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ANALYTICAL TREATMENT OF SYSTEM OF ABEL INTEGRAL EQUATIONS BY HOMOTOPY ANALYSIS METHOD

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Abstract. Abel equation has important applications in describing the least time for an object which is sliding on surface without friction in uniform gravity, and the classical theory of elasticity of materials is modeled by a system of Abel integral equations. In this manuscript, the homotopy analysis method is presented for obtaining analytical solutions of a system of Abel integral equations as fractional equations. The applied method has lessened the size of calculation and improved the accuracy of solution in the case of the singular Abel integral equation. The illustrated examples and numerical results have proved the assertion.

Key words: system of Abel integral equations, homotopy analysis method, *h*-curve, fractional calculus.

1. INTRODUCTION

Abel equation is one of the integral equations that arise from physical or mechanical models without passing through a differential equation [1]. Systems of singular Volterra integral equations are used in many branches of science, like astronomy, quantum mechanics, optics and so on. The approximate solutions of systems of singular integral equations and numerical methods for elastostatic problem are presented in Refs. [2, 3]. The one dimensional singular integral equation with the Cauchy and Dirichlet problems has been explained in details in Ref. [4]. The linear and nonlinear integral equations have been solved by adequate approximate and numerical methods [5]. The stable, approximate inversion of Abel integral equation applying the Taylor expansion gives a simple and closed form of approximate Abel inversion, which is executed by symbolic computation [6]. The generalized version of the Abel integral equation upon a finite segment was studied by Zeilon [7]. Homotopy perturbation method (HPM) is picked out to solve a system of generalized Abels integral equations [8]. The general form of this integral equations is considered as the following

$$\delta(x)y(x)+\lambda\int_a^x k(x-t)y(t)\,\mathrm{d}t = f(x), \quad 0\le x\le 1. \tag{1}$$

where

$$y(x) = (y_1(x), y_2(x), \ldots, y_n(x))^T,$$
$$f(x) = (f_1(x), f_2(x), \ldots, f_n(x))^T,$$
$$k(x-t) = (\frac{1}{(x-t)^{\alpha_{ii}}})_{ii},\ \ \lambda = (\lambda_{ii})_{ii},\ \ \delta(x) = (\delta_{ij}(x))_{ij}$$

for $i, j = 1, 2, \ldots, n$.
Some methods for solving the system of Abel integral equation have been studied [9–11]. Homotopy analysis method (HAM) as an alternative method, and has been widely used for solving systems of singular Volterra integral equations [12]. The main property of the method is its flexibility and ability to solve integral equations accurately and conveniently. Recently, *homotopy analysis method* (HAM) has been successfully employed to solve many types of nonlinear problems [13–21]. In this work HAM is presented for obtaining useful closed form solutions of a system of Abel integral equations, which is an application of fractional differential equations [22–28].

2. BASIC IDEA OF HAM

We consider the following differential equation

$$N[u(\tau)] = 0, \tag{2}$$

where N is a nonlinear operator, τ is an independent variable, and $u(\tau)$ is an unknown function, see Ref. [16]. One of the solutions of Eq. (2) is

$$u(\tau) = u_0(\tau) + \sum_{m=1}^{\infty} u_m(\tau), \tag{3}$$

where

$$u_m(\tau) = \frac{1}{m!}\frac{\partial^m \phi(\tau;q)}{\partial q^m}\,|_{q=0}. \tag{4}$$

Here $\phi(\tau;q)$ is the solution of the deformation equation [13–16]. We emphasize that $u_m(\tau)$ for $m \ge 1$ is governed by a linear equation under the linear initial condition that comes from the original problem, which can be easily symbolically solved by the Matlab computer software.

