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CONDENSED MATTER

EDGE MAGNETOPLASMONS OF A HALF-PLANE

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Abstract. We establish the dispersion relation for the edge magnetoplasmons of a semi-infinite half-plane by solving the integral equations for the oscillation amplitudes of the electrons.

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1. INTRODUCTION

Edge magnetoplasmons (EMP) were experimentally detected in 1983 [1] during the investigation of light absorption in a GaAs-AlGaAs system of disks, and next in 1985 [2] in experiments on electrons trapped on the surface of liquid helium placed in a perpendicular magnetic field. Then it was identified an unusual gapless magnetoplasma mode which had not been previously predicted by the theory, which is characterized by a very small damping in high magnetic field and whose frequency is proportional to the Hall velocity and therefore decreases with the magnetic field. Theoretically, EMP were first studied in [2]–[6]. The authors of Refs.[2, 3, 6] worked within the framework of classical hydrodynamics and used approximate mathematical methods for finding the dispersion relation. The exact dispersion relation for EMP propagating along the boundary of a half-plane was calculated in [4, 5] using a model of sharp-edge for the electron density.

The authors of Ref. [5] solved the integral equation of the self-consistent EMP potential for a half-sheet of conductivity σ_{ij} (which is to be introduced phenomenologically) and found the dispersion relation of the EMP after imposing the finiteness of the electrostatic (internal) potential at the edge. We present in Sec. 2 an other approach for calculating the magnetoplasmon modes of a half-sheet. It is based on the method introduced in Ref. [7], which consists in solving the integral equations of the oscillations amplitudes of the electrons. We neglect in this paper the retardation effects. The end section contains a summary of our results and in the Appendix we give the expressions of the split functions and some hints for their simplification.

2. SOLVING THE ELECTRON'S EQUATION OF MOTION

We consider the half-sheet $x \geq 0$, $y = 0$. We assume that the matter consists of elementary charges (mass m and electrical charge $-e$) moving in a neutralizing rigid uniform background. Denoting by $\mathbf{u}(\mathbf{r},\mathbf{t})$ the displacement field of the elementary charges, the polarization charge density is given by

$$\rho(\mathbf{r},t) = ne\,\mathrm{div}\,\mathbf{u}(\mathbf{r},\mathbf{t}), \tag{1}$$

where n is the equilibrium (two-dimensional) electron density. Note that throughout this paper, n has the dimensionality $1/length^2$. The electrons move according to the Newton's law of motion, under the action of the following forces: the electrostatic polarization force $-e\mathbf{E}$ ($\mathrm{div}\,\mathbf{E}(\mathbf{r},t) = 4\pi\rho(\mathbf{r},t)$), the electrostatic force $+e\,\mathrm{grad}\,\phi_0(\mathbf{r},t)$ given by an external applied potential $\phi_0(\mathbf{r},t)$, a dissipation force proportional to the relaxation constant γ and the force given by an external constant magnetic field parallel to the y axis $\mathbf{H_0} = H_0\mathbf{e}_y$:

$$m\ddot{\mathbf{u}} = -e\mathbf{E} + e\,\mathrm{grad}\,\phi_0 - m\gamma\dot{\mathbf{u}} - \frac{e}{c}\dot{\mathbf{u}} \times \mathbf{H_0}, \tag{2}$$

where dot means time derivative. We use the notation $\omega_c \equiv \frac{eH_0}{mc}$. Equations (1), (2) are all the hypotheses we make about the electrons motion. From these equations we are able to extract the magnetoplasmons dispersion relation, without imposing any supplementary boundary condition at the edge $x = 0$.

Before starting the calculation itself, we note that, if we solve Eq.(2) for an infinite sheet, we obtain the magnetoplasmons resonance $\Omega^4 - \omega^2\omega_c^2 - \frac{k}{2}\Omega^2\omega_p^2 = 0$, where $\Omega^2 = \omega^2 + i\gamma\omega$, $k = |\vec{k}|$ and $\vec{k}$ is the wave vector; also, if we calculate the conductivity tensor of the infinite sheet using the formula: $j_\alpha(x,z) = \sigma_{\alpha\beta}E_\beta^{tot}(x,z)$, where $\alpha, \beta \in \{x,z\}$, $\mathbf{j}(x,z) = -ne\dot{\mathbf{u}}(x,z)$ is the polarization current and $\mathbf{E}^{tot}(x,z)$ is the total (polarization + external) electric field inside the sheet, we obtain

$$\sigma_{xx} = \sigma_{yy} = \frac{i\omega ne^2\Omega^2}{m(\Omega^4 - \omega^2\omega_c^2)},\ \sigma_{xz} = -\sigma_{zx} = -\frac{\omega^2 ne^2\omega_c}{m(\Omega^4 - \omega^2\omega_c^2)}. \tag{3}$$

Now we start solving Eq.(2) for the half-sheet. We put $\mathbf{u}(x,z;t) = \mathbf{u}(x,z;t)\theta(x)$, take the Fourier transforms $\mathbf{u}(x,z;t) = \int dt e^{-i\omega t}\sum_{k_z} e^{ik_z z}\mathbf{u}(x,k_z;\omega)$, $\phi_0(x,z;t) = \int dt e^{-i\omega t}\sum_{k_z} e^{ik_z z}\phi_0(x,k_z;\omega)$ and use the well-known expansion:

$$\frac{1}{\sqrt{(\mathbf{r}-\mathbf{r}')^2}} = \sum_{k_x,k_z} \frac{2\pi}{\sqrt{k_x^2+k_z^2}} e^{-\sqrt{k_x^2+k_z^2}|y-y'|} e^{ik_x(x-x')} e^{ik_z(z-z')}. \tag{4}$$

