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A NOTE ON PROPER PROJECTIVE COLLINEATION IN SPECIAL NON-STATIC SPHERICALLY SYMMETRIC SPACE-TIMES

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Abstract. In this paper a study of proper projective collineation in special non-static spherically symmetric space-times is given by using real eigenvalues and eigenbivectors of the Riemann tensor, direct integration and algebraic techniques. From the above study we have shown that when the above space-time admits proper projective collineation they become a very special class of static spherically symmetric space-times.

Key words: real eigenvalues and eigenbivectors, direct integration and algebraic techniques, proper projective collineation.

1. INTRODUCTION

The aim of this paper is to find the existance of proper projective collineation in special non-static spherically symmetric space-times. In this paper an approach, which is given in [1] is used to study the projective collineation for the above space-times. Doubtless, this approach is lengthy but it will definitely tell the existance of a proper projective collineation. Throughout M represents a four dimensional, connected, Hausdorff space-time manifold with Lorentz metric g of signature (–, +, +, +). The curvature tensor associated with g_{ab}, through the Levi-Civita connection, is denoted in component form by $R^a{}_{bcd}$, and the Ricci tensor components are $R_{ab} = R^c{}_{acb}$. The usual covariant, partial and Lie derivatives are denoted by a semicolon, a comma and the symbol L, respectively. Round and square brackets denote the usual symmetrization and skew-symmetrization, respectively. The space-time M will be assumed non flat in the sense that the curvature tensor does not vanish over any non empty open subset of M.

Any vector field X on M can be decomposed as

$$X_{a;b} = \frac{1}{2}h_{ab} + F_{ab}, \tag{1}$$

where $h_{ab}(=h_{ba})=L_X g_{ab}$ and $F_{ab}(=-F_{ba})$ are symmetric and skew symmetric tensors on M, respectively. Such a vector field X is called projective if the local diffeomorphisms ψ_t (for appropriate t) associated with X map geodesics into geodesics. This is equivalent to the condition that h_{ab} satisfies [2]

$$h_{ab;c}=2g_{ab}\eta_c+g_{ac}\eta_b+g_{bc}\eta_a, \tag{2}$$

for some smooth closed 1-form on M with local components η_a. Thus η_a is locally gradient because the connection is metric and will, where appropriate, be written as $\eta_a=\eta_{,a}$ for some function η on some open subset of M. If X is a projective collineation and $\eta_{a;b}=0$ then X is called a special projective collineation on M. The statement that h_{ab} is covariantly constant on M is, from (2), equivalent to η_a being zero on M and is, in turn equivalent to X being an affine vector field on M (so that the local diffeomorphisms ψ_t preserve not only geodesics but also their affine parameters). If X is projective but not affine then it is called proper projective collineation [3]. The vector field X is said to be proper special projective collineation, if X is not affine and $\eta_{a;b}=0$. Further if X is affine and $h_{ab}=2cg_{ab}, c\in R$ then X is homothetic (otherwise proper affine). If X is homothetic and $c\neq 0$ it is proper homothetic while if $c=0$ it is Killing.

The second order skew symmetric tensor F_{ab} is called a bivector (at p). Regarding F_{ab} as a skew matrix, its rank is therefore an even number 0, 2 or 4. If it is 0 then $F_{ab}=0$. Suppose if the rank of F_{ab} is 2 then it is called simple bivector otherwise it is called non-simple (for more detalis see [3]). Here, at $p\in M$ one may choose a orthonormal tetrad $(t,\ r,\ \theta,\ \phi)$ satisfying $-t^a t_a=r^a r_a=\theta^a\theta_a=\phi^a\phi_a=1$ (with all others inner products zero). Since at p, the set of bivectors at p is a six-dimensional vector space which can spanned by the six bivectors given by [3]

$$^1F_{ab}=2t_{[a}r_{b]},\quad ^2F_{ab}=2t_{[a}\theta_{b]},\quad ^3F_{ab}=2t_{[a}\phi_{b]},$$
$$^4F_{ab}=2r_{[a}\theta_{b]},\quad ^5F_{ab}=2r_{[a}\phi_{b]},\quad ^6F_{ab}=2\theta_{[a}\phi_{b]}.$$

