Romanian Reports in Physics, Vol. 66, No. 4, P. 923–928, 2014

MATHEMATICAL PHYSICS

LIE SYMMETRY ANALYSIS AND EXACT SOLUTIONS FOR A VARIABLE COEFFICIENT GENERALISED KURAMOTO-SIVASHINSKY EQUATION

RAJEEV KUMAR[1], R.K. GUPTA[2], S.S. BHATIA[3]

[1]Department of Mathematics,
Maharishi Markandeshwar University,
Mullana, Ambala-133001(Haryana), India
E-mail: rajeevkumarbudhiraja@gmail.com
[2,3]School of Mathematics and Computer Applications,
Thapar University, Patiala-147004 (Punjab), India

Received December 6, 2013

Abstract. In this paper a variable-coefficient Generalised Kuramoto-Sivashinsky Equation is considered. By using the Lie classical symmetry analysis method symmetries for this equation are obtained. Some new exact solutions for the considered PDE are obtained.

Key words: Exact solution, symmetry analysis, travelling wave solutions.

PACS: 04.20.Jb, 02.30.Jr, 02.20.Sv, 11.30.J

1. INTRODUCTION

The investigation of solutions to nonlinear evolution equations plays an important role in the study of nonlinear physical phenomena. The wave phenomena are observed in physics, mechanics, biology, etc. The effort to find these solutions is significant for the understanding of many physical phenomena, thus they may give more insight into the physical aspects of the problems [1–8]. This paper is devoted to study the solutions of the variable coefficient Kuramoto-Sivashinsky Equation [9–11] which is given by

$$\frac{\partial}{\partial t}u(x,t)+\left(\frac{\partial}{\partial x}u(x,t)\right)(u(x,t))^m+a(t)\frac{\partial^2}{\partial x^2}u(x,t) + b(t)\frac{\partial^3}{\partial x^3}u(x,t)+c(t)\frac{\partial^4}{\partial x^4}u(x,t)=0, \quad (1)$$

where $a(t)$, $b(t)$ and $c(t)$ are function of t.

Some special cases of (1) with constant coefficients were also considered

$$\frac{\partial}{\partial t}u(x,t)+\left(\frac{\partial}{\partial x}u(x,t)\right)(u(x,t))^m+a\frac{\partial^2}{\partial x^2}u(x,t)$$
$$+b\frac{\partial^3}{\partial x^3}u(x,t)+c\frac{\partial^4}{\partial x^4}u(x,t)=0 \quad (2)$$

$$\frac{\partial}{\partial t}u(x,t)+\alpha\left(\frac{\partial}{\partial x}u(x,t)\right)(u(x,t))^m+\beta\frac{\partial^2}{\partial x^2}u(x,t)+k\frac{\partial^4}{\partial x^4}u(x,t)=0 \quad (3)$$

Nonlinear evolution equation (1) at $m=1$ has been studied by a number of authors from various viewpoints. Exact solutions of (3) in the form of the solitary wave at $m=1$ and at $\beta=0$ were obtained in [12]. The rest of this paper is arranged as follows. In section 2, we apply the Lie classical symmetry analysis method [13]-[18] to (1). In section 3, the Series solution method [19] and the Extended Tanh method [20, 21] are applied to find the solutions of the equation. Some new exact solutions of (1) are obtained.

2. CLASSICAL SYMMETRIES

To apply the Lie classical symmetry analysis method to (1) we consider the one-parameter Lie group of infinitesimal transformations in x, t and u given by

$$x^*=x+\epsilon\xi(x,t,u),\quad t^*=t+\epsilon\tau(x,t,u),\quad u^*=u+\epsilon\eta(x,t,u), \quad (4)$$

where ϵ is the group parameter. We require that the set of solutions of (1) be invariant under this transformation. This yields an overdetermined system of linear equations for the infinitesimals $\xi(x,t,u)$, $\tau(x,t,u)$ and $\eta(x,t,u)$. The associated Lie algebra of infinitesimal symmetries is the set of infinitesimal generators of the form

$$v=\xi(x,t,u)\frac{\partial}{\partial x}+\tau(x,t,u)\frac{\partial}{\partial t}+\eta(x,t,u)\frac{\partial}{\partial u} \quad (5)$$