3. HAM FOR THE SYSTEM OF ABEL INTEGRAL EQUATIONS

In this section we apply HAM for the system of generalized Abel integral equations of the type

$$y_i(x) = f_i(x) + \lambda_i \int_0^x \frac{\sum_{j=1}^n \beta_{ij} y_j(t)}{(x-t)^{\alpha_{jj}}}\,\mathrm{d}t,\ \ 1 \le i \le n. \tag{5}$$

According to the HAM [13–16], we have

$$N[\phi_i(x,q)] = \phi_i(x,q) - f_i(x) - \lambda_i \int_0^x \frac{\sum_{j=1}^n \beta_{ij} \phi_j(t,q)}{(x-t)^{\alpha_{jj}}}\,\mathrm{d}t,\ \ 1 \le i \le n,$$

the corresponding mth-order deformation equation reads

$$L[u_{im}(x) - \chi_m u_{im-1}(x)] = hH(x)R_{im}(\vec{u}_{im-1}(x)),\ \ 1 \le i \le n, \tag{6}$$

where

$$R_{im}(\vec{u}_{im-1}(x)) = u_{im}(x) - (1-\chi_m) f_i(x) - \lambda_i \int_0^x \frac{\sum_{j=1}^n \beta_{ij} u_j(t)}{(x-t)^{\alpha_{jj}}}\,\mathrm{d}t,\ \ 1 \le i \le n. \tag{7}$$

We take an initial guess $u_{i0}(x) = f_i(x)$, an auxiliary linear operator $Lu = u$, and the nonzero auxiliary function $H(x) = 1$. These are substituted into (6) to give the recurrence relation

$$u_{i0}(x) = f_i(x),$$
$$u_{im}(x) = \chi_m u_{im-1}(x) + hR_{im}(\vec{u}_{im-1}(x)),\ \ m \ge 1.$$

The corresponding series solution is

$$u_i(x) = \sum_{m=0}^{\infty} u_{im}(x),\ \ 1 \le i \le n. \tag{8}$$

4. NUMERICAL RESULTS

In this section we have applied some examples to illustrate the ability and the accuracy of this method.

Example 1. Consider the following system of Abel integral equations, with the exact solutions $y_1(x) = x$ and $y_2(x) = \sqrt{x}$.

$$\begin{cases} y_1(x) + \int_0^x \frac{y_2(t)}{\sqrt{x-t}}\,\mathrm{d}t = x + \frac{\pi}{2}x, \\ y_2(x) + \frac{1}{2}\int_0^x \frac{y_1(t)+y_2(t)}{\sqrt{x-t}}\,\mathrm{d}t = \sqrt{x} + \frac{2}{3}x^{3/2} + \frac{\pi}{4}x. \end{cases} \tag{9}$$

We get some first terms of HAM series as follows

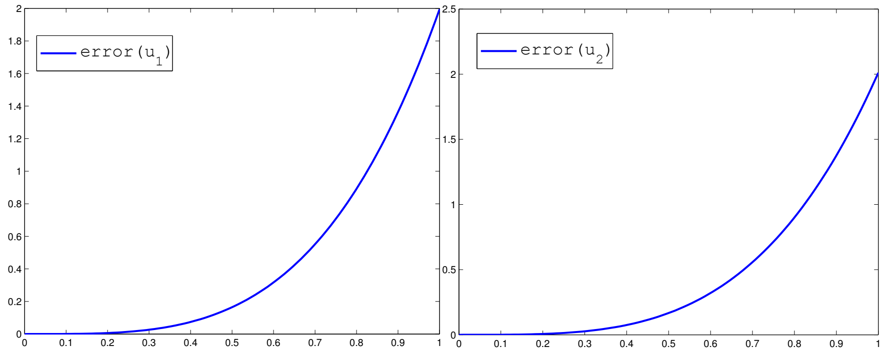

Fig. 1 – The absolute error $u_1(x)$ and $u_2(x)$ for example 1.