In general, however, equation (2) is difficult to handle directly and alternative techniques are needed. One such technique arises from the following result. Let X be a projective collineation on M so that (1) and (2) hold and let F be a real curvature eigenbivector at $p \in M$ with eigenvalue $\lambda \in R$ (so that $R^{ab}{}_{cd} F^{cd} = \lambda F^{ab}$ at p) then at p one has [4]

$$P_{ac} F^c{}_b + P_{bc} F^c{}_a = 0 \qquad (P_{ab} = \lambda h_{ab} + 2\eta_{a;b}). \tag{3}$$

Equation (3) gives a relation between $F^a{}_b$ and P_{ab} (a second order symmetric tensor) at p and reflects the close connection between h_{ab}, $\eta_{a;b}$ and the algebraic structure of the curvature at p. If F is simple then the blade of F (a two dimensional subspace of T_pM) consists of eigenvectors of P with same eigenvalue. Similarly, if F is non-simple then it has two well defined orthogonal timelike and spacelike blades at p each of which consists of eigenvectors of P with same eigenvalue but with possibly different eigenvalues for the two blades [3].

2. MAIN RESULTS

Consider a non static spherically symmetric space-time in the usual coordinate system (t, r, θ, ϕ) (labeled by (x^0, x^1, x^2, x^3), respectively) with line element [5]

$$\mathrm{d}s^2 = -\mathrm{e}^{A(t,r)}\mathrm{d}t^2 + \mathrm{e}^{B(t,r)}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \tag{4}$$

The above space-time (4) admits three linearly independent Killing vector fields which are

$$\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\varphi}, \quad \sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}, \quad \frac{\partial}{\partial\phi}. \tag{5}$$

The non-zero independent components of the Riemann tensor are

$$R^{01}{}_{01} = -\frac{1}{4}\mathrm{e}^{-A(t,r)-B(t,r)}\begin{bmatrix}\mathrm{e}^{A(t,r)}(A_r^2(t,r) + 2A_{rr}(t,r)) - \mathrm{e}^{B(t,r)}(B_t^2(t,r) + 2B_{tt}(t,r) - \\ A_t(t,r)B_t(t,r)) - \mathrm{e}^{A(t,r)}A_r(t,r)B_r(t,r)\end{bmatrix} \equiv \alpha_1,$$

$$R^{02}{}_{02} = R^{03}{}_{03} = -\frac{1}{2r}\mathrm{e}^{-B(t,r)}A_r(t,r) \equiv \alpha_2, \; R^{12}{}_{12} = R^{13}{}_{13} = \frac{1}{2r}B_r(t,r)\mathrm{e}^{-B(t,r)} \equiv \alpha_3,$$

$$R^{23}{}_{23} = \frac{1}{r^2}(1 - \mathrm{e}^{-B(t,r)}) \equiv \alpha_4, \; R^{02}{}_{12} = R^{03}{}_{13} = -\frac{1}{2r}B_t(t,r)\mathrm{e}^{-A(t,r)} \equiv \alpha_5,$$

$$R^{12}{}_{02} = R^{13}{}_{03} = -\frac{1}{2r}B_t(t,r)\mathrm{e}^{-B(t,r)} \equiv \alpha_6, \tag{6}$$

where α_1, α_2, α_3, α_4, α_5 and α_6 are real functions of t and r only. It also follows that $\alpha_6 = e^{A-B}\alpha_5$. One can write the above equation (6) as

$$\begin{aligned} R^{ab}{}_{cd}\,{}^{1}F^{cd} &= \alpha_1{}^{1}F^{ab}, & R^{ab}{}_{cd}\,{}^{2}F^{cd} &= \alpha_2{}^{2}F^{ab} + \alpha_6{}^{4}F^{ab}, \\ R^{ab}{}_{cd}\,{}^{3}F^{cd} &= \alpha_2{}^{3}F^{ab} + \alpha_6{}^{5}F^{ab}, & R^{ab}{}_{cd}\,{}^{4}F^{cd} &= \alpha_3{}^{4}F^{ab} + \alpha_5{}^{2}F^{ab}, \\ R^{ab}{}_{cd}\,{}^{5}F^{cd} &= \alpha_3{}^{5}F^{ab} + \alpha_5{}^{3}F^{ab}, & R^{ab}{}_{cd}\,{}^{6}F^{cd} &= \alpha_4{}^{6}F^{ab}. \end{aligned} \tag{7}$$

