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NOETHER CONSTANTS FOR THE ACOUSTIC FIELD IN THE ROBERTSON-WALKER SPACE-TIME

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Abstract. The *acoustic space-time* for sound waves in the isotropic and homogeneous universe inherits the background Robertson-Walker isometries. Noether constants associated with the six-parameter isometry group of this space defines components of the momentum, hyperbolic momentum and angular momentum of sound. These constants are exact and can replace the *approximate constants of motion* which are currently in use.

Key words: Robertson-Walker space-time.

1. INTRODUCTION

Acoustic field which originates due to small perturbation of the Robertson-Walker space-time contributes to the microwave background temperature fluctuations [1]. Perturbation formalisms describing this phenomenon, were developed by many authors and expressed in different ways [1–10]. Discussion of concurrent formalisms and further references can be found in [11].

The theory of sound in non static media has developed in roughly the same period. Research refers both to theoretical [12, 13] and engineering [14–16] aspects. The key feature that distinguishes the acoustics of fluids in motion is the absence of the conserved quantities: the energy-momentum and the angular momentum [17] of the sound. Non-conservation of the acoustic energy or momentum leads to the idea of the energy-momentum interchange between waves and the background flows. The more radical formulations suggest that no momentum at all can be assigned to sound wave propagating in non-static environment (see critical review [17]).

The acoustics of non static media has several tangent points with the perturbation theory in the Robertson-Walker space-time. In both cases:

a) the environment evolves, and the energy is not conserved,
b) density perturbations propagate as waves,
c) inhomogeneities are small, the linear theory is adequate,
d) splitting the solutions into the background component and the perturbation is not unique (the gauge problem).

In both cases the questions *what is the momentum of the sound wave* [18] arise, and to our best knowlegde still remains open.

Having at our disposal the six parameter isometry group [19] of Robertson-Walker space-time we identify the resulting Noether constants with the momentum and angular momentum components. Following Unruh [20, 21], Visser [22], and continuing [23–25] we construct the *acoustic space-time*, that is the auxiliary pseudoriemannian space in which the perturbations formally behave as the scalar field (the propagation equations have the d'Alembert equation form). Consequently, we introduce the energy momentum tensor for this auxiliary field, and following classical methods [26], we reconstruct the conserved currents. Conservation laws are exact and independent of the perturbation scale.

Concentrating on adiabatic perturbations, we do not specify any particular equation of state of the matter content. We claim however that $p = p(\epsilon) > 0$ and $\delta p/\delta\epsilon > 0$ (the speed of sound $c_{\rm s}$ is positive). We adopt the synchronous system of reference, where the gauge freedom is limited by the constraint $\delta g_{0\mu} = 0$. This constraint is a good compromise between full gauge freedom and completely fixed-gauge theory. The residual gauge freedom still admits variations of the constant time hypersurfaces*. The freedom is enough to introduce the hypersurface-independent variables and the time-independent perturbation spectrum. On the other hand, the synchronous constraint makes the symbolic computations more efficient.

Throughout this paper we use the convention $8\pi G = 1$, $c = 1$. The following notation is adopted: $a(\eta)$ — the scale factor, η — conformal time, $x^k = \{\mathsf{x}, \mathsf{y}, \mathsf{z}\}$ — Cartesian 3-space coordinates, $x^\mu = \{\eta, \mathsf{x}, \mathsf{y}, \mathsf{z}\}$ — Cartesian space-time coordinates, ${}^{(3)}g_{mn}$ — the metric of the maximally symmetric 3-dimensional space. The curvature K is an arbitrary real number. Where the confusion is unlikely, we abbreviate the notation by writing $\mathbf{x}$ for x^μ and $\mathtt{x}$ for x^k. Dot in $\mathtt{x}{\cdot}\mathtt{x}$ stands for $\mathsf{x}^2 + \mathsf{y}^2 + \mathsf{z}^2$. We do not explicitly express the time or space dependence of the metric tensor g (we hope, this is obvious), but we always keep this dependence in the metric corrections $\mathcal{C}(\mathbf{x})$ and $\mathcal{E}(\mathbf{x})$.

*The gauge-invariance evoked below means invariance against this residual freedom.