We take external applied scalar potential $\phi_0(x,k_z) = \phi_0(k_z)e^{i\kappa_0 x}$, where κ_0 has a small positive imaginary part. In explicit component form, Eq.(2) reads:

$$\begin{cases} u_x(x,k_z) & = -\frac{2ne^2}{m\Omega^2}\partial_x^2\int_0^\infty dx' K_0(k_z|x-x'|)u_x(x',k_z) - \\ & \frac{2ne^2 ik_z}{m\Omega^2}\partial_x\int_0^\infty dx' K_0(k_z|x-x'|)u_z(x',k_z) - \frac{i\kappa_0 e}{m\Omega^2}\phi_0(k_z)e^{i\kappa_0 x} + \\ & \frac{i\omega\omega_c}{\Omega^2}u_z(x,k_z) \\ u_z(x,k_z) & = -\frac{2ne^2 ik_z}{m\Omega^2}\partial_x\int_0^\infty dx' K_0(k_z|x-x'|)u_x(x',k_z) + \\ & \frac{2ne^2 k_z^2}{m\Omega^2}\partial_x\int_0^\infty dx' K_0(k_z|x-x'|)u_z(x',k_z) - \frac{ik_z e}{m\Omega^2}\phi_0(k_z)e^{i\kappa_0 x} - \\ & \frac{i\omega\omega_c}{\Omega^2}u_x(x,k_z), \\ & \qquad\qquad x\in(0,\infty), \end{cases} \tag{5}$$

where $K_0(x)$ is the modified Bessel function of the second kind (MacDonald function) [10]. We solve the system Eqs. (5) by using the Wiener-Hopf technique [9]. We define the functions:

$$U_x(x,k_z) = \begin{cases} u_x(x,k_z), & x>0 \\ 0, & x<0 \end{cases}, \quad U_z(x,k_z) = \begin{cases} u_z(x,k_z), & x>0 \\ 0, & x<0 \end{cases},$$

$$\phi_x(x,k_z) = \begin{cases} 0, & x>0 \\ -\frac{2ne^2}{m\Omega^2}\partial_x^2\int_0^\infty dx' K_0(x-x';k_z)u_x(x',k_z) - & \\ \frac{2ne^2 ik_z}{m\Omega^2}\partial_x\int_0^\infty dx' K_0(x-x';k_z)u_z(x',k_z), & x<0 \end{cases},$$

$$\phi_z(x,k_z) = \begin{cases} 0, & x>0 \\ -\frac{2ne^2 ik_z}{m\Omega^2}\partial_x\int_0^\infty dx' K_0(x-x';k_z)u_x(x',k_z) + & \\ \frac{2ne^2 k_z^2}{m\Omega^2}\int_0^\infty dx' K_0(x-x';k_z)u_z(x',k_z), & x<0 \end{cases},$$

$$\psi_x(x,k_z) = \begin{cases} -\frac{i\kappa_0 e}{m\Omega^2}\phi_0(k_z)e^{i\kappa_0 x}, & x>0 \\ 0, & x<0 \end{cases}, \psi_z(x,k_z) = \begin{cases} -\frac{ik_z e}{m\Omega^2}\phi_0(k_z)e^{i\kappa_0 x}, & x>0 \\ 0, & x<0 \end{cases}.$$

Then, the system Eq.(5) reads:

$$\begin{cases} U_x(x,k_z) + \phi_x(x,k_z) = -\frac{\omega_p^2}{2\pi\Omega^2}\partial_x^2\int_{-\infty}^\infty dx' K_0(k_z|x-x'|)U_x(x',k_z) - \\ \qquad \frac{\omega_p^2}{2\pi\Omega^2}ik_z\partial_x\int_{-\infty}^\infty dx' K_0(k_z|x-x'|)U_z(x',k_z) + \\ \qquad \frac{i\omega\omega_c}{\Omega^2}U_z(x,k_z) + \psi_x(x,k_z) \\ U_z(x,k_z) + \phi_z(x,k_z) = -\frac{\omega_p^2}{2\pi\Omega^2}ik_z\partial_x\int_{-\infty}^\infty dx' K_0(k_z|x-x'|)U_x(x',k_z) - \\ \qquad \frac{\omega_p^2}{2\pi\Omega^2}(ik_z)^2\int_{-\infty}^\infty dx' K(k_z|x-x'|)U_z(x',k_z) \\ \qquad -\frac{1\omega\omega_c}{\Omega^2}U_x(x,k_z) + \psi_z(x,k_z), \\ \qquad\qquad x\in(-\infty,\infty), \end{cases} \tag{6}$$

where $\omega_p^2 = 4\pi ne^2/m$ is the (two-dimensional) plasma frequency. Note that throughout this paper ω_p^2 has the dimensionality $\frac{\text{length}}{\text{time}^2}$. Taking the Fourier transform

$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}dxe^{i\alpha x}$, we obtain:

$$\begin{cases}\phi_x^-(\alpha,k_z) = & \left[\frac{\omega_p^2}{2\Omega^2}\frac{\alpha^2}{\sqrt{\alpha^2+k_z^2}}-1\right]U_x^+(\alpha,k_z)+ \\ & \left[-\frac{\omega_p^2}{2\Omega^2}\frac{k_z\alpha}{\sqrt{\alpha^2+k_z^2}}+\frac{i\omega\omega_c}{\Omega^2}\right]U_z^+(\alpha,k_z)+\frac{\kappa_0 e}{\sqrt{2\pi}m\Omega^2}\frac{\phi_0(k_z)}{\alpha+\kappa_0}, \\ \phi_z^-(\alpha,k_z) = & \left[-\frac{\omega_p^2}{2\Omega^2}\frac{k_z\alpha}{\sqrt{\alpha^2+k_z^2}}-\frac{i\omega\omega_c}{\Omega^2}\right]U_x^+(\alpha,k_z)+ \\ & \left[\frac{\omega_p^2}{2\omega^2}\frac{k_z^2}{\sqrt{\alpha^2+k_z^2}}-1\right]U_z^+(\alpha,k_z)+\frac{k_z e}{\sqrt{2\pi}m\Omega^2}\frac{\phi_0(k_z)}{\alpha+\kappa_0},\end{cases} \tag{7}$$

where the superscripts $\pm$ indicate an upper/lower half-plane of convergence. We work in the proper sheet of the square root $\sqrt{\alpha^2+k_z^2}$ ([11]) (its phase for $\alpha\to\infty$ along the real axis is zero, and the branch cuts are taken from $\pm ik_z$ to $\pm\infty$ respectively, along the imaginary axis). One can easily see that the strip of analyticity for the system Eq.(7) is $k_z-\epsilon<\operatorname{Im}\alpha<k_z$, where ϵ is some positive quantity.