Invariance of (1) under a Lie group of point transformations with infinitesimal generator (5) leads to a set of determining equations. Solving this system we will obtain:

$$\eta=k_1u,\quad \xi=k_2x+k_3,\quad \tau=(k_2-mk_1)t+k_4.$$

The coefficient functions are governed by the following relations:

$$a=c_1\left[(k_2-mk_1)t+k_4\right]^{(k_2+mk_1)/(k_2-mk_1)} \quad (6)$$

$$b=c_2\left[(k_2-mk_1)t+k_4\right]^{(2k_2+mk_1)/(k_2-mk_1)} \quad (7)$$

$$c=c_3\left[(k_2-mk_1)t+k_4\right]^{(3k_2+mk_1)/(k_2-mk_1)} \quad (8)$$

where, c_1, c_2, c_3 , k_1 , k_2 , k_3 and k_4 are arbitrary constants. The associated infinitesimal generators are given by:

$$v_1 = u\frac{\partial}{\partial u} - mt\frac{\partial}{\partial t}, \quad v_2 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}, \quad v_3 = \frac{\partial}{\partial x}, \quad v_4 = \frac{\partial}{\partial t}. \tag{9}$$

Considering the infinitesimal generator $v_1 + \lambda v_2 + \mu v_4$ we obtain the surface condition

$$((\lambda - m)t + \mu)\frac{\partial u}{\partial t} + \lambda x\frac{\partial u}{\partial x} = u, \tag{10}$$

which when solving will lead to the similarity transformation

$$u = x^{\frac{1}{\lambda}} f(\xi), \quad \xi = x\left[(\lambda - m)t + \mu\right]^{\frac{\lambda}{m-\lambda}} \tag{11}$$

when substituting from (11) into (1) we'll obtain the following ordinary differential equation

$$-f(\xi) - \lambda\xi f'(\xi) + f^m f'(\xi) + c_3 f''\xi + c_2 f'''\xi + c_1 f''''\xi = 0 \tag{12}$$

Considering the infinitesimal generator $v_3 + \omega v_4$ we obtain the surface condition

$$\frac{\partial u}{\partial x} + \omega\frac{\partial u}{\partial t} \tag{13}$$

which leads to the similarity transformation

$$\xi = (x - \omega t), \quad u = f(\xi), \tag{14}$$

when substituting from (14) into (1) we will obtain the following ordinary differential equation

$$c_4 f''''(\xi) + c_5 f'''(\xi) + c_6 f''(\xi) + f'(\xi) f^m(\xi) - \omega f'(\xi) = 0, \tag{15}$$

where c_4, c_5 and c_6 are constants.

3. ANALYSIS OF REDUCED ORDINARY DIFFERENTIAL EQUATION

Our main goal is to derive exact solutions of (1). As exact solutions are helpful for mathematical as well as physical description of nature. In this Section, we will look for some exact solutions of (1) by the reduced equations in Section 2.

3.1. SOLUTION OF EQUATION (12)

We seek the solutions of (12) in the following form

$$f(\xi) = A\xi^p, \tag{16}$$

where A and p are constants to be found out. We need to equate the exponents of ξ suitably such that their respective coefficients become zero. Put $\lambda = 1$ in (12) and integrate we get

$$c_1 f^{'''}(\xi) + c_2 f^{''}(\xi) + c_3 f^{'}(\xi) + \frac{f^{m+1}(\xi)}{(m+1)} - \xi f(\xi) = 0, \tag{17}$$

where constant of integration is taken as zero. By equating the exponents $(p-3)$ and pm, we have $p = (-3/m)$. On substituting (16) into (17), we get the following solution of (1).

$$u = \left[\frac{-m^3}{c_1(1+m)(6m^2+27m+27)}\right]^{\frac{-1}{m}} x^{\frac{-3}{m}} \left[(1-m)t+\mu\right]^{\frac{-(m+3)}{m(m+1)}} \tag{18}$$

3.2. SOLUTION OF EQ. 15

On integrating (15) we get

$$c_4 f^{'''}(\xi) + c_5 f^{''}(\xi) + c_6 f^{'}(\xi) + \frac{f^{(m+1)}(\xi)}{(m+1)} - \omega f(\xi) = 0 \tag{19}$$

where constant of integration is taken as zero.

