RELATIVISTIC BOUND STATES IN A PSEUODOHARMONIC OSCILLATOR
VIA LAPLACE TRANSFORM

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Received July 3, 2013

Abstract. The two-dimensional Klein-Gordon equation with equal scalar and vector pseudo-harmonic oscillator is studied via the Laplace transform method. The exact bound state energy eigenvalues and wave functions are calculated in terms of chemical potential parameter and magnetic quantum number by requiring good behaviour of wave function at the origin and at infinity. The non-relativistic solution of the pseudo-harmonic oscillator solution is also found in three-dimensional and D-dimensional space.

Key words: Klein-Gordon equation, two-dimensional pseudo-harmonic oscillator, Laplace transform method, bound states.

PACS: 03.65.-w; 03.65.Pm; 03.65.Ge

1. INTRODUCTION

The quantum pseudo-harmonic oscillator (PHO) is one of the most important models in quantum mechanics used to describe the bound state of the interaction systems and has been applied for both classical and modern physics. It plays a basic role in chemical and molecular physics since it can be used to calculate the molecular vibration-rotation energy spectrum of linear and non-linear systems [1-3]. It is considered as an intermediate between harmonic oscillator (HO) and Morse-type potentials which are more realistic anharmonic potentials. In the non-relativistic quantum mechanics, the PHO can be solved in closed form and it can be useful for generating approximate or exact solutions to various problems. This potential is usually solved in the framework of Schrödinger and Klein-Gordon (KG) equations in with one-dimensional (1D), two-dimensional (2D), three-dimensional (3D) and even in D-dimensional space.

In recent times, the discrete (bound) and continuous (scattering) energy spectra
of the PHO have been investigated by the SU(1,1) spectrum generating algebra [4]. The PHO potential was obtained depending on the dimension, the angular momentum and the zero-point energy in two-dimensions. The stationary radial Schrödinger equation in 3D space with PHO was solved for any orbital quantum number \( l \) using the polynomial method [5]. Recently, solutions of the Schrödinger and KG equations with PHO potential have been investigated using various methods [6-10]. A realization of the creation and annihilation (ladder) operators for the solution to the Schrödinger equation with a PHO in 2D was examined in Refs. [11, 12] where the operators satisfy the commutation relations of an SU(1,1) group. The exact solution of the Schrödinger equation with a PHO in D-dimansional space for the PHO potential have been presented by means of the ansatz method and the energy eigenvalues were calculated from the eigenfunction ansatz [13, 14]. Also the exact bound-state solutions of the KG equation with equal scalar and vector PHO potential have been obtained using the supersymmetric quantum mechanics, shape invariance and other alternative methods [15]. The Dirac equation with PHO was solved in the presence of spin and pseudospin symmetries [16].

The spectral properties in a 2D charged particle (electron or hole) confined by a 2D PHO potential in the presence of external strong uniform magnetic field \( \vec{B} \) along the \( z \) direction and Aharonov-Bohm (AB) flux field created by a solenoid have been studied [17]. The spectral properties of quantum dots influenced by a confining potential model within the exact analytical iteration method have been found [18]. The spectra of cylindrical quantum dots with 2D parabolic harmonic oscillator under the effect of electrical and magnetic field together with Aharonov-Bohm flux field are also studied [19]. So, it is natural that the relativistic effects for a charged particle under the action of this potential could become important, especially for a strong coupling. Recently, we studied the exact analytical bound state energy eigenvalues and normalized wave functions of the spinless relativistic equation with equal scalar and vector PHO interaction under the effect of external uniform magnetic field and AB flux field in the framework of the NU method [20]. The non-relativistic limit was found by making an appropriate mapping of parameters.

Another operational method useful in quantum mechanics is the Laplace transform method. It was used during the first years of quantum mechanics by Schrödinger when discussing the radial eigenfunction of the hydrogen atom [21] and later Enlefield approached the Schrödinger equation with the Coulomb, oscillator, exponential and Yamauchi potentials [22]. The hydrogen atom was studied using the Laplace transform method [23]. Some years ago, the \( 1/x \) [24], Morse [25], N-dimensional harmonic oscillator [26], 3D pseudoharmonic and Mie-type [27], one-dimensional harmonic oscillator [28] and double Dirac delta [29] potentials were solved by means of Laplace transform (LT).

Very recently, the exact solutions to the relativistic spinless particle under the
effect of a particular kind of Morse potential are studied [30]. The energy levels and normalized eigenfunctions are calculated exactly by using the Laplace transform method and the parametric Nikiforov-Uvarov (pNU) method.

The aim of this work is to study the bound state solution of spinless particle under the pseudoharmonic oscillator by means of the LT method. Therefore, the structure of this paper is as follows. We study the bound state solutions of the relativistic spinless particle under equal scalar and vector PHO interaction by means of Laplace transform technique in Section 2. Finally, we end with conclusion in Section 3.