We write the system Eqs.(7) in the form:

$$\begin{cases}\alpha\phi_x^-(\alpha,k_z)-k_z\phi_z^-(\alpha,k_z)=\left[\frac{\omega_p^2}{2\Omega^2}\sqrt{\alpha^2+k_z^2}-1\right]\cdot \\ \qquad \Big(\alpha U_x^+(\alpha,k_z)-k_zU_z^+(\alpha,k_z)\Big)+ \\ \qquad \frac{i\omega\omega_c}{\Omega^2}\Big(k_zU_x^+(\alpha,k_z)+\alpha U_z^+(\alpha,k_z)\Big)+\frac{e\phi_0(k_z)}{\sqrt{2\pi}m\Omega^2}\frac{\kappa_0\alpha-k_z^2}{\alpha+\kappa_0}, \\ k_z\phi_x^-(\alpha,k_z)+\alpha\phi_z^-(\alpha,k_z)=-\frac{i\omega\omega_c}{\Omega^2}\Big(\alpha U_x^+(\alpha,k_z)-k_zU_z^+(\alpha,k_z)\Big)- \\ \qquad \Big(k_zU_x^+(\alpha,k_z)+\alpha U_z^+(\alpha,k_z)\Big)+\frac{k_ze\phi_0(k_z)}{\sqrt{2\pi}m\Omega^2}.\end{cases} \tag{8}$$

We eliminate $k_zU_x^+(\alpha,k_z)+\alpha U_z^+(\alpha,k_z)$ from the first of Eqs.(8) and obtain the following equivalent system:

$$\begin{cases}\Big(\alpha+\frac{i\omega\omega_c}{\Omega^2}k_z\Big)\phi_x^-(\alpha,k_z)+(-k_z+\frac{i\omega\omega_c}{\Omega^2}\alpha\Big)\phi_z^-(\alpha,k_z)= \\ \left[\frac{\omega_p^2}{2\Omega^2}\sqrt{\alpha^2+k_z^2}+\frac{\omega^2\omega_c^2}{\Omega^4}-1\right]\Big(\alpha U_x^+(\alpha,k_z)-k_zU_z^+(\alpha,k_z)\Big)+ \\ \frac{e\phi_0(k_z)}{\sqrt{2\pi}m\Omega^2}\Big(\frac{\kappa_0\alpha-k_z^2}{\alpha+\kappa_0}+\frac{i\omega\omega_c}{\Omega^2}k_z\Big), \\ k_z\phi_x^-(\alpha,k_z)+\alpha\phi_z^-(\alpha,k_z)=-\frac{i\omega\omega_c}{\Omega^2}\Big(\alpha U_x^+(\alpha,k_z)-k_zU_z^+(\alpha,k_z)\Big)- \\ \Big(k_zU_x^+(\alpha,k_z)+\alpha U_z^+(\alpha,k_z)\Big)+\frac{k_ze\phi_0(k_z)}{\sqrt{2\pi}m\Omega^2}.\end{cases} \tag{9}$$

While in the second equation of the system (9), the "+" functions are separated from the "−" functions, in the first equation we have to factorize the kernel $\left[\frac{\omega_p^2}{2\Omega^2}\sqrt{\alpha^2+k_z^2}+\frac{\omega^2\omega_c^2}{\Omega^4}-1\right]$.

Before continue our calculation, we introduce the notation $l\equiv-\frac{1}{2}\frac{\Omega^2\omega_p^2}{\Omega^4-\omega^2\omega_c^2}$ and we first consider the case $k_z>0$. We note that l has dimension of length. Our l coincides with l from Ref.[5], Eq.(10) (which was introduced phenomenologically as being the width of the strip near the edge in which the charge accumulate) if we put $\tilde{\kappa}=1$ and $\sigma_{xx}=\frac{i\omega ne^2\Omega^2}{m(\Omega^4-\omega^2\omega_c^2)}$ from our Eq.(3). We write :

$$\Big[\frac{\omega_p^2}{2\Omega^2}\sqrt{\alpha^2+k_z^2}+\frac{\omega^2\omega_c^2}{\Omega^4}-1\Big]=\frac{i}{k_z}\Big(1-\frac{\omega^2\omega_c^2}{\Omega^4}\Big)\sqrt{(ik_z-\alpha)(ik_z+\alpha)}\cdot K(\alpha),$$

where

$$K(\alpha)=\Big(-\frac{k_z}{2}\frac{\Omega^2\omega_p^2}{\Omega^4-\omega^2\omega_c^2}+\frac{k_z}{\sqrt{k_z^2+\alpha^2}}\Big)$$

and then use the factorization formulas from Appendix.