It is important to note that the method we are going to follow is given in [1]. Define ${}^{1}W_{ab} = {}^{4}F_{ab} + \lambda\,{}^{2}F_{ab}$ and ${}^{2}W_{ab} = {}^{5}F_{ab} + \eta\,{}^{3}F_{ab}$. We are interested to find that whether these bivectors ${}^{1}W_{ab}$ and ${}^{2}W_{ab}$ are the real eigenbivectors of the Riemann tensor for the real choice of λ and η. Equation (7) gives

$$R^{ab}{}_{cd}\,{}^{1}W^{cd} = (\alpha_3 + \lambda\alpha_6)\left({}^{4}F^{ab}\right) + \frac{(\alpha_5 + \lambda\alpha_2)}{(\alpha_3 + \lambda\alpha_6)}\,{}^{2}F^{ab})$$

and

$$R^{ab}{}_{cd}\,{}^{2}W^{cd} = (\alpha_3 + \lambda\alpha_6)\left({}^{5}F^{ab} + \frac{\alpha_5 + \eta\alpha_2}{\alpha_3 + \eta\alpha_6}\right){}^{3}F^{ab}).$$

${}^{1}W_{ab}$ and ${}^{2}W_{ab}$ are the eigenbivectors of the Riemann tensor for the choice of $\lambda_1 = \eta_1 = \frac{(\alpha_2 - \alpha_3) + \rho}{2\alpha_6}$ and $\lambda_2 = \eta_2 = \frac{(\alpha_2 - \alpha_3) - \rho}{2\alpha_6}$, where $\rho^2 = (\alpha_3 - \alpha_2)^2 - 4\,\alpha_5\alpha_6$ and $\alpha_6 \neq 0$. The case when $\alpha_6 = 0$ will be discuss later. One can easily see that in general λ_1, η_1, λ_2 and η_2 are not real. Hence the eigenbivectors of the Riemann tensor are not real in general. Here, we are only interested in real eigenbivectors of the Riemann tensor with real eigenvalues. There exist the following possibilities $\rho^2 \geq 0$ or $(\alpha_3 - \alpha_2)^2 - 4\alpha_5\alpha_6 \geq 0$. Substituting the above information back we get

$$R^{ab}{}_{cd}\left({}^{4}F^{cd} + \lambda_1{}^{2}F^{cd}\right) = \frac{(\alpha_3 + \alpha_2 + \rho)}{2}\left({}^{4}F^{ab} + \lambda_1{}^{2}F^{ab}\right),$$

$$R^{ab}{}_{cd}\left({}^{5}F^{cd} + \lambda_2{}^{3}F^{cd}\right) = \frac{(\alpha_3 + \alpha_2 - \rho)}{2}\left({}^{5}F^{ab} + \lambda_2{}^{3}F^{ab}\right).$$