2. SCALAR PERTURBATIONS

Consider the Robertson-Walker space-time as the ground state,

$$g_{\mu\nu} = a^2(\eta)\mathrm{diag}(-1, {}^{(3)}g_{mn}), \tag{1}$$

where ${}^{(3)}g_{mn}$ stands for the maximally symmetric 3-dimensional space with the metric tensor expressed in Cartesian coordinates as follows

$${}^{(3)}g_{mn} = \delta_{mn} + \frac{K}{1 - K\mathbf{x}\cdot\mathbf{x}} x_m x_n. \tag{2}$$

Adopting the synchronous gauge $\delta g_{\mu 0} = 0$ we introduce the small correction $\delta g_{\mu\nu}$

$$\delta g_{mn} = \nabla_m \nabla_n \mathcal{E}(\mathbf{x}) + \frac{1}{3}\left(\mathcal{C}(\mathbf{x}) - \triangle \mathcal{E}(\mathbf{x})\right) g_{mn} \quad \text{and} \quad \delta g_{\mu 0} = 0. \tag{3}$$

∇_m and $\triangle$ denote the covariant derivative and the Laplacian, respectively — both are calculated in the static space ${}^{(3)}g_{mn}$. $\mathcal{C}(\mathbf{x})$ and $\mathcal{E}(\mathbf{x})$ are arbitrary functions dependent on space and time. The correction (3) is the most general formula for the scalar perturbations [27] in the synchronous system of reference.

Einstein equations with the hydrodynamic energy momentum tensor are expanded to the first perturbation order with respect to the metrics corrections. The zero-order equations reproduce Friedman equations

$$3\frac{a'^2(\eta)}{a^4(\eta)} + 3\frac{K}{a^2(\eta)} = \epsilon(\eta) \quad \text{and} \quad -2\frac{a''(\eta)}{a^3(\eta)} + \frac{a'^2(\eta)}{a^4(\eta)} - \frac{K}{a^2(\eta)} = p(\eta), \tag{4}$$

which define the dynamics of the background, and the first perturbation order leads to

$$\begin{aligned} \frac{\partial^2}{\partial\eta^2}\mathcal{E}_{|m}{}^{|n} + 2aH\frac{\partial}{\partial\eta}\mathcal{E}_{|m}{}^{|n} + \mathcal{E}_{|ipml}\, g^{in} g^{pl} - \frac{2}{3}\mathcal{E}_{|iplm}\, g^{ip} g^{ln} \\ + \mathcal{E}(\mathbf{x})_{|impl}(g^{il}g^{np} - g^{in}g^{pl}) - \frac{1}{3}\mathcal{C}_{|m}{}^{|n} - 4K\mathcal{E}_{|m}{}^{|n} = 0, \text{ for } m \neq n, \end{aligned} \tag{5}$$

$$\begin{aligned} \frac{\partial^2}{\partial\eta^2}\mathcal{C} + (2 + 3c_{\mathrm{s}}^2)aH\frac{\partial}{\partial\eta}\mathcal{C} + (1 + 3c_{\mathrm{s}}^2) \\ \times \left[\frac{1}{2}\mathcal{E}_{|lhsm}(g^{lm}g^{hs} - \frac{1}{3}g^{lh}g^{sm}) - \frac{1}{3}\left(\mathcal{C}_{|r}{}^{|r} + 3K\mathcal{C}\right)\right] = 0. \end{aligned} \tag{6}$$

The stroke stands for the covariant space-derivative ∇_n ($T_{|n} = \nabla_n T$) and c_{s} stands for the sound velocity: $c_{\mathrm{s}}^2 = p'(\eta)/\epsilon'(\eta)$. $H = a'(\eta)/a^2(\eta)$ is the Hubble parameter. The identities

$$\mathcal{E}_{|ipml}(g^{in}g^{pl} - g^{ip}g^{nl}) + \mathcal{E}_{|impl}(g^{il}g^{np} - g^{in}g^{pl}) - 4K\mathcal{E}_{|m}{}^{|n} = 0, \tag{7}$$

$$\mathcal{E}_{|lhsm}\left(g^{lm}g^{sh} - g^{lh}g^{sm}\right) - 2K\mathcal{E}_{|m}{}^{|m} = 0, \tag{8}$$