$$\begin{cases} u_{10}(x) = x + \frac{\pi}{2}x, \\ u_{20}(x) = \sqrt{x} + \frac{2}{3}x^{3/2}. \end{cases}$$

$$\begin{cases} u_{11}(x) = \frac{\pi}{2}hx + \frac{\pi}{4}hx^2 + \frac{\pi}{3}hx^{3/2}, \\ u_{21}(x) = \frac{2}{3}hx^{3/2} + \frac{\pi}{2}hx^{3/2} + \frac{\pi}{4}hx + \frac{\pi}{8}hx^2. \end{cases}$$

$$\begin{cases} u_{12}(x) = \frac{\pi}{2}hx + \frac{\pi}{4}hx^2 + \frac{\pi}{3}hx^{3/2} + \frac{\pi}{2}h^2x + \frac{\pi}{4}h^2x^2 + \frac{\pi}{3}h^2x^{3/2} \\ \qquad + \frac{\pi}{4}h^2x^2 + \frac{3\pi^2}{16}h^2x^2 + \frac{\pi}{3}h^2x^{3/2} + \frac{2\pi}{15}h^2x^{5/2}, \\ u_{22}(x) = \frac{2}{3}hx^{3/2} + \frac{\pi}{2}hx^{3/2} + \frac{\pi}{4}hx + \frac{\pi}{8}hx^2 + \frac{2}{3}h^2x^{2/3} + \frac{\pi}{2}h^2x^{3/2} + \frac{\pi}{4}h^2x + \\ \qquad \frac{\pi}{8}h^2x^2 + \frac{\pi}{2}h^2x^{3/2} + \frac{\pi}{5}h^2x^{5/2} + \frac{5\pi^2}{32}h^2x^2 + \frac{\pi}{8}h^2x^2. \end{cases}$$

Table 1

The admissible values of h derived from Figs. 2–3.

$u_1(x)$	$-1.15 \leq h_1 \leq -0.4$
$u_2(x)$	$-1.11 \leq h_2 \leq -0.4$

It is important to ensure that solution series (8) is convergent. Note that the solution series (8) contains the auxiliary parameter h, which helps us to adjust and control the convergence of the series solution. By suitable choice of the auxiliary parameter h, one can acquire common sense solution for equation (9). Thus the auxiliary parameter h plays an important role within the frame of the HAM. We have plotted the h-curve of $u_1(0.1)$ and $u_2(0.1)$ of the HAM in Figs. 2–3. For better presentation, the valid regions have been listed in Table 1. The absolute error $E(x) = u_{exact} - u_{HAM}$ is represented in Fig. 1. Also for a special case when $h = -1$,

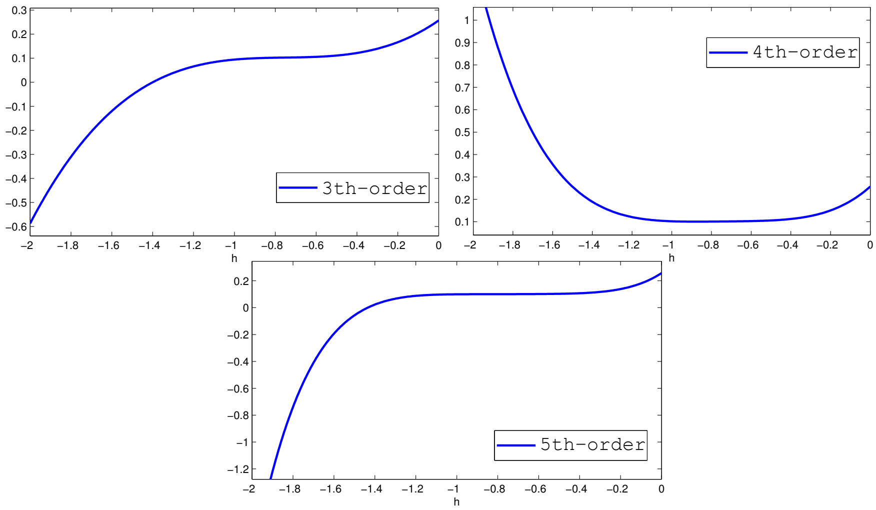

Fig. 2 – The HAM approximate solutions $u_1(x)$ with $x=0.1$.