The bivectors ${}^{1}W^{ab} = ({}^{4}F^{ab} + \lambda^{12}F^{ab})$ and ${}^{2}W^{ab} = ({}^{5}F^{ab} + \lambda_2\ {}^{3}F^{ab})$ are simple. Now define $\gamma_1 = \frac{1}{2}(\alpha_3 + \alpha_2 + \rho)$ and $\gamma_2 = \frac{1}{2}(\alpha_3 + \alpha_2 - \rho)$. Here, at $p \in M$ one can choose the tetrad (t, r, θ, ϕ) satisfying $-t^a t_a = r^a r_a = \theta^a\theta_a = \phi^a\phi_a = 1$ (with all other inner products zero). Here the vector fields are chosen as $t_a = e^{\frac{A}{2}}\delta_a^0$, $r_a = e^{\frac{B}{2}}\delta_a^1$, $\theta_a = r\delta_a^2$ and $\phi_a = r\sin\theta\,\delta_a^3$. It is important to note that we are using (t, r, θ, ϕ) as both coordinates and vector fields. The eigenbivectors ${}^1W_{ab}$ and ${}^2W_{ab}$ of the curvature tensor at $p \in M$ are simple with blades spanned by the vector pairs $(r+\lambda_1 t, \theta)$ each with eigenvalue $\gamma_1(p)$ and $(r+\lambda_2 t, \phi)$ each with eigenvalue $\gamma_2(p)$. We consider the open sub region where γ_1 and γ_1 are nowhere equal. The rest will be considered latter. It is important to note that we are considering the case when $\rho^2 > 0$. The case when $\rho = 0 \Rightarrow \gamma_1 = \gamma_2$ which gives contradiction to our assumption that γ_1 and γ_1 are nowhere equal. The case when $\gamma_1 = \gamma_2$ will consider later. At p, the second order symmetric tensors is a 10-dimensional vector space which can spanned by the 10 basis symmetric tensors given by: L_aL_b, S_aS_b, $\theta_a\theta_b$, $\phi_a\phi_b$, $2L_{(a}S_{b)}$, $2L_{(a}\theta_{b)}$, $2L_{(a}\phi_{b)}$, $2S_{(a}\theta_{b)}$, $2S_{(a}\phi_{b)}$ and $2\theta_{(a}\phi_{b)}$, where $L_a \equiv r_a + \lambda_1 t_a$ and $S_a \equiv r_a + \lambda_2 t_a$. It also follows that $L_aS^a = 1-\lambda_1\lambda_2$, $L_aL^a = 1-\lambda_1^2$, $S_aS^a = 1-\lambda_2^2$ and λ_1 and λ_2 are nowhere equal. Now at p, the symmetric tensor $P_{ab}(=\gamma_1 h_{ab} + 2\eta_{a;b})$ can be written as a linear combination of the basis members of the 10-dimensional vector space and use the fact that $P_{ab} = \gamma_1 h_{ab} + 2\eta_{a;b}$ has eigenvectors $r+\lambda_1 t, \theta$ with same eigenvalue, say ζ_1 and similarly for the symmetric tensor $P_{ab}(=\gamma_2 h_{ab} + 2\eta_{a;b})$ can be written as a linear combination of the basis members of the 10-dimensional vector space and use the fact that $P_{ab} = \gamma_2 h_{ab} + 2\eta_{a;b}$ has eigenvectors $r+\lambda_2 t, \phi$ with same eigenvalue, say ζ_2. Hence on M one has after the use of completeness relation ($g_{ab} = \gamma_3 L_aL_b + \gamma_4 S_aS_b + \theta_a\theta_b + \phi_a\phi_b$)

$$\begin{aligned} \gamma_1 h_{ab} + 2\eta_{a;b} &= \zeta_1 g_{ab} + a_1 S_aS_b + b_1\phi_a\phi_b + c_1 S_a\phi_b + c_1 S_b\phi_a, \\ \gamma_2 h_{ab} + 2\eta_{a;b} &= \zeta_2 g_{ab} + a_2 L_aL_b + b_2\theta_a\theta_b + c_2 L_a\theta_b + c_2 L_b\theta_a, \end{aligned} \tag{8}$$

where a_1, a_2, b_1, b_2, c_1 and c_2 are real functions on M. Since $\gamma_1 \neq \gamma_2$ equation (8) gives

$$\begin{aligned} h_{ab} &= a_3 g_{ab} + a_4 S_aS_b + a_5 L_aL_b + a_6\phi_a\phi_b + a_7\theta_a\theta_b + 2a_8 S_{(a}\phi_{b)} + 2a_9 L_{(a}\theta_{b)}, \\ \eta_{a;b} &= b_3 g_{ab} + b_4 S_aS_b + b_5 L_aL_b + b_6\phi_a\phi_b + b_7\theta_a\theta_b + 2b_8 S_{(a}\phi_{b)} + 2b_9 L_{(a}\theta_{b)}, \end{aligned} \tag{9}$$

where $a_3, a_4, a_5, a_6, a_7, a_8, a_9, b_3, b_4, b_5, b_6, b_7, b_8$ and b_9 are functions on some open subregion of M. Now we are interested in finding the projective vector fields by using the relation

$$L_X g_{ab} = h_{ab}, \qquad \forall \quad a,\ b = 0,\ 1,\ 2,\ 3. \tag{10}$$

Writing equation (10) explicitly and using first equation of (9) and (4) we get ten coupled non linear equations. In order to find the projective vector field we need to solve ten equations. After some tedious and lengthy calculation one finds that $\eta_a = 0$. Hence no proper projective collineations exist in this case. Projective collineations in this case are Killing vector fields.