assure that system (5) and (6) reduces to

$$\frac{\partial^2 \triangle\mathcal{E}}{\partial\eta^2} = -2\frac{a'}{a}\frac{\partial\triangle\mathcal{E}}{\partial\eta} - \frac{1}{3}\triangle\left(\triangle\mathcal{E} - \mathcal{C}\right), \tag{9}$$

$$\frac{\partial^2 \mathcal{C}}{\partial\eta^2} = -(3c_\mathrm{s}^2 + 2)\frac{a'}{a}\frac{\partial\mathcal{C}}{\partial\eta} - \left(\frac{1}{3} + c_\mathrm{s}^2\right)\left(\triangle + 3K\right)\left(\triangle\mathcal{E} - \mathcal{C}\right). \tag{10}$$

These equations are the partial differential analogue to the Lifshitz–Khalatnikov *ODE* system. Neither the metric potentials nor the density contrast evaluated from them

$$\delta\epsilon(\mathbf{x}) = \frac{1}{3a^2}\left[3\frac{a'}{a}\frac{\partial\mathcal{C}}{\partial\eta} + (\triangle + 3K)(\triangle\mathcal{E} - \mathcal{C})\right] \tag{11}$$

are observables. All of them are ambiguous due to the existing gauge freedom.

3. ACOUSTIC SPACE-TIME

The gauge freedom is guaranteed by two arbitrary space dependent functions $\mathcal{G}_1(x^k)$ and $\mathcal{G}_2(x^k)$. The gauge solutions for $\triangle\mathcal{E}$ and $\mathcal{C}$ are

$$\triangle\mathcal{E}_g(x^\mu) = -\mathcal{G}_1(x^k) - 2\int\frac{1}{a(\eta)}\mathrm{d}\eta\,\triangle\mathcal{G}_2(x^k), \tag{12}$$

$$\mathcal{C}_g(x^\mu) = \mathcal{G}_1(x^k) + 6\frac{a'(\eta)}{a^2(\eta)}\mathcal{G}_2(x^k) + 2\int\frac{1}{a(\eta)}\mathrm{d}\eta\,\triangle\mathcal{G}_2(x^k). \tag{13}$$

Not eliminating this freedom, we look for quantities of physical interest, which are independent of $\mathcal{G}_1(x^k)$ and $\mathcal{G}_2(x^k)$. Below we abbreviate the notation by writing $\mathbf{x}$ for x^μ and x for x^k. Observing that the pure-gauge density perturbation

$$\delta\epsilon_g(\mathbf{x}) = \mathcal{G}_2(\mathrm{x})\frac{\epsilon'(\eta)}{a(\eta)} = \mathcal{G}_2(\mathrm{x})\dot{\epsilon}(t) \tag{14}$$

is the product of $\mathcal{G}_2(\mathrm{x})$ and the background quantity $\dot{\epsilon}(t)$, we define the associated gauge-invariant variable $\gamma(\mathbf{x})$ as follows

$$\gamma(\mathbf{x}) = \dot{a}^2(t)\left[\frac{\delta\epsilon(\mathbf{x})}{\dot{\epsilon}(t)}\right]^{\cdot}. \tag{15}$$

Restoring the conformal time parametrization one obtains

$$\frac{\partial^2\gamma(\mathbf{x})}{\partial\eta^2} + \left[2\frac{\mathfrak{a}'(\eta)}{\mathfrak{a}(\eta)} - \frac{c_\mathrm{s}'(\eta)}{c_\mathrm{s}(\eta)}\right]\frac{\partial\gamma(\mathbf{x})}{\partial\eta} - c_\mathrm{s}^2(\eta)\triangle\gamma(\mathbf{x}) = 0, \tag{16}$$

with

$$\mathfrak{a}(\eta) = a(\eta)\sqrt{\frac{p(\eta) + \epsilon(\eta)}{3c_\mathrm{s}(\eta)H^2(\eta)}}. \tag{17}$$