2. LAPLACE TRANSFORM TO KG-PSEUDOHARMONIC PROBLEM

The KG equation is a wave equation mostly used in describing particle dynamics in relativistic quantum mechanics. Nonetheless, physically this equation describes a scalar particle (spin 0) [20, 31, 32]. It can be written for a 2D single particle as (in units $\hbar = c = 1$)

$$\left[ p^2 - (E - V(\vec{r}))^2 + (M + S(\vec{r}))^2 \right] \psi(\vec{r}, \phi) = 0, \tag{1}$$

where $\vec{p} = -i\hbar \vec{\nabla}$ is the momentum, $M$ is the mass and $E$ is the binding energy. Further, $V(\vec{r})$ and $S(\vec{r})$ are the vector and scalar potentials, respectively. The 2D cylindrical wave function $\psi(\vec{r}, \phi)$ is defined as [20]

$$\psi(\vec{r}, \phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi} R(r), \quad R(r) = r^{-1/2} g(r), \quad m = 0, \pm 1, \pm 2, \ldots, \tag{2}$$

where $m$ is the magnetic quantum number. The scalar-like potential coupling is added to the scalar mass so that in case when $S(\vec{r}) = \pm V(\vec{r})$, the KG equation could always be reduced to a Schrödinger-type second order differential equation as follows [20]

$$\left[ p^2 + 2(E \pm M)V(\vec{r}) + M^2 - E^2 \right] \psi(\vec{r}, \phi) = 0. \tag{3}$$

The potential $V(\vec{r})$ is taken as the repulsive PHO potential [1-3, 11, 20]:

$$V(\vec{r}) = V_0 \left( \frac{r}{r_0} - \frac{r_0}{r} \right)^2, \tag{4}$$

where $r_0$ and $V_0$ are the zero point (effective radius) and the chemical potential, respectively. Hence, the bound-state solutions of the two cases in Eq. (3) are to be treated separately as follows.

The positive energy states (corresponding to $S(\vec{r}) = +V_{\text{conf}}(\vec{r})$) in the non-relativistic limit (taking $E - M \to E$ and $E + M \to 2\mu/\hbar^2$, where $M$ is an effective
mass and $|E| \ll M$ are solutions of
\[
\left[ \frac{p^2}{2\mu} + 2V(\vec{r}) - E \right] \psi(\vec{r}, \phi) = 0,
\] (5)
where $\psi(\vec{r}, \phi)$ stands for either $\psi^+(\vec{r}, \phi)$ or $\psi^{(KG)}(\vec{r}, \phi)$. This is the Schrödinger equation for the potential $2V(\vec{r})$. Thus, the choice $S(\vec{r}) = +V(\vec{r})$ produces a nontrivial non-relativistic limit with a potential function $2V(\vec{r})$, and not $V(\vec{r})$. Accordingly, it would be natural to scale the potential term in Eq. (3) and Eq. (5) so that in the non-relativistic limit the interaction potential becomes $V$, not $2V$. Thus, we need to recast Eq. (3) for the $S(\vec{r}) = V(\vec{r})$ as [20, 31-33]
\[
\left[ \nabla^2 + 2(E + M) V(\vec{r}) \right] \psi(\vec{r}, \phi) = (E^2 - M^2) \psi(\vec{r}, \phi),
\] (6)
and in order to simplify Eq. (6) we introduce new parameters $\lambda_1 = E + M$ and $\lambda_2 = E - M$ so that it can be reduced to the form
\[
\left[ \frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} \right) - \frac{m^2}{r^2} + \lambda_1 (\lambda_2 - V(\vec{r})) \right] \psi(\vec{r}, \phi) = 0.
\] (7)
Now, inserting Eqs. (2) and (4) into the KG equation (7) and further introducing a change of variable $s = r^2$, that maps $r \in (0, \infty)$ to $s \in (0, \infty)$, we obtain second-order differential equation satisfying $g(s)$,
\[
\frac{d^2 g(s)}{ds^2} + \frac{1}{s} \frac{dg(s)}{ds} - \frac{1}{s^2} \left( \varepsilon_{nm}s^2 + a_2^2 s + a_1^2 \right) g(s) = 0, \ g(0) = 0 \text{ and } g(\infty) = 0,
\] (8)
where we have employed the following identifications
\[
a_2^2 = -\frac{\lambda_1}{4} (\lambda_2 + 2V_0), \ \varepsilon_{nm}^2 = \frac{V_0 \lambda_1}{4r_0^2}, \ a_1^2 = \frac{1}{4} \left( m^2 + r_0^2 V_0 \lambda_1 \right), \ m = 1, 2, 3, \ldots,
\] (9)
with $m$ is the magnetic quantum number. Letting
\[
g(s) = s^A f(s), \ A = -a_1,
\] (10)
with $A$ is a constant and after inserting into Eq. (8) leads to equation
\[
\frac{d^2 f(s)}{ds^2} - (2a_1 - 1) \frac{df(s)}{ds} - (\varepsilon_{nm}^2 s + a_2^2) f(s) = 0, \ f(0) = 0 \text{ and } f(\infty) = 0.
\] (11)
Now, we will use the Laplace transform method to solve Eq. (8) [26-30, 34-36]. It is an integral transform and comprehensively useful in physics and engineering: Recently, LT method has been used by many authors to solve the Schrödinger equation for different potential models [21, 23, 25, 28, 29, 34-36]. The advantage of LT is that it converts the second-order differential equation into a first-order differential equation whose solutions may be obtained easily. Applying the Laplace transforms
defined by [25, 26, 34-36]