Now consider the case when $\gamma_1 = \gamma_2$. Equation $\gamma_1 = \gamma_2 \Rightarrow \rho = 0$. Substituting back and again writing equation (10) explicitly in to ten equations. After some lengthy calculation one finds that $\eta_a = 0$ which implies in this case no proper projective collineations exists. Projective collineations in this case are Killing vector fields.

Now consider the case when $\alpha_6 = 0 \Rightarrow B_t(t,\ r) = 0 \Rightarrow B = B(r)$ and also we have $\alpha_5 = 0$. It is important to mention here that throughout in this paper we have $B = B(r)$. Substituting $\alpha_6 = 0$ and $\alpha_5 = 0$ in equation (7) we get

$$R^{ab}{}_{cd}\ {}^{1}F^{cd} = \alpha_1\,{}^{1}F^{ab}, \qquad R^{ab}{}_{cd}\ {}^{2}F^{cd} = \alpha_2\,{}^{2}F^{ab},$$
$$R^{ab}{}_{cd}\ {}^{3}F^{cd} = \alpha_2\,{}^{3}F^{ab}, \qquad R^{ab}{}_{cd}\ {}^{4}F^{cd} = \alpha_3\,{}^{4}F^{ab},$$
$$R^{ab}{}_{cd}\ {}^{5}F^{cd} = \alpha_3\,{}^{5}F^{ab}, \qquad R^{ab}{}_{cd}\ {}^{6}F^{cd} = \alpha_4\,{}^{6}F^{ab}.$$

Here, at $p \in M$ one can choose the tetrad (t, r, θ, ϕ) satisfying $-t^a\, t_a = r^a\, r_a = \theta^a\, \theta_a = \phi^a\, \phi_a = 1$ (with all other inner products zero) such that the eigenbivectors of the curvature tensor at $p \in M$ are all simple with blades spanned by the vector pairs (t,θ), (t,φ) each with eigenvalue $\alpha_2(p)$ and (r,θ), (r,φ) each with eigenvalue $\alpha_3(p)$. Here the vector fields are chosen as $t_a = \mathrm{e}^{\frac{A}{2}}\delta^0_a$, $r_a = \mathrm{e}^{\frac{B}{2}}\delta^1_a$, $\theta_a = r\delta^2_a$ and $\phi_a = r\sin\theta\,\delta^3_a$. We are considering the open sub region where α_2 and α_3 are nowhere equal and $\alpha_2 \neq 0$. The rest will be considered latter. It is important to note that we are using (t, r, θ, ϕ) as both coordinates and vector fields. Thus at p, the tensor $P_{ab} = \alpha_2\ h_{ab} + 2\ \eta_{a;b}$ has eigenvectors t, θ, φ with same eigenvalue, say β_1 and $P_{ab} = \alpha_3\ h_{ab} + 2\ \eta_{a;b}$ has eigenvectors r, θ, φ with same eigenvalue, say β_2. Hence on M one has after the use of completeness relation $(g_{ab} = -t_a t_b + r_a r_b + \theta_a \theta_b + \phi_a \phi_b)$

$$\alpha_2 h_{ab} + 2\eta_{a;b} = \beta_1\, g_{ab} + \beta_3\, r_a r_b, \qquad \alpha_3 h_{ab} + 2\eta_{a;b} = \beta_2 g_{ab} + \beta_4 t_a t_b, \tag{11}$$

where $\beta_1, \beta_2, \beta_3$ and β_4 are real functions on *M*. Since $\alpha_2 \neq \alpha_3$ then it follows from (6) that

$$h_{ab} = Q\, g_{ab} + D\, r_a r_b + E\, t_a t_b,\ \eta_{a;b} = F\, g_{ab} + G\, r_a r_b + K\, t_a t_b, \tag{12}$$

where $D, E, F, G,\ K$ and Q are functions on some open subregion of M. Next one substitutes the first equation of (12) in (2), we get

$$\begin{aligned} &Q_c \quad g_{ab} + D_c \quad r_a r_b + D \quad r_{a;c} r_b + D \quad r_{b;c} r_a + E_c \quad t_a t_b + E \quad t_{a;c} t_b + E \quad t_{b;c} t_a = \\ &\quad = 2 \quad g_{ab} \quad \eta_c + g_{ac} \quad \eta_b + g_{bc} \quad \eta_a . \end{aligned} \tag{13}$$

