}x_0\int_V \mathrm{d}^3\mathbf{x}\left[\psi_f^\dagger(x)\tau_2 J_f^*(x)-J_f^T(x)\tau_2\psi_f(x)\right]=0. \tag{74}$$

Therefore it is not necessary to separate $J_f(x)$ from $J_f^*(x)$ in the Lagrangian density of interaction, and the equation (67) can also be derived from the following Lagrangian density

$$\mathcal{L}(x)=\mathcal{L}_0(x)+\mathcal{L}_{\text{int}}(x), \tag{75}$$

where the Lagrangian density of interaction is given by

$$\mathcal{L}_{\text{int}}(x)=i\hbar c\left[\psi_f^\dagger(x)\tau_2 J_s(x)-J_s^\dagger(x)\tau_2\psi_f(x)\right], \tag{76}$$

with

$$J_s(x)=J_f(x)+J_f^*(x). \tag{77}$$

According to the relation between the photon field $\psi_f(x)$ and the four-vector potential $A_f(x)$, the Lagrangian density of interaction $\mathcal{L}_{\text{int}}(x)$ can also be written as

$$\mathcal{L}_{\text{int}}(x)=-A_f^\dagger(x)j(x)-j(x)A_f(x), \tag{78}$$

where $A_f(x)$ is the "positive frequency" part of $A(x)$:

$$\int \mathrm{d}^4x\, e^{ikx}A_f(x)=\theta(kp)\int \mathrm{d}^4x\, e^{ikx}A(x). \tag{79}$$

One may observe that if $j(x)$ commutes with $A_f(x)$ and $A_f^\dagger(x)$, the Lagrangian density of interaction $\mathcal{L}_{\text{int}}(x)$ would become $-j(x)A(x)$, as in the classical electrodynamics.

The equation (67) can be solved. We have

$$\psi_f(x) = \psi_f^0(x) + \int \mathrm{d}^4 x' G_f(x - x') J_f(x'), \tag{80}$$

where $\psi_f^0(x)$ is the free photon field given by expressions (57), and $G_f(x)$ is the Green function for the photon field:

$$G_f(x) = \frac{1}{(2\pi)^4} \int \mathrm{d}^4 k \, \frac{-i \exp(-ikx)}{k^2 + i\varepsilon} \left(k_0 + \boldsymbol{\alpha}_w \cdot \mathbf{k} - \frac{i\beta_-}{\hbar c} \right). \tag{81}$$

6. CONCLUSION

I introduced a spinor field theory for the photon. The spinor equation for the photon field is equivalent to Maxwell's equations together with the relations between the four-vector potential and electric and magnetic fields, and the Lorentz gauge condition for the 4-vector potential. The quantization of free photon field is done, and only transverse photons are obtained. The vacuum energy divergence does not occur in this theory. The solution for the photon field in the presence of the electric current and charge is found, and a covariant "positive frequency" condition is introduced for separating the photon field from its complex conjugate.

APPENDIX A: INVARIANCE OF SPINOR EQUATIONS FOR ELECTROMAGNETIC FIELD AND POTENTIAL UNDER LORENTZ TRANSFORMATIONS

By direct verification, one may find the following relations for matrices $\mathbf{s}$ and $\mathbf{l}$:

$$[s_n, \alpha_{em}] = i \sum_{p=1}^{3} \varepsilon_{nmp} \alpha_{ep}, \quad n, m = 1, 2, 3, \tag{A1}$$

and

$$\alpha_{en} l_m \alpha_{en} = l_m - (1 - \delta_{nm}) \alpha_{em}, \quad n, m = 1, 2, 3. \tag{A2}$$

Let's consider a Lorentz transformation

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \\ x'_0 \end{pmatrix} = \begin{pmatrix} \cosh\varphi & 0 & 0 & -\sinh\varphi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\varphi & 0 & 0 & \cosh\varphi \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_0 \end{pmatrix}. \tag{A3}$$

We have

$$\frac{\partial}{\partial x_1} = \cosh\varphi \frac{\partial}{\partial x'_1} - \sinh\varphi \frac{\partial}{\partial x'_0}, \quad \frac{\partial}{\partial x_0} = \cosh\varphi \frac{\partial}{\partial x'_0} - \sinh\varphi \frac{\partial}{\partial x'_1} \tag{A4}$$

and

$$\begin{aligned} j_1(x) &= \cosh\varphi j'_1(x') + \sinh\varphi c\rho'(x'), \\ c\rho(x) &= \cosh\varphi c\rho'(x') + \sinh\varphi j'_1(x'). \end{aligned} \tag{A5}$$