\[
\mathcal{L} \{ f(s) \} = F(t) = \int_0^\infty f(s) e^{-st} ds, \tag{12a}
\]

\[
\mathcal{L} \{ f^{(n)}(s) \} = s^n \mathcal{L} \{ f(s) \} - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0), \tag{12b}
\]

\[
\mathcal{L} \{ t^n f(s) \} = (-1)^n F^{(n)}(t). \tag{12c}
\]

\[
\mathcal{L} \{ sf(s) \} = -\frac{dF(t)}{dt}, \tag{12d}
\]

\[
\mathcal{L} \{ sf''(s) \} = -2tF(t) - t^2 \frac{dF(t)}{dt} + f(0); \quad f(0) = 0, \tag{12e}
\]

\[
\mathcal{L} \{ sf'(s) \} = tF(t). \tag{12f}
\]

Inserting Eq. (12) in Eq. (11), it reduces to the following first-order differential equation

\[
(t^2 - \varepsilon_{nm}^2) \frac{dF(t)}{dt} + [(2a_1 + 1)^2 + a_2^2] F(t) = 0, \tag{13}
\]

or equivalently

\[
\int_0^\infty \frac{dF(t)}{F(t)} = - \int_0^\infty \frac{[(2a_1 + 1)^2 + a_2^2]}{(t - \varepsilon_{nm}) (t + \varepsilon_{nm})} dt. \tag{14}
\]

The solution of the above equation is

\[
\ln(F(t)) = \ln \left( \frac{t - \varepsilon_{nm}}{t + \varepsilon_{nm}} \right)^{-a_2^2/2\varepsilon_{nm}} + \ln \left( t^2 - \varepsilon_{nm}^2 \right)^{-(2a_1 + 1)/2}, \tag{15}
\]

giving

\[
F(t) = N \left( t + \varepsilon_{nm} \right)^{-(2a_1 + 1)} \left( \frac{t - \varepsilon_{nm}}{t + \varepsilon_{nm}} \right)^{-a_2^2/2\varepsilon_{nm} - (2a_1 + 1)/2}, \tag{16}
\]

where \( N \) is the normalization constant. Noting that \( \left( \frac{t - \varepsilon_{nm}}{t + \varepsilon_{nm}} \right)^{-a_2^2/2\varepsilon_{nm} - (2a_1 + 1)/2} \) is a multi-valued function and the wave functions are required to be single-valued, we must take

\[
-\frac{a_2^2}{2\varepsilon_{nm}} - \frac{(2a_1 + 1)}{2} = n, \quad n = 0, 1, 2, \ldots, \tag{17}
\]

which is the equation used to obtain the energy spectrum.
Our further goal is to obtain \( f(s) \) which is possible by taking the inverse Laplace transform of Eq. (16), one can expand Eq. (16) into the simple series

\[
F(t) \sim \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!k!} (2\varepsilon_{nm})^k (t + \varepsilon_{nm})^{-(2a_1+1)-k},
\]

which provides \( f(s) \) as

\[
f(s) \sim s^{2a_1} e^{-\varepsilon_{nm}s} \sum_{k=0}^{n} \frac{(-1)^k n!\Gamma(2a_1+1)}{(n-k)!k!\Gamma(2a_1+1+k)} (2\varepsilon_{nm}s)^k,
\]

and hence the radial wave function \( g(r) \) becomes

\[
g(r) = N r^{2a_1+1} \exp(-\varepsilon_{nm}r^2) F\left(-n,2a_1+1;2\varepsilon_{nm}r^2\right),
\]

where the confluent hypergeometric function is defined by

\[
F\left(-n,\sigma; x\right) = \sum_{p=0}^{n} \frac{(-1)^p n!\Gamma(\sigma)}{(n-p)!p!\Gamma(\sigma+p)} x^p.
\]