1) Case $\Omega^2<\omega\omega_c$**,** $k_z>0$

After we factorize the kernel according to the formulas from Appendix and separate the "+" functions from the "-" functions in the first equation of the system (9), we find the following equivalent system:

$$\begin{cases} \frac{k_zK_-(\alpha)}{i\sqrt{ik_z-\alpha}}\Bigg[\Big(\alpha+\frac{i\omega\omega_c}{\Omega^2}k_z\Big)\phi_x^-(\alpha,k_z)+\Big(-k_z+\frac{i\omega\omega_c}{\Omega^2}\alpha\Big)\phi_z^-(\alpha,k_z)\Bigg]- \\ \frac{e\phi_0(k_z)\omega\omega_ck_z^2}{\sqrt{2\pi}m\Omega^4}\frac{K_-(\alpha)}{\sqrt{ik_z-\alpha}}- \\ \frac{ek_z\phi_0(k_z)}{i\sqrt{2\pi}m\Omega^2}\frac{1}{\alpha+\kappa_0}\Bigg[\frac{\kappa_0\alpha-k_z^2}{\sqrt{ik_z-\alpha}}K_-(\alpha)+(\kappa_0-ik_z)\sqrt{\kappa_0-ik_z}K_-(-\kappa_0)\Bigg]= \\ \Big(1-\frac{\omega^2\omega_c^2}{\Omega^4}\Big)\sqrt{ik_z+\alpha}K_+(\alpha)\Big[\alpha U_x^+(\alpha,k_z)-k_zU_z^+(\alpha,k_z)\Big]- \\ \frac{ek_z\phi_0(k_z)}{i\sqrt{2\pi}m\Omega^2}\frac{(\kappa_0-ik_z)\sqrt{\kappa_0+ik_z}}{\alpha+\kappa_0}K_-(-\kappa_0), \\ k_z\phi_x^-(\alpha,k_z)+\alpha\phi_z^-(\alpha,k_z)=-\frac{i\omega\omega_c}{\Omega^2}\Big(\alpha U_x^+(\alpha,k_z)-k_zU_z^+(\alpha,k_z)\Big)- \\ \Big(k_zU_x^+(\alpha,k_z)+\alpha U_z^+(\alpha,k_z)\Big)+\frac{k_ze\phi_0(k_z)}{\sqrt{2\pi}m\Omega^2}. \end{cases} \tag{10}$$

After we study the asymptotic behaviour of the equations (10) (using the asymptotic behaviour of the functions $K_+(\alpha)$ and $K_-(\alpha)$ from the Appendix) and apply the generalized Liouville theorem, we find that each equation of the above system must be equal to a constant in the whole complex plane. Writing:

$$\begin{cases} \Big(1-\frac{\omega^2\omega_c^2}{\Omega^4}\Big)\sqrt{ik_z+\alpha}K_+(\alpha)\Big[\alpha U_x^+(\alpha,k_z)-k_zU_z^+(\alpha,k_z)\Big]- \\ \frac{ek_z\phi_0(k_z)}{i\sqrt{2\pi}m\Omega^2}\frac{(\kappa_0-ik_z)\sqrt{\kappa_0+ik_z}}{\alpha+\kappa_0}K_-(-\kappa_0)=A, \\ +\frac{i\omega\omega_c}{\Omega^2}\Big(\alpha U_x^+(\alpha,k_z)-k_zU_z^+(\alpha,k_z)\Big)+\Big(k_zU_x^+(\alpha,k_z)+\alpha U_z^+(\alpha,k_z)\Big)- \\ \frac{k_ze\phi_0(k_z)}{\sqrt{2\pi}m\Omega^2}=B, \end{cases} \tag{11}$$

we find $U_x^+(\alpha,k_z)$ and $U_z^+(\alpha,k_z)$ in terms of two unknown constants A, B:

$$\begin{cases} (\alpha^2+k_z^2)U_x^+(\alpha,k_z) = \frac{\Omega^2\alpha - i\omega\omega_c k_z}{(\Omega^4-\omega^2\omega_c^2)\sqrt{ik_z+\alpha}K_+(\alpha)}\cdot \\ \quad \Big[-\frac{iek_z\phi_0(k_z)}{\sqrt{2\pi}m}\frac{(\kappa_0-ik_z)\sqrt{\kappa_0+ik_z}K_-(-\kappa_0)}{\alpha+\kappa_0}+A\Omega^2\Big]+\frac{k_z^2 e\phi_0(k_z)}{\sqrt{2\pi}m\Omega^2}+Bk_z, \\ (\alpha^2+k_z^2)U_z^+(\alpha,k_z) = \frac{-\Omega^2 k_z - i\omega\omega_c\alpha}{(\Omega^4-\omega^2\omega_c^2)\sqrt{ik_z+\alpha}K_+(\alpha)}\cdot \\ \quad \Big[-\frac{iek_z\phi_0(k_z)}{\sqrt{2\pi}m}\frac{(\kappa_0-ik_z)\sqrt{\kappa_0+ik_z}K_-(-\kappa_0)}{\alpha+\kappa_0}+A\Omega^2\Big]+\frac{k_z e\phi_0(k_z)}{\sqrt{2\pi}m\Omega^2}\alpha+B\alpha. \end{cases} \tag{12}$$

As $U_{x,z}^+(\alpha,k_z)$ have to be analytic for $\mathrm{Im}\,\alpha > k_z-\epsilon$, we impose that the r.h.s. of Eqs.(12) should be zero for $\alpha = +ik_z$. Thus we obtain the following connection between the constants A, B:

$$\frac{\sqrt{ik_z}}{\sqrt{2}(\Omega^2+\omega\omega_c)K_+(ik_z)}\Big[-\frac{iek_z\phi_0(k_z)}{\sqrt{2\pi}m}\frac{(\kappa_0-ik_z)K_-(-\kappa_0)}{\sqrt{\kappa_0+ik_z}}+A\Omega^2\Big]+ \frac{k_z^2 e\phi_0(k_z)}{\sqrt{2\pi}m\Omega^2}+Bk_z=0. \tag{13}$$