The last relations can be written in the terms of $j_e(x)$ and $j'_e(x')$:

$$j_e(x) = \exp\left(\varphi\left(l_1 - \alpha_{e1}\right)\right) j'_e(x'). \tag{A6}$$

By using the relations (A4), the equation (7) can then be written as the following:

$$\begin{aligned} &\left(\cosh\varphi - \sinh\varphi\alpha_{e1}\right)\frac{\partial}{\partial x'_0}\psi_{em}(x) + \\ &\left[\left(\cosh\varphi\alpha_{e1} - \sinh\varphi\right)\frac{\partial}{\partial x'_1} + \alpha_{e2}\frac{\partial}{\partial x'_2} + \alpha_{e3}\frac{\partial}{\partial x'_3}\right]\psi_{em}(x) + j_e(x) = 0. \end{aligned} \tag{A7}$$

Because

$$\exp\left(-\varphi\alpha_{e1}\right) = \cosh\varphi - \sinh\varphi\alpha_{e1}, \tag{A8}$$

so

$$\begin{aligned} &\exp\left(-\varphi\alpha_{e1}\right)\frac{\partial}{\partial x'_0}\psi_{em}(x) = \\ &= -\left[\exp\left(-\varphi\alpha_{e1}\right)\alpha_{e1}\frac{\partial}{\partial x'_1} + \alpha_{e2}\frac{\partial}{\partial x'_2} + \alpha_{e3}\frac{\partial}{\partial x'_3}\right]\psi_{em}(x) = \\ &= -\exp\left(\varphi\left(l_1 - \alpha_{e1}\right)\right) j'_e(x'). \end{aligned} \tag{A9}$$

But

$$\alpha_{e1} l_1 = l_1 \alpha_{e1}, \tag{A10}$$

we then have

$$\begin{aligned} \exp\left(-\varphi l_1\right)\frac{\partial}{\partial x'_0}\psi_{em}(x) = -\Bigg[\alpha_{e1}\frac{\partial}{\partial x'_1} + \exp\left(-\varphi\left(l_1 - \alpha_{e1}\right)\right)\times \\ \times\left(\alpha_{e2}\frac{\partial}{\partial x'_2} + \alpha_{e3}\frac{\partial}{\partial x'_3}\right)\exp\left(\varphi l_1\right)\Bigg]\exp\left(-\varphi l_1\right)\psi_{em}(x) - j'_e(x') = 0. \end{aligned} \tag{A11}$$

According to the relation (A2), we have

$$\alpha_{em} l_1^n = \left(l_1 - \alpha_{e1}\right)^n \alpha_{em}, \quad m = 2,3, \tag{A12}$$

therefore

$$\begin{aligned} &\alpha_{em} \exp\left(\varphi l_1\right) = \exp\left(\varphi\left(l_1 - \alpha_{e1}\right)\right)\alpha_{em}, \\ &\exp\left(\varphi l_1\right)\alpha_{em} = \alpha_{em} \exp\left(\varphi\left(l_1 - \alpha_{e1}\right)\right), \quad m = 2,3, \end{aligned} \tag{A13}$$

and the equation (A11) becomes

$$\frac{\partial}{\partial x'_0} \psi'_{em}(x') = -\boldsymbol{\alpha}_e \cdot \nabla \psi'_{em}(x') - j'_e(x'), \tag{A14}$$

with

$$\psi'_{em}(x') = \exp\left(-\varphi l_1\right) \psi_{em}(x). \tag{A15}$$

The equation (A14) in the new reference frame has exactly the same form as Eq. (7), this means the spinor equation for the electromagnetic field is invariant under Lorentz transformations. The invariance of Eq. (14) can be shown in a similar way. We have

$$\begin{aligned} \exp\left(\varphi\alpha_{e1}\right)\frac{\partial}{\partial x'_0}\psi_a(x) = \Bigg[& \alpha_{e1} \exp\left(\varphi\alpha_{e1}\right)\frac{\partial}{\partial x'_1} + \\ & + \alpha_{e2} \frac{\partial}{\partial x'_2} + \alpha_{e3} \frac{\partial}{\partial x'_3} \Bigg] \psi_a(x) - \frac{1}{\hbar c} \exp\left(\varphi l_1\right) \psi'_{em}(x'). \end{aligned} \tag{A16}$$