Note that the wave function (20) is finite and satisfying the standard asymptotic analysis for \( r = 0 \) and \( r \to \infty \). On the other hand, the energy states \( E \) can be obtained via Eq. (17) and using Eq. (9) as

\[
\frac{(E+M)}{2}(E-M+2V_0) = \left(1 + 2n + \sqrt{m^2 + V_0r_0^2(E+M)}\right) \sqrt{\frac{V_0(E+M)}{r_0^2}},
\]

where the quantum numbers \( n = 0,1,2,\ldots \) and \( m = 0,\pm1,\pm2,\ldots \). We may find solution to the above transcendental equation as \( E = E_{KG}^{(+)}. \) The above energy equation (22) is identical to energy equation of Ref. [20] when one inserts the magnetic field and Aharonov-Bohm flux field to zero, i.e., when \( \omega_c = \xi = 0. \)

![Fig. 1 – (Color online) Variation of the energy states \( E \) of the pseudoharmonic oscillator versus the magnetic quantum number \( m \) for various values of principal quantum number, \( n = 0,1,2,3,4. \)](image)

In Fig. 1, we plot the energy states \( E \) of the pseudoharmonic oscillator versus the magnetic quantum numbers \( m \) for various values of principal quantum numbers, \( n = 0,1,2,3,4. \) Further, in Table 1, we calculate the relativistic energy levels of the pseudoharmonic oscillator for various values of \( n \) and \( m \) taking the parameter values, \( V_0 = 5.0, r_0 = 2.4 \) and \( M = 10.0. \)
Table 1

Relativistic energy levels of spinless particle under the pseudoharmonic oscillator potential for various values of \( n \) and \( m \) taking the parameter values, \( V_0 = 5.0 \), \( r_0 = 2.4 \) and \( M = 10.0 \).

<table>
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<th>( m/n )</th>
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<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<td>11.213714</td>
<td>11.986968</td>
<td>12.735576</td>
<td>13.462272</td>
</tr>
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</tr>
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</table>

3. NON-RELATIVISTIC MODEL

In this section we examine the non-relativistic model considering various dimensional space. The non-relativistic limit can be obtained when \( E + M \to 2M \) and \( E - M \to E_{nm} \). Hence, the nonrelativistic energy equation (22) in two-dimensional space can be simply written as

\[
E_{nm} = -2V_0 + \frac{1}{M} \left( 1 + 2n + \sqrt{m^2 + 2MV_0r_0^2} \right) \sqrt{\frac{2MV_0}{r_0^2}},
\]

(23)

where \( m = 0, 1, 2, \ldots \) and \( n = 0, 1, 2, \ldots \).

The above equation can be expressed in three-dimensional space by making the change \( m \to l + 1/2 \). Therefore, the energy formula for vibrational-rotational quantum states becomes

\[
E_{nl} = -2V_0 + \frac{1}{M} \left( 1 + 2n + \sqrt{(l + 1/2)^2 + 2MV_0r_0^2} \right) \sqrt{\frac{2MV_0}{r_0^2}},
\]

\[l = 0, 1, 2, \ldots \]

(24)

The above equation is found to be identical to energy formula (15) of Ref. [37].
Therefore, taking the set of parameters given in Table 1 of Ref. [37], we can obtain the energy states displayed in Table 2 of Ref. [37] for various diatomic molecules. It is necessary to state that we can also obtain the bound state solutions in any arbitrary dimension $D$ by making the following replacement $l \rightarrow l + (D - 3)/2$ throughout this work [37]. We can also make replacement to the radial wave function in $D$-dimensional space, see Eq.(2), as $R(r) = r^{-1-(D-3)/2}g(r)$ [38].

4. CONCLUSION

To sum up, in this work, we have obtained bound state energies and wave functions of the relativistic spinless particle subject to a PHO interaction using Laplace transform method. We explored the solution of the spinless KG energy states. The Schrödinger bound state solution is found as non-relativistic limit of the present model. It is noticed that the solution with equal mixture of scalar and vector potentials can be easily reduced into the well-known non-relativistic solution for a particle with an interaction potential field and a free field, respectively. We have considered the solution for any vibrational-rotational quantum state by making appropriate transformations. This procedure should be helpful for students on a mathematical physics course.

REFERENCES

8. Ikhdair S M and Sever R 2009 Int. J. Mod. Phys. C 20 361
29. de Castro A S 2012 Rev. Bras. Ens. Fis. 34 4301
30. Ikhdair S M 2013 Exact bound states to the relativistic spinless particle by a special Morse potential and its non-relativistic limit, Mod. Phys. Lett. A (submitted)
33. Xu Y, He S and Jia C S 2010 Phys. Scr. 81 045001