Case I

put $m = 3$ we get in (19)

$$c_4 f^{'''}(\xi) + c_5 f^{''}(\xi) + c_6 f^{'}(\xi) + \frac{f^{(4)}(\xi)}{(4)} - \omega f(\xi) = 0 \tag{20}$$

We find the solution of above non linear ODE by using Extended Tanh Method. Balancing $f^{'''}$ with f^4 in (20) we get $n = 1$. Introducing new independent variable $y = \tanh\xi$, By Extended Tanh Method we get

$$f(\xi) = a_0 + a_1 y + a_2 y^{-1} \tag{21}$$

Substituting (21) in (20) and equating the coefficient of powers of y we obtain the system of algebraic equations for a_0, a_1, a_2 and ω. On solving the algebraic equations we get

$a_0 = 0$, $a_1 = \sqrt[3]{24c_4}$, $a_2 = \frac{(c_6-8c_4)}{(24c_4)^{\frac{2}{3}}}$, $\omega = -2c_5$

$a_0 = \frac{-1}{12}\frac{c_5\sqrt[3]{24}}{c_4^{2/3}}$, $a_1 = 0$, $a_2 = \sqrt[3]{24c_4}$, $\omega = -\frac{1}{288}\frac{3456\,c_6 c_4^3 - 6912\,c_4^4 + c_5^4}{c_5 c_4^2}$

$a_0 = \frac{-1}{6}\frac{c_5\sqrt[3]{3}}{c_4^{2/3}}$, $a_1 = 2\sqrt[3]{3c_4}$, $a_2 = 0$, $\omega = -\frac{1}{216}\frac{c_5\left(c_5^2\sqrt[3]{3} + 216\,a_2 c_4^{5/3} + 144\sqrt[3]{3}c_4^2\right)3^{2/3}}{c_4^2}$

Solutions

$$u = \sqrt[3]{24c_4}\tanh(x-\omega t) + \frac{(c_6-8c_4)}{(24c_4)^{\frac{2}{3}}}\tanh^{-1}(x-\omega t) \tag{22}$$

$$u = \frac{-c_5}{12}\sqrt[3]{\frac{24}{c_4^2}} + \sqrt[3]{24c_4}\tanh^{-1}(x-\omega t) \tag{23}$$

$$u=\frac{-1}{6}\frac{c_5\sqrt[3]{3}}{{c_4}^{2/3}}+\left(2\sqrt[3]{3}\sqrt[3]{c_4}\right)\tanh\left(x-\omega t\right) \tag{24}$$

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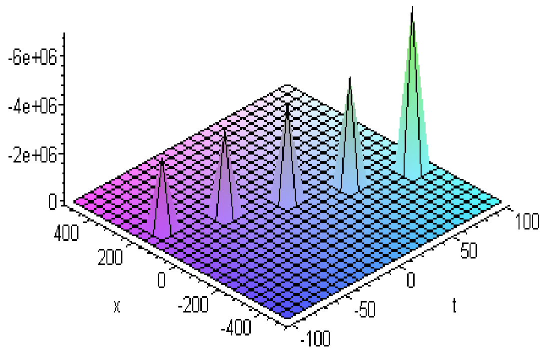

Fig. 1 – Wave solutions of (22) by putting $c_4=1, c_5=1, c_6=1$.

Case II

put $m=1$ we get in (19).

$$c_4 f^{'''}\left(\xi\right)+c_5 f^{''}\left(\xi\right)+c_6 f^{'}\left(\xi\right)+\frac{f^{(2)}\left(\xi\right)}{(2)}-\omega f\left(\xi\right)=0 \tag{25}$$

Balancing $f^{'''}$ with f^2 in (25) we get $n=3$. By Extended Tanh Method we get

$$f\left(\xi\right)=a_0+a_1 y+a_2 y^2+a_3 y^3+a_4 y^{-1}+a_5 y^{-2}+a_6 y^{-3} \tag{26}$$

Subsituting (26) in (25) and equating the coefficient of powers of y then we obtain the system of algebraic equations for $a_i(i=\overline{0,6})$, and ω.