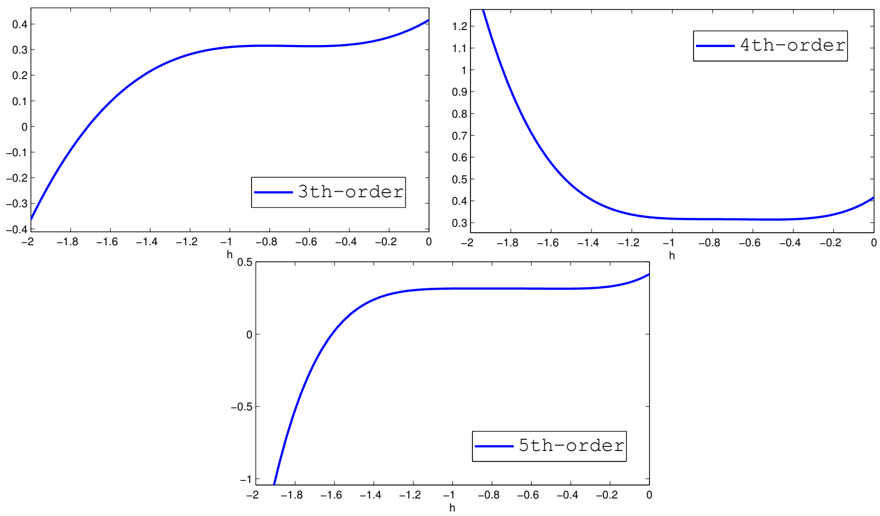

Fig. 3 – The curves are presented using HAM approximate solutions of $u_2(x)$ with $x=0.1$.

we have

$$u_1(x)=\sum_{i=0}^{\infty}u_{1i}(x)=\begin{cases}\sum_{i=0}^{n}u_{1i}(x)+O(x^{m+\frac{3}{2}})=x+O(x^{m+\frac{3}{2}}),\ n=2m\\ \sum_{i=0}^{n}u_{1i}(x)+O(x^{m+2})=x+O(x^{m+2}),\ n=2m+1\end{cases}$$

$$u_2(x)=\sum_{i=0}^{\infty}u_{2i}(x)=\begin{cases}\sum_{i=0}^{n}u_{2i}(x)+O(x^{m+\frac{3}{2}})=\sqrt{x}+O(x^{m+\frac{3}{2}}),\ n=2m\\ \sum_{i=0}^{n}u_{2i}(x)+O(x^{m+2})=\sqrt{x}+O(x^{m+2}),\ n=2m+1\end{cases}$$

as $n \to \infty$

$$\begin{cases} u_1(x) = x, \\ u_2(x) = \sqrt{x}. \end{cases} \qquad (10)$$

Example 2. In this example, we considered the following system of Abel integral equations of second kind as

$$\begin{cases} y_1(x) + \frac{1}{4}\int_0^x \frac{y_2(t)-y_1(t)}{\sqrt{x-t}}\,\mathrm{d}t = \sqrt{x} + \frac{3\pi}{32}x^2 - \frac{\pi}{8}x, \\ y_2(x) + 2\int_0^x \frac{y_1(t)}{\sqrt{x-t}}\,\mathrm{d}t = x^{3/2} + \pi x, \end{cases} \qquad (11)$$

with the exact solutions $y_1(x) = \sqrt{x}$ and $y_2(x) = x^{3/2}$.