But

$$\exp\left(-\varphi l_1\right)\alpha_{em} = \alpha_{em} \exp\left(\varphi\left(\alpha_{e1} - l_1\right)\right), \quad m = 2,3, \tag{A17}$$

thus Eq. (A16) can be reduced to

$$\frac{\partial}{\partial x'_0} \psi'_a(x') = \boldsymbol{\alpha}_e \cdot \nabla \psi'_a(x') - \frac{1}{\hbar c} \psi'_{em}(x'), \tag{A18}$$

with

$$\psi'_a(x') = \exp\left(\varphi\left(\alpha_{e1} - l_1\right)\right) \psi_a(x). \tag{A19}$$

This equation has exactly the same form as Eq. (14).

APPENDIX B: INVARIANCE OF SPINOR EQUATIONS FOR ELECTROMAGNETIC FIELD AND POTENTIAL UNDER SPACE ROTATION

Let's consider an infinitesimal space rotation

$$x'_0 = x_0, \quad x'_n = x_n - \sum_{m,p=1}^{3} \varepsilon_{nmp} \delta_m x_p, \quad n = 1,2,3. \tag{B1}$$

We have

$$\frac{\partial}{\partial x_n} = \frac{\partial}{\partial x'_n} + \sum_{m,p=1}^{3} \varepsilon_{nmp} \delta_m \frac{\partial}{\partial x'_p}, \quad n = 1,2,3, \tag{B2}$$

and

$$\rho(x) = \rho'(x'), \quad j_n(x) = j'_n(x') + \sum_{m,p=1}^{3} \varepsilon_{nmp} \delta_m j'_p(x'), \quad n = 1,2,3. \tag{B3}$$

The last relation is equivalent to

$$j_e(x) = \left(1 - i\boldsymbol{\delta} \cdot \mathbf{s}\right) j'_e(x'). \tag{B4}$$

The equation (7) can be written as

$$\begin{aligned} &\left(1 + i\boldsymbol{\delta} \cdot \mathbf{s}\right) \frac{\partial}{\partial x'_0} \psi_{em}(x) = \\ &= -\left(1 + i\boldsymbol{\delta} \cdot \mathbf{s}\right) \left(\alpha_{en} - \sum_{l,m=1}^{3} \varepsilon_{nlm} \delta_l \alpha_{em} \right) \frac{\partial}{\partial x'_n} \psi_{em}(x) - j'_e(x'). \end{aligned} \tag{B5}$$

But

$$\sum_{m=1}^{3} \varepsilon_{nlm} \alpha_{em} = i s_l \alpha_{en} - i \alpha_{en} s_l, \quad l,n = 1,2,3, \tag{B6}$$

so

$$\sum_{l,m=1}^{3} \varepsilon_{nlm} \delta_l \alpha_{em} = i\boldsymbol{\delta} \cdot \mathbf{s} \alpha_{en} - i \alpha_{en} \boldsymbol{\delta} \cdot \mathbf{s}, \quad n = 1,2,3. \tag{B7}$$

Then Eq. (B5) becomes

$$\frac{\partial}{\partial x'_0} \left(1 + i\boldsymbol{\delta} \cdot \mathbf{s}\right) \psi_{em}(x) = -\sum_{n=1}^{3} \alpha_{en} \frac{\partial}{\partial x'_n} \left(1 + i\boldsymbol{\delta} \cdot \mathbf{s}\right) \psi_{em}(x) - j'_e(x'). \tag{B8}$$

On the other hand, we have $\psi'_{em}(x') = \left(1 + i\boldsymbol{\delta}\cdot\mathbf{s}\right)\psi_{em}(x)$, so we rewrite Eq. (B8) as

$$\frac{\partial}{\partial x'_0}\psi'_{em}(x') = -\boldsymbol{\alpha}_e \cdot \nabla\psi'_{em}(x') - j'_e(x'), \tag{B9}$$

which has exactly the same form as Eq. (7). So the spinor equation for the electromagnetic field is invariant under space rotations. The invariance of Eq. (14) can be demonstrated exactly in the same way.

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