Returning to the system (10) and using Eq.(13), we obtain the functions $\Phi_{x,z}^-(\alpha)$ in terms of the constant A:

$$\begin{cases} (\alpha^2+k_z^2)\Phi_x^-(\alpha,k_z) = -\frac{\alpha\sqrt{\alpha-ik_z}}{k_z\Omega^2 K_-(\alpha)}\cdot \\ \quad \Big[-\frac{iek_z\phi_0(k_z)}{\sqrt{2\pi}m}\frac{(\kappa_0-ik_z)\sqrt{\kappa_0+ik_z}K_-(-\kappa_0)}{\sqrt{\alpha+\kappa_0}}+A\Omega^2\Big]+ \\ \quad \Big(k_z-\frac{i\omega\omega_c}{\Omega^2}\alpha\Big)\frac{\sqrt{ik_z}}{\sqrt{2}k_z(\Omega^2+\omega\omega_c)K_+(ik_z)}\cdot \\ \quad \Big[-\frac{iek_z\phi_0(k_z)}{\sqrt{2\pi}m}\frac{(\kappa_0-ik_z)K_-(-\kappa_0)}{\sqrt{\kappa_0+ik_z}}+A\Omega^2\Big]+\frac{e\phi_0(k_z)\kappa_0}{\sqrt{2\pi}m\Omega^2}\frac{\alpha^2+k_z^2}{\alpha+\kappa_0}, \\ (\alpha^2+k_z^2)\Phi_z^-(\alpha,k_z) = \frac{\sqrt{\alpha-ik_z}}{\Omega^2 K_-(\alpha)}\cdot \\ \quad \Big[-\frac{iek_z\phi_0(k_z)}{\sqrt{2\pi}m}\frac{(\kappa_0-ik_z)\sqrt{\kappa_0+ik_z}K_-(-\kappa_0)}{\sqrt{\alpha+\kappa_0}}+A\Omega^2\Big]+ \\ \quad \Big(\alpha+\frac{i\omega\omega_c}{\Omega^2}k_z\Big)\frac{\sqrt{ik_z}}{\sqrt{2}k_z(\Omega^2+\omega\omega_c)K_+(ik_z)}\cdot \\ \quad \Big[-\frac{iek_z\phi_0(k_z)}{\sqrt{2\pi}m}\frac{(\kappa_0-ik_z)K_-(-\kappa_0)}{\sqrt{\kappa_0+ik_z}}+A\Omega^2\Big]+\frac{e\phi_0(k_z)k_z}{\sqrt{2\pi}m\Omega^2}\frac{\alpha^2+k_z^2}{\alpha+\kappa_0}. \end{cases} \tag{14}$$

As the functions $\Phi_{x,z}(\alpha,k_z)$ have to be analytic for $\mathrm{Im}\,\alpha < k_z$, we impose that the r.h.s. of Eqs.(14) should be zero for $\alpha = -ik_z$. We obtain:

$$A = \frac{iek_z\phi_0(k_z)}{\sqrt{2\pi}m\Omega^2}\frac{K_-(-\kappa_0)}{\sqrt{\kappa_0+ik_z}}\cdot \frac{2(\kappa_0+ik_z)(\Omega^2+\omega\omega_c)K_+(ik_z)+(\kappa_0-ik_z)(\Omega^2-\omega\omega_c)K_-(-ik_z)}{2(\Omega^2+\omega\omega_c)K_+(ik_z)+(\Omega^2-\omega\omega_c)K_-(-ik_z)} \tag{15}$$

and, after taking into account Eq.(13),

$$B = \frac{2k_z\sqrt{ik_z}e\phi_0(k_z)}{\sqrt{\pi}m}\frac{K_-(-\kappa_0)}{\sqrt{\kappa_0+ik_z}}\cdot \frac{1}{2(\Omega^2+\omega\omega_c)K_+(ik_z)+(\Omega^2-\omega\omega_c)K_-(-ik_z)} - \frac{k_z e\phi_0(k_z)}{\sqrt{2\pi}m\Omega^2}. \tag{16}$$

Having determined the constants A and B, the problem is completely solved. We note that A, B have resonances for :

$$\frac{\Omega^2-\omega\omega_c}{\Omega^2+\omega\omega_c} = -2\frac{K_+(ik_z)}{K_-(-ik_z)}, \tag{17}$$

which is the dispersion relation of the EMP.

2) Case $\Omega^2 > \omega\omega_c$, $k_z > 0$

In this case, the kernel of the first of Eqs.(9) has two zeros $\alpha_{1,2} = \pm ip$, where $p = \frac{i}{l}\sqrt{1-k_z^2l^2}$ and we have to use the second factorization formula from the Appendix. Because we always have $|\operatorname{Im}\alpha_{1,2}| < k_z$, we can solve the system (9) in the narrower strip $Im(ip) < \operatorname{Im}(\alpha) < k_z$. For this, we have to properly choose the "+" and the "−" functions for the factorization of the kernel. We take: $\left[\frac{\omega_p^2}{2\Omega^2}\sqrt{\alpha^2+k_z^2}+\frac{\omega^2\omega_c^2}{\Omega^4}-1\right] = \frac{f_+(\alpha)}{f_-(\alpha)}$, where $f_+(\alpha) = \left(1-\frac{\omega^2\omega_c^2}{\Omega^4}\right)\sqrt{ik_z+\alpha}\frac{l^2}{k_zl-1}(\alpha-ip)(\alpha+ip)\sigma_+(\alpha)$, $f_-(\alpha) = \frac{k_z}{i\sqrt{ik_z-\alpha}}\sigma_-(\alpha)$, and $\sigma_\pm(\alpha)$ are those from Appendix. $f_+(\alpha)$ is analytic and free of zeros for $Im(\alpha) > \operatorname{Im}(ip)$ and $f_-(\alpha)$ is analytic and free of zeros for $\operatorname{Im}(\alpha) < k_z$. After separating the "−" functions from the "+" functions, the system (9) reads:

$$\begin{cases} \frac{k_z\sigma_-(\alpha)}{i\sqrt{ik_z-\alpha}}\Big[\Big(\alpha+\frac{i\omega\omega_c k_z}{\Omega^2}k_z\Big)\phi_x^-(\alpha,k_z)+\Big(-k_z+\frac{i\omega\omega_c}{\Omega^2}\alpha\Big)\phi_z^-(\alpha,k_z)\Big] - \\ \frac{e\phi_0(k_z)\omega\omega_c k_z^2}{\sqrt{2\pi}m\Omega^4}\frac{\sigma_-(\alpha)}{\sqrt{ik_z-\alpha}} - \frac{k_z e\phi_0(k_z)}{i\sqrt{2\pi}m\Omega^2}\frac{1}{\alpha+\kappa_0}\cdot \\ \Big[(\kappa_0\alpha-k_z^2)\frac{\sigma_-(\alpha)}{\sqrt{ik_z-\alpha}}+(\kappa_0-ik_z)\sqrt{\kappa_0+ik_z}\sigma_-(-\kappa_0)\Big] = \\ \big(1-\frac{\omega^2\omega_c^2}{\Omega^4}\big)\sqrt{ik_z+\alpha}\frac{l^2(\alpha^2+p^2)}{k_zl-1}\sigma_+(\alpha)\Big(\alpha U_x^+(\alpha,k_z)-k_zU_z^+(\alpha,k_z)\Big) \\ +\frac{ik_z e\phi_0(k_z)}{\sqrt{2\pi}m\Omega^2}\frac{1}{\alpha+\kappa_0}(\kappa_0-ik_z)\sqrt{\kappa_0+ik_z}\sigma_-(-\kappa_0), \\ k_z\phi_x^-(\alpha,k_z)+\alpha\Phi_z^-(\alpha,k_z) = -\frac{i\omega\omega_c}{\Omega^2}\Big(\alpha U_x^+(\alpha,k_z)-k_zU_z^+(\alpha,k_z)\Big) - \\ \Big(k_zU_x^+(\alpha,k_z)+\alpha U_z^+(\alpha,k_z)\Big)+\frac{k_z e\phi_0(k_z)}{\sqrt{2\pi}m\Omega^2}. \end{cases} \tag{18}$$

We apply the generalized Liouville's theorem (using the asymptotic behaviour of the functions $\sigma_\pm(\alpha)$ from the Appendix) and find that the first equation of the system (18) can be at most a polynomial of the first degree in α and the second equation can be at most a constant in the whole complex plane. Thus we have:

$$\begin{cases} \left(1-\frac{\omega^2\omega_c^2}{\Omega^4}\right)\sqrt{ik_z+\alpha}\frac{l^2(\alpha^2+p^2)}{k_zl-1}\sigma_+(\alpha)\left(\alpha U_x^+(\alpha)-k_zU_z^+(\alpha)\right)+ \\ \frac{ik_ze\phi_0(k_z)}{\sqrt{2\pi}m\Omega^2}\frac{1}{\alpha+\kappa_0}(\kappa_0-ik_z)\sqrt{\kappa_0+ik_z}\sigma_-(-\kappa_0)=A\alpha+B, \\ \frac{k_ze\phi_0(k_z)}{\sqrt{2\pi}m\Omega^2}-\frac{i\omega\omega_c}{\Omega^2}\left(\alpha U_x^+(\alpha)-k_zU_z^+(\alpha)\right)-\left(k_zU_x^+(\alpha)+\alpha U_z^+(\alpha)\right)=C, \end{cases} \tag{19}$$

where A, B and C are constants to be determined. From Eqs.(19), we find $U_x^+(\alpha)$ and $U_z^+(\alpha)$ in terms of A, B, C:

$$\begin{cases} (\alpha^2+k_z^2)U_x^+(\alpha)=\frac{1}{\Omega^4-\omega^2\omega_c^2}\frac{k_zl-1}{l^2}\frac{\Omega^2\alpha-i\omega\omega_ck_z}{\sqrt{ik_z+\alpha}(\alpha^2+p^2)\sigma_+(\alpha)}\cdot \\ \quad\left[-\frac{ik_ze\phi_0(k_z)}{\sqrt{2\pi}m}\frac{(\kappa_0-ik_z)\sqrt{\kappa_0+ik_z}\sigma_-(-\kappa_0)}{\alpha+\kappa_0}+(A\alpha+B)\Omega^2\right]+\frac{k_z^2e\phi_0(k_z)}{\sqrt{2\pi}m\Omega^2}-Ck_z, \\ (\alpha^2+k_z^2)U_z^+(\alpha)=\frac{1}{\Omega^4-\omega^2\omega_c^2}\frac{k_zl-1}{l^2}\frac{-\Omega^2\alpha k_z-i\omega\omega_c\alpha}{\sqrt{ik_z+\alpha}(\alpha^2+p^2)\sigma_+(\alpha)}\cdot \\ \quad\left[-\frac{ik_ze\phi_0(k_z)}{\sqrt{2\pi}m}\frac{(\kappa_0-ik_z)\sqrt{\kappa_0+ik_z}\sigma_-(-\kappa_0)}{\alpha+\kappa_0}+(A\alpha+B)\Omega^2\right]+\frac{k_ze\phi_0(k_z)\alpha}{\sqrt{2\pi}m\Omega^2}-C\alpha. \end{cases} \tag{20}$$

As $U_{x,z}^+(\alpha)$ have to be analytic for $\mathrm{Im}(\alpha)>\mathrm{Im}(ip)$, the right-hand sides of Eqs.(20) have to cancel for $\alpha=ik_z$. Thus we obtain the following connection between the constants A, B, C:

$$\begin{aligned} C=&\frac{i(k_zl-1)}{(\Omega^2+\omega\omega_c)l^2\sqrt{2ik_z}(-k_z^2+p^2)\sigma_+(ik_z)}\cdot \\ &\left[-\frac{ik_ze\phi_0(k_z)}{\sqrt{2\pi}m}\frac{(\kappa_0-ik_z)\sigma_-(-\kappa_0)}{\sqrt{\kappa_0+ik_z}}+(Aik_z+B)\Omega^2\right]+\frac{k_ze\phi_0(k_z)}{\sqrt{2\pi}m\Omega^2}. \end{aligned} \tag{21}$$