On solving the algebraic equations we get

$a_0=c_6, a_1=2c_6, a_2=0, a_3=0, a_4=0, a_5=-12c_5, a_6=0, \omega=c_6$

$a_0=0, a_1=0, a_2=0, a_3=0, a_4=\frac{60}{19}(c_6-20c_4), a_5=0, a_6=120c_4, \omega=-2c_6$

$a_0=2c_6, a_1=2c_6, a_2=0, a_3=0, a_4=0, a_5=-12c_5, a_6=120c_4, \omega=2c_6$

$a_0=0, a_1=0, a_2=\frac{-1}{4}\frac{c_5c_6}{c_4}, a_3=0, a_4=0, a_5=15c_5, a_6=120c_4, \omega=-2c_5$

$a_0=0, a_1=2c_6, a_2=\frac{-c_6c_5}{(c_6-3c_4)}, a_3=0, a_4=2c_6-12c_4, a_5=0, a_6=0, \omega=-2c_5$

Solutions

$$u=c_6+2c_6\tanh\left(x-\omega t\right)-12c_5\tanh^{-2}\left(x-\omega t\right) \tag{27}$$

$$u=\left(\frac{60}{19}c_6-\frac{1200}{19}c_4\right)\tanh^{-1}\left(x-\omega t\right)+120c_4\tanh^{-3}\left(x-\omega t\right) \tag{28}$$

$$u=2c_6+2c_6\tanh\left(x-\omega t\right)-12c_5\tanh^{-2}\left(x-\omega t\right)+120c_4\tanh^{-3}\left(x-\omega t\right) \tag{29}$$

$$u = \frac{-1}{4}\frac{c_5 c_6}{c_4}\tanh^2(x-\omega t) + 15c_5\tanh^{-2}(x-\omega t) + 120c_4\tanh^{-3}(x-\omega t) \quad (30)$$

$$u = \frac{-c_6 c_5}{(c_6-3c_4)}\tanh^2(x-\omega t) + 2c_6\tanh(x-\omega t) + \frac{(2c_6-12c_4)}{\tanh(x-\omega t)} \quad (31)$$

4. CONCLUSION

Using the classical symmetry analysis method we obtained similarity transformations (11) and (14) that, for arbitrary $a(t)$, $b(t)$, $c(t)$, transform the original nonlinear PDE (1) into ODEs (12) and (15) respectively. Then we used the Extended Tanh Method to obtain many solutions for (1). To our best knowledge, the set of solutions u are new solutions of (1) and are not shown in the current literature until now. The availability of computer systems like Mathematica or Maple facilitates the tedious algebraic calculations.

REFERENCES

1. H. Triki *et al.*, Rom. J. Phys. **57**, 1029–1034 (2012).
2. A. Biswas *et al.*, Proc. of the Romanian Acad. A **13**, 32–41 (2012).
3. S. Crutcher *et al.*, Rom. Rep. Phys. **64**, 367–380 (2012).
4. L. Girgis *et al.*, Rom. Rep. Phys. **64**, 663–671 (2012).
5. B. S. Ahmed, E. Zerrad and A. Biswas, Proc. of the Romanian Acad. A. **14**, 281–286 (2013).
6. G. Ebadi *et al.*, Rom. Rep. Phys. **64**, 915–932 (2012).
7. H. Triki *et al.*, Rom. Rep. Phys. **64**, 672–684 (2012).
8. G. Ebadi *et al.*, Rom. J. Phys. **58**, 3–14 (2013).
9. R. Conte and M. Musette, J. Phys. A: Math. and Gen. **22**, 169–177 (1989).
10. Y. Kuramoto and T. Tsuzuki, Prog. of Theo. Phys. **55**, 356–369 (1976).
11. N.A. Kudryashov, D.I. Sinelschikov, I.L. Chernyavsky, Rus. J. Nonlin. Dyn. **4**, 69–86 (2008).
12. N.A. Kudryashov, Regular and Chaotic Dyna. **13**, 234–238 (2008).
13. G.W. Bluman and S.C. Anco, Appl. Math. Sci. **154** (2002).
14. P.J. Olver, *Applications of Lie Groups to Differential Equations* (Springer-Verlag, 1993).
15. K. Singh and R.K. Gupta, Int. J. Eng. Sci. **44**, 241–255 (2006).
16. R.K. Gupta and K. Singh, Comm. in Nonlinear Sci. and Num. Simu. **16**, 4189–4196 (2011).
17. G. Ebadi *et al.*, Rom. Rep. Phys. **65**, 27–62 (2013).
18. A. Biswas *et al.*, Rom. Rep. Phys. **65**, 1125–1137 (2013).
19. A. Esfahani, Comm. in Theo. Phys. **55**, 381–386 (2011).
20. A.H. Khater *et al.*, Chaos, Solitons and Fractals **14**, 513–522 (2002).
21. A.M. Wazwaz, App. Math. and Comp. **154**, 713–723 (2004).