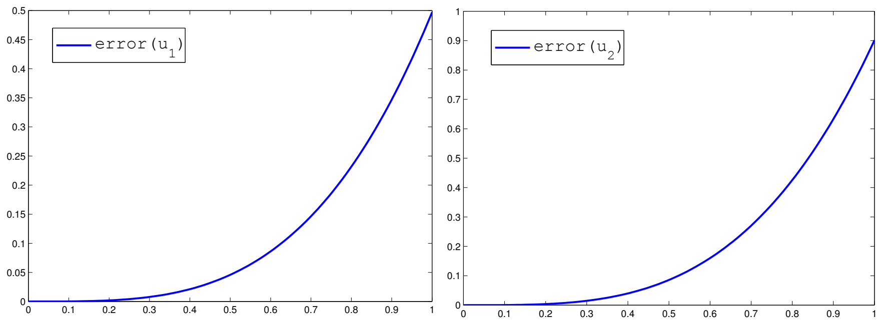

Fig. 4 – The absolute error $u_1(x)$ and $u_2(x)$ for example 2.

Thus, some first terms of HAM series read as

$$\begin{cases} u_{10}(x) = \sqrt{x} + \frac{3\pi}{32}x^2 - \frac{\pi}{8}x, \\ u_{20}(x) = x^{3/2} + \pi x. \end{cases}$$

$$\begin{cases} u_{11}(x) = -\frac{\pi}{8}hx + \frac{3\pi}{32}hx^2 + \frac{3\pi}{8}hx^{3/2} - \frac{\pi}{40}hx^{5/2}, \\ u_{21}(x) = \pi hx + \frac{\pi}{5}hx^{5/2} - \frac{\pi}{3}hx^{3/2}. \end{cases}$$

$$\begin{cases} u_{12}(x) = -\frac{\pi}{8}hx + \frac{3\pi}{32}hx^2 + \frac{3\pi}{8}hx^{3/2} - \frac{\pi}{40}hx^{5/2} - \frac{\pi}{8}h^2x + \frac{3\pi}{32}h^2x^2 + \frac{3\pi}{8}h^2x^{3/2} - \\ \qquad \frac{\pi}{40}h^2x^{5/2} + \frac{3\pi}{8}h^2x^{3/2} + \frac{9\pi^2}{512}h^2x^3 - \frac{\pi}{40}h^2x^{5/2} - \frac{17\pi^2}{256}h^2x^2 \\ u_{22}(x) = \pi hx + \frac{\pi}{5}hx^{5/2} - \frac{\pi}{3}hx^{3/2} + \pi h^2 + \frac{\pi}{5}h^2x^{5/2} - \frac{\pi}{3}h^2x^{3/2} + \\ \qquad \frac{9\pi^2}{32}h^2x^2 - \frac{\pi}{3}h^2x^{3/2} + \frac{\pi}{5}h^2x^{5/2} - \frac{\pi^2}{64}h^2x^3. \end{cases}$$

$$\vdots$$

Also for a special case when $h = -1$, we have

$$u_1(x) = \sum_{i=0}^{\infty} u_{1i}(x) = \begin{cases} \sum_{i=0}^{n} u_{1i}(x) + O(x^{m+2}) = \sqrt{x} + O(x^{m+2}), \ n = 2m \\ \sum_{i=0}^{n} u_{1i}(x) + O(x^{m+\frac{5}{2}}) = \sqrt{x} + O(x^{m+\frac{5}{2}}), \ n = 2m+1 \end{cases}$$

$$u_2(x) = \sum_{i=0}^{\infty} u_{2i}(x) = \begin{cases} \sum_{i=0}^{n} u_{2i}(x) + O(x^{m+2}) = x^{3/2} + O(x^{m+2}), \ n = 2m \\ \sum_{i=0}^{n} u_{2i}(x) + O(x^{m+\frac{5}{2}}) = x^{3/2} + O(x^{m+\frac{5}{2}}), \ n = 2m+1 \end{cases}$$

as $n \to \infty$

$$\begin{cases} u_1(x) = \sqrt{x}, \\ u_2(x) = x^{3/2}. \end{cases} \tag{12}$$

In general, by means of the so-called h-curve, it is straightforward to choose an appropriate range for h which ensures the convergence of the solution series. To study the influence of h on the convergence of solution, the h-curves of $u_1(0.1)$ and $u_2(0.1)$ are sketched, as shown in Figs. 5–6. For better presentation, the valid regions have been listed in Table 2. The absolute error $E(x) = u_{exact} - u_{HAM}$ is shown in Fig. 4.