Equation (16) describes the acoustic waves propagating in the expanding environment with variable sound velocity $c_{\mathrm{s}}(\eta)$. Proof of the equation (16) consists in straightforward reduction of (4), (9), (10), (11), (15) and (16). Redefinition of the time parameter

$$\zeta = \int^{\zeta} c_{\mathrm{s}}(\eta)\,\mathrm{d}\eta \tag{18}$$

completes the construction of the *acoustic space-time* [20–22], and effectively removes the time-dependence of the sound velocity. The result is the d'Alembert equation

$$\mathfrak{g}_{\mu\nu}\nabla^{\mu}\nabla^{\nu}\gamma(\mathbf{x}') = 0, \tag{19}$$

in the space-time $\mathbf{x}' = \{\zeta, \mathsf{x}, \mathsf{y}, \mathsf{z}\}$ with the metric $\mathfrak{g}_{\mu\nu} = \mathfrak{a}^2(\zeta)\mathrm{diag}\,(-1, {}^{(3)}g_{mn})$ and the scale factor $\mathfrak{a}(\eta(\zeta)) = a\sqrt{\frac{p+\epsilon}{3c_{\mathrm{s}}H^2}}$. According to equation (19) the sound in the Robertson-Walker space-time $\mathcal{M}$ with the scale factor $a(\eta)$, propagates as massless scalar field γ in Robertson-Walker space-time $\mathcal{M}'$ with the scale factor $\mathfrak{a}(\eta(\zeta))$. Equation (19) with appropriate identification of the quantities involved may be considered as the generalization of the *Sachs-Wolfe theorem* to the case of spatially curved Robertson-Walker models and to arbitrary equation of state of the barotropic form $p = p(\epsilon)$.

In the forthcoming section we drop the sign prime. The coordinates $\{\zeta, \mathsf{x}, \mathsf{y}, \mathsf{z}\}$ we briefly denote by the same letter $\mathbf{x}$.

4. SYMMETRIES AND CONSERVED MOMENTA

The propagation equation (19) can be obtained as the Lagrange equation for the Lagrangian $\mathcal{L} = -\frac{1}{2}\mathfrak{g}_{\rho\sigma}\partial^{\rho}\gamma(\mathbf{x})\,\partial^{\sigma}\gamma(\mathbf{x})$, or equivalently, it follows from the identity $\nabla_{\nu}\mathcal{T}^{\mu\nu} = 0$, with the energy momentum tensor

$$\mathcal{T}^{\mu\nu} = \partial^{\mu}\gamma(\mathbf{x})\,\partial^{\nu}\gamma(\mathbf{x}) - \frac{1}{2}\mathfrak{g}^{\mu\nu}\mathfrak{g}_{\rho\sigma}\partial^{\rho}\gamma(\mathbf{x})\,\partial^{\sigma}\gamma(\mathbf{x}). \tag{20}$$

Both $\mathcal{L}$ and $\mathcal{T}^{\mu\nu}$ are gauge-invariant, therefore, all the conserved quantities constructed by means of them have the same property. The procedure below differs from that of Katz *et al.* [28], (see also [29, 30]) in the definition of the background space-time, and in that the perturbation field $\gamma(\mathbf{x})$ is already gauge-invariant. We consider the conserved currents $\mathcal{J}^{(i)}_{\mu} = \mathcal{T}_{\mu\nu}K^{(i)\nu}$. The six space-like isometries $K^{(i)}$ provide the six Noether integrals $\pi(K^{(i)})$

$$\pi(K^{(i)}) = \int_{\Sigma}\mathcal{J}^{(i)}_{\mu}\mathrm{d}\Sigma^{\mu} = \int_{\Sigma}\mathcal{T}_{\mu\nu}K^{(i)\nu}\,\mathrm{d}\Sigma^{\mu} = \mathrm{const}^{(i)}, \tag{21}$$

where Σ^{μ} is an arbitrary Cauchy surface. Killing algebra, and consequently, the interpretation of constants (21) involves the sign of curvature. For Killing basis chosen as $K^{(i)} = \{K_T^{(i)}, K_L^{(i)}\}$

$$K_T^{(i)j} = \delta^{ij}\sqrt{1 - K\mathtt{x}\cdot\mathtt{x}}, \qquad K_L^{(i)j} = \sum_{k=1}^{3} \epsilon^{ijk} x^k, \qquad K_L^{(i)0} = K_T^{(i)0} = 0 \tag{22}$$

generators of infinitesimal isometries

$$T^{(i)} = -\mathrm{i}\, K_T^{(i)j}\partial_j, \quad L^{(i)} = -\mathrm{i}\, K_L^{(i)j}\partial_j \tag{23}$$