Contracting the above equation with $\theta^a \phi^b$, and then comparing both sides, we have $\eta_a \theta^a = \eta_a \phi^a = 0$ which implies $\eta = \eta(t, r)$. Now contracting equation (13) with $\theta^a \theta^b$ we get $Q_{,c} = 2\eta_c$ which implies $Q = Q(t, r)$. Once again contracting equation (13) with $t^a t^b$ and $r^a r^b$, we get $E = E(t)$ and $D = D(r)$, respectively. Now consider the first equation of (12) and using (4), we get the following non-zero components of h_{ab}

$$\begin{aligned} &h_{00} = [E(t) - Q(t,r)] \ \mathrm{e}^{A(t,r)}, \qquad &&h_{11} = [Q(t,r) + D(r)] \ \mathrm{e}^{B(r)}, \\ &h_{22} = Q(t,r) \ r^2, &&h_{33} = Q(t,r) \ r^2 \ \sin^2\theta. \end{aligned} \tag{14}$$

Now we are interested in finding the projective vector fields by using the relation (10). Writing equation (10) explicitly and using (4) and (14) we get

$$A_t\,(t,r) X^0 + A_r(t,r) X^1 + 2X^0_{,0} = Q(t,r) - E(t) \tag{15}$$

$$\mathrm{e}^{B(r)} X^1_{,0} - \mathrm{e}^{A(t,r)} X^0_{,1} = 0 \tag{16}$$

$$r^2 X^2_{,0} - \mathrm{e}^{A(t,r)} X^0_{,2} = 0 \tag{17}$$

$$r^2 \sin^2\theta X^3_{,0} - \mathrm{e}^{A(t,r)} X^0_{,3} = 0 \tag{18}$$

$$B_r(r) X^1 + 2\,X^1_{,1} = Q(t,r) + D(r) \tag{19}$$

$$r^2 X^2_{,1} + \mathrm{e}^{B(r)} X^1_{,2} = 0 \tag{20}$$

$$r^2 \sin^2\theta X^3_{,1} + \mathrm{e}^{B(r)} X^1_{,3} = 0 \tag{21}$$

$$2X^1 + 2r\,X^2_{,2} = r\,Q(t,r) \tag{22}$$

$$\sin^2\theta X^3_{,2} + X^2_{,3} = 0 \tag{23}$$

$$2X^1 + 2r\cot\theta\, X^2 + 2\,r\,X^3_{,3} = r\,Q(t,r). \tag{24}$$

Considering equations (17) and (18) and differentiating with respect to ϕ and θ, respectively and subtracting them we get

$$-\left[\sin^2\theta\, X^3_{,0}\right]_{,2} + X^2_{,03} = 0. \tag{25}$$

Differentiating equation (23) with respect to t we get

$$\sin^2\theta\, X^3_{,02} + X^2_{,03} = 0. \tag{26}$$

Subtracting equation (25) from equation (26) and upon integrating we get

$$X^3 = \operatorname{cosec}\theta \int E^1(t,r,\phi)\mathrm{d}t + E^2(r,\theta,\phi),$$

where $E^1(t,r,\phi)$ and $E^2(r,\theta,\phi)$ are functions of integration. Using the above information in equation (18) one has

$$X^0 = \mathrm{e}^{-A(t,r)} r^2 \sin\theta \int E^1(t,r,\phi)\mathrm{d}\phi + E^3(t,r,\theta),$$

where $E^3(t,r,\theta)$ is a function of integration. Substituting the value of X^0 in equations (16) and (17) we get

$$X^1 = r^2\sin\theta\mathrm{e}^{-B(r)}\int\left[\int E^1_r(t,r,\phi)\mathrm{d}\phi\right]\mathrm{d}t + 2r\sin\theta\;\mathrm{e}^{-B(r)}\int\left[\int E^1(t,r,\phi)\;\mathrm{d}\phi\right]\mathrm{d}t -$$
$$- r^2\sin\theta\,\mathrm{e}^{-B(r)}\int\left[A_r\int E^1(t,r,\phi)\mathrm{d}t\right]\mathrm{d}t + \int \mathrm{e}^{A(t,r)-B(r)} E^3_r(t,r,\theta)\mathrm{d}t + E^4(r,\theta,\phi),$$