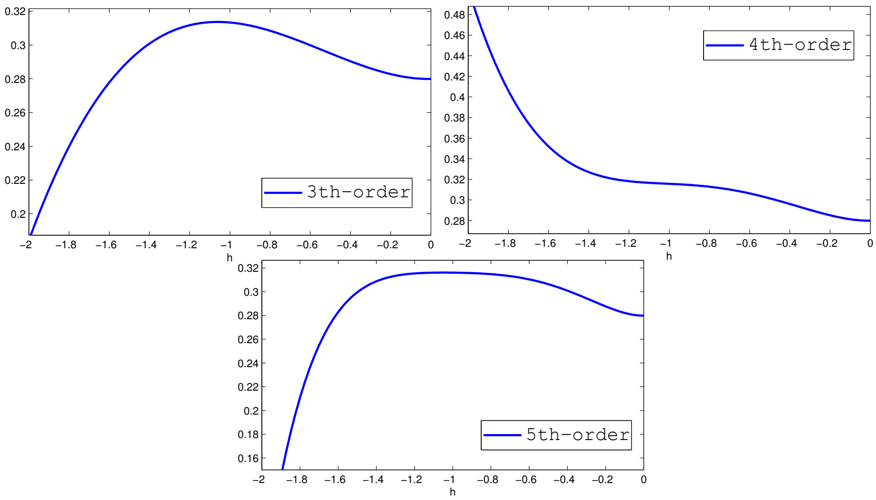

Fig. 5 – The curves are obtained from the HAM approximate solutions of $u_1(x)$ with $x = 0.1$.

5. CONCLUSIONS

In this paper, the system of Abel integral equations of the second kind has been studied using the HAM. The HAM is more suitable than other analytic methods, be-

Table 2

The admissible values of h derived from Figs. 5-6.

$u_1(x)$	$-1.2 \leq h_1 \leq -0.8$
$u_2(x)$	$-1.1 \leq h_2 \leq -0.95$

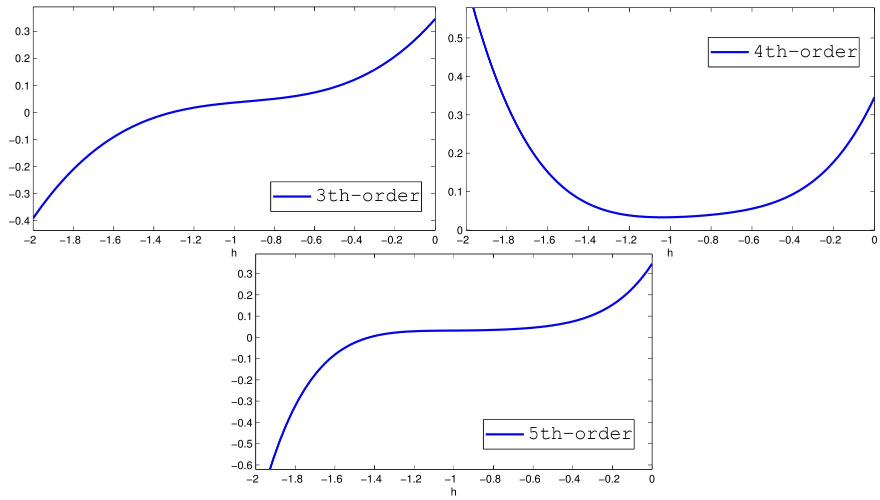

Fig. 6 – The curves are obtained from the HAM approximate solutions of $u_2(x)$ with $x = 0.1$.

cause this procedure provides us with a convenient way to control the convergence of an approximate series, which is a fundamental qualitative difference between HAM and other methods. The approximate solution of this system is calculated in the form of series and its components are computed by applying a recursive relation. The results indicate that the solution obtained by this method converges rapidly to an exact one. The plotted graphs confirm the obtained results.

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