satisfy the commutation relations

$$\begin{aligned} \left[T^{(i)}, T^{(j)}\right] &= \mathrm{i}K \sum_{k=1}^{3} \epsilon^{ijk} L^{(k)}, \\ \left[L^{(i)}, L^{(j)}\right] &= \mathrm{i} \sum_{k=1}^{3} \epsilon^{ijk} L^{(k)}, \\ \left[L^{(i)}, T^{(j)}\right] &= \mathrm{i} \sum_{k=1}^{3} \epsilon^{ijk} T^{(k)}. \end{aligned} \tag{24}$$

For $K > 0$, both $K_T^{(i)}$ and $K_L^{(i)}$ assure the angular momenta conservation (six constants of motion). For $K = 0$ translations $K_T^{(i)}$ conserve the momentum (3 constants of motion), while the rotations $K_L^{(i)}$ conserve the angular momentum (the next 3 constants of motion). For $K < 0$ vectors $K_T^{(i)}$ correspond to hyperbolic momentum [31–33]. The absence of the time isometry† breaks down the energy conservation. Operators $T^{(i)}$ and Δ commute

$$\left[T^{(i)}, \Delta\right] = 0 \tag{25}$$

hence, each pair $\{T^{(i)}, \Delta\}$, $i = 1, \ldots, 3$ has common eigenfunctions. There are no common eigenfunctions for pairs $\{T^{(i)}, T^{(j)}\}$ with $i \neq j$ and $K \neq 0$.

Solutions $u_k(x^\mu, n^i)$ to equation (19) which lie in the kernel of operator $n_i L^{(i)}$, and simultaneously are eigenfunctions of the operator $n_i T^{(i)}$

$$n_i L^{(i)} u_k(x^\mu, n^i) = 0, \tag{26}$$

$$n_i T^{(i)} u_k(x^\mu, n^i) = \lambda_k u_k(x^\mu, n^i) = (k - \mathrm{i}\sqrt{-K}) u_k(x^\mu, n^i), \tag{27}$$

define *plane waves* of the wavenumber k, propagating in the direction n^i. On the strength of (25) $u_k(x^\mu, n^i)$ are also the eigenfunctions of the Laplace operator

$$\Delta u_k(x^\mu, n^i) = -(k^2 - K) u_k(x^\mu, n^i), \tag{28}$$

†Conformal Killing vectors in both spaces $\mathcal{M}$ and $\mathcal{M}'$ are not identical. The conserved quantities related to conformal isometries [28–30] of the space $\mathcal{M}'$ are a separate issue, not discussed in this paper.

and hence, are orthogonal on constant time hypersurfaces. When separated they read

$$u_k = \frac{1}{\mathfrak{a}(\zeta)} \chi_k(\zeta) F(\mathtt{x}, \mathfrak{n}, k), \tag{29}$$

with the evolution equation implicitly depend on the equation of state

$$K - k^2 = \frac{\chi_k''(\zeta)}{\chi_k(\zeta)} + 2\frac{\mathfrak{a}'(\zeta)}{\mathfrak{a}(\zeta)}\frac{\chi_k'(\zeta)}{\chi_k(\zeta)}. \tag{30}$$

The space-dependent solutions $F(\mathtt{x}, \mathfrak{n}, k)$ to the Helmholtz equation (28) are

$$F(\mathtt{x}, \mathfrak{n}, k) \propto \left(\sqrt{-K}\, \mathfrak{n}\cdot\mathtt{x} + \sqrt{1 - K\mathtt{x}\cdot\mathtt{x}}\right)^{-1 + \frac{\mathrm{i}\,k}{\sqrt{-K}}}. \tag{31}$$

In the $K = 0$ limit $F(\mathtt{x}, \mathfrak{n}, k)$ tend to $e^{\mathrm{i}\,k(\mathfrak{n}\cdot\mathtt{x})}$. For the negative space curvature functions $F(\mathtt{x}, \mathfrak{n}, k)$ are Shapiro functions [34]. *Principal series* characterised by positive wavenumbers $k > 0$ consists of functions orthogonal and complete in L^2 [35, 36]. *Supplementary series* of regular, bounded, non-oscillating and non-orthogonal functions $F(\mathtt{x}, \mathfrak{n}, k)$ with imaginary wavenumber $k \in (0, \sqrt{K})$ are redundant to expand the square integrable perturbations, although, they contribute to Fourier decomposition of weakly homogeneous stochastic processes‡. In both cases, flat $(K = 0)$ or open $(K < 0)$ universes, the Fourier bases are denumerable, and the spectrum is continuous. For positive curvature the functions $F(\mathtt{x}, \mathfrak{n}, k)$ coincide with Sherman–Volobuyev functions [39, 40]. The spectrum of Beltrami–Laplace operator is numerable.