$$X^2 = \cos\theta\int E^1(t,r,\phi)\mathrm{d}\phi\,\mathrm{d}t + \frac{1}{r^2}\int \mathrm{e}^{A(t,r)} E^3_\theta(t,r,\theta)\;\mathrm{d}t + E^5(r,\theta,\phi),$$

where $E^4(r,\theta,\phi)$ and $E^5(r,\theta,\phi)$ are functions of integration. In order to find the projective vector field X we are interested to find the all unknown functions $E^1(t,r,\phi), E^2(r,\theta,\phi)$, $E^3(t,r,\theta)$, $E^4(r,\theta,\phi)$ and $E^5(r,\theta,\phi)$. To avoid lengthy details we shall write only the results. From the lengthy and tedious calculations there exists only one case when the above space-time admits proper projective collineation which is in this case $A = \ln(b\,r^2)$ and $B = \ln\dfrac{c}{b\,r^2 + a}$, where $a,\ b,\ c \in \mathbb{R}(b \neq 0,\ c \neq 0)$. The space-time (4), after a rescaling of t, takes the form

$$\mathrm{d}s^2 = -r^2\mathrm{d}t^2 + \frac{c}{br^2 + a}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right). \tag{27}$$

The above space-time (27) admits four linearly independent Killing vector fields which are $\frac{\partial}{\partial t}, \cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\varphi}, \sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}, \frac{\partial}{\partial\phi}$.

Proper projective collineation after subtracting Killing vector fields is

$$X = (0,\ \frac{r}{2}(br^2 + a),\ 0,\ 0) \tag{28}$$

and one form is $\eta_a = (r\,b)\,r_a$. The above space-time (27) becomes special class of static spherically symmetric space-time.

Now consider the case when $\alpha_2 = 0$. From equation (6) one can see that the rank of the 6×6 Riemann tensor is 3 or $R^a{}_{bcd}t^d = 0$, where t^a is a timelike vector field and unique solution of $R^a{}_{bcd}t^d = 0$. Here, it is important to mention here that $B = B(r)$ and $\alpha_2 \neq \alpha_3$. The condition $\alpha_2 = 0 \Rightarrow A_r(t, r) = 0$ which gives $A = A(t)$. The line element (4) can, after a rescaling of t, be written in this form

$$ds^2 = -dt^2 + e^{B(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{29}$$

The above space-time is 1+3 decomposable. It follows from [3] that space-time (29) does not admit proper projective collineation. The projective collineation admitted by (29) is a proper affine vector field which is $t\,t^a$.

Consider when $\alpha_2 = \alpha_3$ (and excluding the special case when $A = \text{constant}$ and $B = \text{constant} \neq 0$) in (4). It follows from [3, 6] that projective collineation admitted by (4) are Killing vector fields which are given in equation (5).

Now consider the special case when $A = \text{constant}$ and $B = \text{constant} \neq 0$. The rank of the 6×6 Riemann tensor is 1 and there exists two independent solutions, which are $R^a{}_{bcd}\,t^d = 0$ and $R^a{}_{bcd}\,r^d = 0$, but only one independent covariantly constant vector field $t_a = t_{,a}$ satisfying $t_{a;b} = 0$. Substituting the above information in equation (4) and after a rescaling of t, the line element takes the form

$$ds^2 = -dt^2 + k dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{30}$$

where $k(= e^B) \in \mathbb{R}(k \neq 0 \text{ or } 1)$. The space time is clearly 1+3 decomposable but the rank of the 6×6 Riemann tensor is 1. It follows from [3] that the above space-time (30) which admits proper special projective collineation which is:

$$U = (t^2, tr, 0, 0). \tag{31}$$

3. CONCLUSIONS

A study of special non-static spherically symmetric space-times according to their proper projective symmetry is given by using the direct integration and algebraic techniques and real eigenvalues and eigenbivectors. Using the above mentioned techniques we have proved that the special class of the above space-times (4) (which become the special class of static spherically symmetric space-times) admit proper projective collineation. This is the space-time given in equation (27) and proper projective collineation is given in equation (28). It is important to mention that different approaches [7–33] were adopted to study projective collineations.

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