Similarly, from Eqs.(18), (19) we obtain the functions $\phi_{x,z}^-(\alpha)$ in terms of A, B, C. Imposing their analyticity at $\alpha=-ik_z$, we find a second connection between the unknown constants:

$$\begin{aligned} &Aik_z[(k_zl-1)(\Omega^2-\omega\omega_c)\sigma_-(-ik_z)+2(\Omega^2+\omega\omega_c)\sigma_+(ik_z)]+ \\ &\quad B[(k_zl-1)(\Omega^2-\omega\omega_c)\sigma_-(-ik_z)-2(\Omega^2+\omega\omega_c)\sigma_+(ik_z)]= \\ &\frac{ik_ze\phi_0(k_z)}{\sqrt{2\pi}m}\frac{\sigma_-(-\kappa_0)}{\Omega^2\sqrt{\kappa_0+ik_z}}[-2(\kappa_0+ik_z)(\Omega^2+\omega\omega_c)\sigma_+(ik_z)+ \\ &\qquad\qquad (k_zl-1)(\kappa_0-ik_z)(\Omega^2-\omega\omega_c)\sigma_-(-ik_z)]. \end{aligned} \tag{22}$$

From Eqs.(20), (21), (22) we have the functions $U_{x,z}^+(\alpha)$ expressed in terms of one arbitrary constant. After inverting the Fourier transforms, we see that the parts which correspond to the pole $\alpha=+ip$ are non-physical solutions (which grow exponentially to infinity, or which propagate backwards from infinity). In order to eliminate the non-physical solutions, we impose the analyticity of $U_{x,z}^+(\alpha)$ at $\alpha=+ip$. We thus obtain the third connection between the constants A, B and C:

$$Aip+B=\frac{ik_ze\phi_0(k_z)}{\sqrt{2\pi}m\Omega^2}\frac{(\kappa_0-ik_z)\sqrt{\kappa_0+ik_z}\sigma_-(-\kappa_0)}{\kappa_0-ip}. \tag{23}$$

From Eqs.(21), (22), (23) we obtain A, B, C and we note that they have resonances for

$$\frac{\Omega^2-\omega\omega_c}{\Omega^2+\omega\omega_c}=\frac{2}{1-k_zl}(k_zl-\sqrt{k_z^2l^2-1})^2\frac{\sigma_+(ik_z)}{\sigma_-(-ik_z)}, \tag{24}$$

which is the EMP resonance for $\Omega^2-\omega\omega_c>0$. We note that for $k_zl\in(-1,0)$, p is purely imaginary and Eq.(24) does not have real solutions.

Up until now we have worked with $k_z>0$. For $k_z<0$ the calculation follows in the same way, except when we impose the analyticity conditions at $\pm ik_z$ (because now ik_z is in the lower half plane, and $-ik_z$ is in the upper half plane). We obtain:

$$\frac{\Omega^2-\omega\omega_c}{\Omega^2+\omega\omega_c}=-\frac{K_-(ik_z)}{2K_+(-ik_z)},\quad if\quad \Omega^2<\omega\omega_c,\, k_z<0 \tag{25}$$

and

$$\frac{\Omega^2-\omega\omega_c}{\Omega^2+\omega\omega_c}=\frac{(-k_zl+\sqrt{k_z^2l^2-1})^2K_-(ik_z)}{2K_+(-ik_z)},\quad if\quad \Omega^2>\omega\omega_c,\, k_z<0. \tag{26}$$

3. CONCLUSIONS

We have solved here the problem of plasmon modes for a half-sheet placed in an uniform external magnetic field $\mathbf{H_0}=H_0\mathbf{e}_y$ and an external applied electrostatic potential $\phi_0(\vec{r})$. Using the method indicated in the Appendix, we can write our results Eqs.(17), (24), (25), (26) in the condensed form:

$$\frac{\Omega^2-\omega\omega_c}{\Omega^2+\omega\omega_c}=-e^{\pm 2Y}, \tag{27}$$

where the superscript "+" is for $k_z>0$ and the superscript "−" for $k_z<0$ and

$$Y(|k_z|l)\equiv\frac{1}{\pi}\int_0^{\frac{\pi}{2}}dt\,\ln\left(\frac{|k_z|l+\sin t}{\sin t}\right). \tag{28}$$

Eq. (27) can be written equivalently:

$$\omega=-2k_z\sigma_{xz}F_1(|k_z|l), \tag{29}$$

where $F_1(|k_z|l)=\frac{\pi}{|k_z|l}\tanh Y(|k_z|l)$ is the function introduced in Eq.(40) of Ref.[5] and σ_{xz} is that from our Eq.(3). This result is formally identical to that from Eq.(40) of Ref.[5] if we put $\tilde{\kappa}=1$. For small dissipation $\gamma\ll\omega$, Eq.(27) becomes:

$$\omega=-\omega_c\frac{k_zl}{\pi}\ln\frac{2}{|k_z|l},\;\; l=-\frac{1}{2}\frac{\omega_p^2}{\omega^2-\omega_c^2}. \tag{30}$$

If we take $\phi_0(\vec{r})=0$, the equations for determining the constants A, B, C become homogeneous, and the dispersion relation follows from the compatibility condition of the homogeneous algebraic system. In this case, the functions $U^+_{x,z}(\alpha)$, $\phi^-_{x,z}(\alpha)$ are determined up to an arbitrary multiplying constant. Taking the inverse Fourier transforms, we obtain the internal polarization fields (from $U^+_{x,z}(\alpha)$) and the external field which is generated by the sheet polarization (from $\phi^-_{x,z}(\alpha)$). Using the general properties of the Fourier transforms, we see that the internal polarization field is finite everywhere, while the x-component of the external field is singular near the edge like $\lim_{x\to 0}\frac{1}{\sqrt{x}}$. Such a result was also reported in [5].

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APPENDIX A: FACTORIZATION FORMULAS

Although the method of factorization for the kernel $K(\alpha)=|k_z|l+\frac{|k_z|}{\sqrt{\alpha^2+k_z^2}}$, $l=-\frac{1}{2}\frac{\Omega^2\omega_p^2}{\Omega^4-\omega^2\omega_c^2}$ can be found in many references (*e.g.* [8], [9]), we give here the final results, for making it easier for the reader to follow the calculations.