In what follows, we limit ourselves to non-positive curvature and to positive wave numbers (continuos spectrum and principal series). We consider the acoustic wave $\gamma(x^\mu, n^i)$ propagating in the direction n^i, *i.e.* an arbitrary solution to the equation (19) that can be expanded as

$$\gamma(x^\mu, n^i) = \sum_k \left(a_k u_k(x^\mu, n^i) + a_k^* u_k^*(x^\mu, n^i)\right) \tag{32}$$

in orthogonal basis $u_k(x^\mu, n^i)$

$$(u_k, u_{k'}) = k\lambda_k^{-1}\delta(k - k') \tag{33}$$

with Fourier coefficients

$$a_k = \lambda_k(u_k, \gamma). \tag{34}$$

Symbol $(,)$ means the nondegenerate symplectic form [26]

$$(\phi_1, \phi_2) = \int \mathrm{i}\, W_\mu(\phi_1^*, \phi_2)\, \mathrm{d}\Sigma^\mu \tag{35}$$

‡For the inflation theory analogue see [37, 38])

where W_μ stands for Wronskian. Since the divergence of Wronskian vanishes for arbitrary pair ϕ_1 and ϕ_2 of complex solutions to the equation (19), the integral (35) is independent of the choice of the Σ hypersurface. In particular, the integral (35) and all other quantities defined on this base are invariant under the gauge modifications. Coefficients a_k are constant in time. The normalisation

$$W_\mu(\chi_k, \chi_k^*) = -\mathrm{i}, \tag{36}$$

$$\int F^*(\mathtt{x}, \mathfrak{n}, k) T^{(i)} F(\mathtt{x}, \mathfrak{n}, k')\, \mathrm{d}\Sigma^\mu = k\delta(k - k'), \tag{37}$$

assures the Plancherel formula for the momentum (21)

$$n_i \pi(K_T^{(i)}) = \int \mathrm{d}\Sigma^0 n_i K_T^{(i)\nu} \mathcal{T}_{0\nu} = \int \mathrm{d}\Sigma^0 \partial_0 \gamma(x^\mu, n^i) n_i K_T^{(i)\nu} \partial_\nu \gamma(x^\mu, n^i) \tag{38}$$

and finally one obtains

$$n_i \pi(K_T^{(i)}) = \sum_k k a_k a_k^* \mathrm{i} W_0(\chi_k, \chi_k^*) = \sum_k k a_k a_k^* = \sum_k k P_k. \tag{39}$$

The quantity $P_k = a_k\, a_k^*$ is constant in time and invariant under both: the gauge transformations of the reference system, and the unitary transformations of Fourier bases. kP_k is an observable. According to (39), it can be regarded as the spectral density of the momentum or hyperbolic momentum of sound.

5. CONCLUSIONS

Momenta of sound waves propagating on the expanding homogeneous background can be identified with Noether constants for scalar field in the associated *acoustic space-time*. These momenta are conserved independently of the space-time curvature, topology, equation of state, cosmological parameters, perturbations' scales and their relations to the particle horizon. The conservation comes directly from the equation (19) and the isometry group (22). Conservation law, as formulated above, is exact and form an alternative to the *approximate-constants-of-motion* approach.

This unique, scale-independent technique enables to verify whether the "horizon crossing" can substantially affect the propagation of sound (in our opinion not), and conversely, whether the different mathematical treatment of the different scale perturbations does not result in numerical artefacts in the relevant computer algorithms. In particular, it is possible to numerically check whether the waves simulated by cosmological codes (CMBFAST for instance) do conserve their momenta. Physical relevance of these codes is of particular importance.

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