1) $\Omega^2<\omega\omega_c$

In this case the function $K(\alpha)$ has no zeros (because we are working in the proper sheet of the square-root $\sqrt{\alpha^2+k_z^2}$, whose real part is positive ([11])) and it can be put in the form $K(\alpha)=\frac{K_+(\alpha)}{K_-(\alpha)}$, where

$$\begin{aligned}
K_+(\alpha) &= \frac{\sqrt{i|k_z|}}{\sqrt{\alpha+i|k_z|}}\sqrt{1+l\sqrt{\alpha^2+k_z^2}}\cdot e^X,\\
K_-(\alpha) &= \frac{\sqrt{\alpha-i|k_z|}}{\sqrt{-i|k_z|}}\frac{1}{\sqrt{1+l\sqrt{\alpha^2+k_z^2}}}\cdot e^X,
\end{aligned} \tag{31}$$

and

$$X=\frac{i}{4\pi}\ln\left(\frac{1-\sqrt{1-k_z^2l^2}}{1+\sqrt{1-k_z^2l^2}}\right)\left(\frac{\sqrt{1-k_z^2l^2}+l\alpha}{\sqrt{1-k_z^2l^2}-l\alpha}\right)-\frac{i}{\pi l}\int_0^\alpha d\xi\frac{\xi}{\xi^2+p^2}\frac{1}{\sqrt{\xi^2+k_z^2}}\ln\frac{\xi+\sqrt{\xi^2+k_z^2}}{|k_z|}.$$

”ln” means the principal determination of the logarithm. $K_+(\alpha)$ is analytic and free of zeros in the region $\mathrm{Im}\,\alpha>-|k_z|$ and $K_-(\alpha)$ is analytic and free of zeros in the region $\mathrm{Im}\,\alpha<|k_z|$. These expressions can be further simplified and can be expressed with the aid of Maliuzhinets functions ([14]). For large α, $K_\pm(\alpha)$ are of order zero in α: $K_\pm(\alpha)\sim$ ct. ([14]).

2) $\Omega^2 > \omega\omega_c$

In this case $K(\alpha)$ has two zeros situated in the analyticity strip $-k_z < \mathrm{Im}\,\alpha < k_z$: $\alpha_{1,2} = \pm ip$, $p = \frac{i}{l}\sqrt{1-k_z^2 l^2}$. We factorize $K(\alpha)$ as

$$K(\alpha) = -l(\alpha + ip)(\alpha - ip)\frac{\sigma_+(\alpha)}{\sigma_-(\alpha)},$$

where

$$\begin{aligned} \sigma_+(\alpha) &= \frac{\sqrt{i|k_z|}}{\sqrt{\alpha + i|k_z|}} \frac{\sqrt{1-|k_z|l}}{\sqrt{1-l\sqrt{\alpha^2+k_z^2}}} \cdot e^X, \\ \sigma_-(\alpha) &= \frac{\sqrt{\alpha - i|k_z|}}{\sqrt{-i|k_z|}} \frac{\sqrt{1-l\sqrt{\alpha^2+k_z^2}}}{\sqrt{1-|k_z|l}} \cdot e^X, \end{aligned} \tag{32}$$

and X is that from item 1. $\sigma_+(\alpha)$ is regular and free of zeros for $\mathrm{Im}(\alpha) > -k_z$ and $\sigma_-(\alpha)$ is regular and free of zeros for $\mathrm{Im}(\alpha) < k_z$. For large α, we have $\sigma_+(\alpha) \sim \frac{\mathrm{ct.}}{\alpha}$, $\sigma_-(\alpha) \sim \mathrm{ct.}\alpha$.

In order to solve our dispersion relation, we need to further simplify the expressions of $K_\pm(\pm i|k_z|)$ and $\sigma_\pm(\pm i|k_z|)$. We use $\ln\frac{\xi+\sqrt{\xi^2+k_z^2}}{k_z} = i\ \arcsin(\frac{\xi}{ik_z})$ and change the integration variable: $\xi \to u$, $\arcsin(\frac{\xi}{ik_z}) = u$. We obtain

$$K_+(i|k_z|) = \frac{1}{\sqrt{2}}\exp\left[\frac{i}{4\pi}\ln\frac{1-\sqrt{1-k_z^2l^2}}{1+\sqrt{1-k_z^2l^2}}\ln\frac{\sqrt{1-k_z^2l^2}+i|k_z|l}{\sqrt{1-k_z^2l^2}-i|k_z|l} + \frac{1}{2\pi}\int_0^{\frac{\pi}{2}} \mathrm{d}u \ln\left|\frac{1+|k_z|l\cos(u)}{1-|k_z|l\cos(u)}\right|\right] \tag{33}$$

and similar expressions for $K_-(-i|k_z|)$, $\sigma_\pm(\pm i|k_z|)$. The remaining integral from Eq.(33), $I \equiv \int_0^{\frac{\pi}{2}} \mathrm{d}u \ln\left|\frac{1+k_z l\cos(u)}{1-k_z l\cos(u)}\right|$ can be evaluated by the method indicated in Ref.[13], Sec.VI. One has to consider separately the cases $|k_z l| < 1$ and $|k_z l| > 1$. For $|k_z l| > 1$, we simply denote $\frac{1}{k_z l} \equiv \cos A$ and directly apply Eq.(90) from [13]. For $|k_z l| < 1$, we start the evaluation of I by expanding the integrand with the aid of the formula 5.4.10.13, [12]

$$\ln(z^2 - 2z\cos\phi + 1) = -2\sum_{j=1}^{\infty}\frac{z^j\cos(j\phi)}{j},\ |z| < 1.$$

We note that, if we put in the above formula: $z = e^{-A}$, $A > 0$ and $\phi = \frac{\pi}{2} - t$ and integrate term by term, we obtain Eq.(106) from [13]. Then, the result Eq.(92), [13] can be successfully applied for evaluating the integral I, after we change the integration variable $t \to y$; $t = y - \frac{\pi}{2}$.

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