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CALCULATION OF THE REACTION OF A 3D THIN WALL TO AN EXTERNAL KINK MODE OF ROTATING PLASMA

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Abstract. In this paper we present the response of a 3D thin multiply connected wall to an external kink mode perturbation in axisymmetric tokamak configurations. The necessary computations have been made by using a newly introduced method in the PDE's solving practice: the radial basis functions collocation meshfree method. The wall response is expressed in terms of a stream function of the wall surface currents or by making use of the magnetic vector potential. Both approaches have requested the solving of a diffusion type equation, taking into account the contribution of the wall currents themselves iteratively. Our approach has been applied to the investigation of Resistive Wall Modes in tokamaks.

Key words: Magnetic fusion plasma, tokamak, MHD, instabilities.

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1. INTRODUCTION

One of the major goals in the thermonuclear fusion research is to produce stable high-pressure plasma, preferably at steady state, for the economic production of fusion energy. High power density implies high β but the maximum magnetic field B is limited by practical engineering constraints. Ideal magnetohydrodynamic (MHD) instabilities impose hard limits on the achievable β. In tokamaks, if internal instabilities are avoided by a suitable choice of the plasma current profile, the β limit comes about by the onset of External Kink Modes (EKMs). They cause a deformation of the plasma boundary, grow on the Alfvénic timescale of the order of 10^{-6}s and can terminate the plasma discharge abruptly. In the presence of resistive wall, the plasma configuration becomes unstable, the stability limit being virtually the same as in the case without wall. However, the modes grow much more slowly, namely, on the resistive timescale of the wall, which is typically of the order of 10^{-2}s. These decelerated EKMs are denoted Resistive Wall Modes (RWMs). Therefore, in the presence of a resistive wall, the active feedback stabilization of RWMs by means of magnetic field sensors and a system of additional correction coils becomes technologically feasible.

There are a large number of papers investigating the stabilization of Resistive Wall Modes, connected explicitly with the determination of the eddy current distribution in tokamaks [1] - [18].

In this paper we will refer first to our previous method to calculate the response of a thin 3D multiply connected thin wall with rectangular holes to an EKM [19] - [21]. Next, we will present a new solving method able to treat real walls with arbitrary cross-sections and holes of arbitrary forms. The paper is organized as follows. Section 2 is concerned with the formulation of the model problem. In Section 3 we draw the equations determining the eddy current distribution in a resistive thin wall, by using two possible formulations: with a scalar stream function or with the magnetic vector potential. The numerical solving method we have used, the Radial Basis Function (RBF) collocation meshless method, to solve the diffusion equation in domains corresponding to real tokamak walls with holes of arbitrary geometry, is presented in Section 4. Finally, in Section 5, the results are summarized and an outlook for further investigations is given.

2. FORMULATION OF THE PROBLEM

Classical linear resistive mode theory predicts stability when the linear stability index for the tearing mode Δ', defined as the jump of the logarithmic derivative of the flux function perturbation across the rational surface, is negative and unstable if this index is positive. To determine the value of the stability index, we have to write the expression for the potential energy due to a plasma displacement in terms of the perturbation of the flux function. Performing an Euler minimization, one obtains a system of ordinary coupled differential equations in that perturbation [23]

$$\mathbf{Y}'' = \mathbf{f}^{-1} \cdot (\mathbf{G} \cdot \mathbf{Y} + \mathbf{V} \cdot \mathbf{Y}'), \tag{1}$$

where, **f**, **V** and **G** are matrices and **Y** is the flux function perturbation vector. By solving this system of differential equation with given boundary conditions, the stability index can be determined. The matrix terms $f_{i,j}$, $V_{i,j}$ and $G_{i,j}$ are expressed with the help of metric coefficients, wave numbers of the helical plasma perturbation (m and n) toroidal plasma current density, plasma major and minor radii and toroidal magnetic field. The system of equations (1) describes a tearing mode or an external kink mode (EKM), the later if the resonance surface is situated at the plasma boundary. To solve this system of equations we have considered a "natural" boundary condition just at the plasma boundary [23] and not at infinite as usually. To determinate the vacuum field (outside of the plasma) due to a helical plasma perturbation of EKM type, we have considered a continuous surface distribution of simple sources extending over a not necessarily closed Liapunov surface (the plasma surface delimited by a separatrix is such a surface) generates a simple-layer potential Φ [22], [23]. Thus, the normal component of the magnetic field is given by $\mathbf{B}_n = -\partial\Phi/\partial n$, with $\mathbf{n}$ the external to plasma surface normal. It is this field which acts on the surrounding wall and this, by the induced eddy currents, acts on the boundary conditions of $\mathbf{Y}$ and so on, up to an "equilibrium". In this paper we will present a method to calculate the response of a surrounding thin wall with holes of arbitrary geometry to a helical EKM plasma perturbation.

3. CALCULATION OF THE EDDY CURRENT DISTRIBUTION IN A RESISTIVE THIN WALL

3.1. 1^{st} APPROACH USING A SCALAR STREAM FUNCTION

The eddy current distribution in the thin wall is described by the known diffusion equation [19, 21]

$$\nabla^2 U(t,\theta,\varphi) = d\sigma \frac{\partial \tilde{B}_n(t,\theta,\varphi)}{\partial t}, \tag{2}$$

where $U = U(t,\theta,\varphi))$ is the stream function of the eddy currents introduced by the relation $\mathbf{J} = \nabla U \times \mathbf{n}$ with $\mathbf{J}$ the linear eddy current density, $\mathbf{n}$ the external normal to the wall, $\tilde{B}_n$ the normal to the wall component of the magnetic field, d the wall thickness and σ the electrical conductivity of the wall. In a vacuum gap separating the toroidal plasma from the wall and other current-carrying elements, the perturbed magnetic field can be expressed as $\tilde{\mathbf{B}} = \tilde{\mathbf{B}}^{pl} + \tilde{\mathbf{B}}^{w} + \tilde{\mathbf{B}}^{ext}$, where each term corresponds to the plasma contribution, to the wall contributions and to the electrical currents flowing outside the wall, respectively. On the plasma surface, the perturbed magnetic field produced by the plasma (a real value) has been considered as excited by many modes.

We will use the same the curvilinear coordinate system (u,v,w) defined in Ref. [21], considered by us the best suited to describe phenomena in a 3D thin wall

$$\mathbf{r}_u \equiv \frac{\partial \mathbf{r}}{\partial u},\ \mathbf{r}_v \equiv \frac{\partial \mathbf{r}}{\partial v},\ \mathbf{r}_w \equiv \frac{\partial \mathbf{r}}{\partial w} \equiv d\frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}, \tag{3}$$

where the two first covariant basis vectors are tangential to the wall surface $\mathbf{r}_u \cdot \mathbf{n} = 0$, $\mathbf{r}_v \cdot \mathbf{n} = 0$, $\mathbf{r}$ denote the vector from an arbitrary origin to a variable point, and $\mathbf{n}$ is the external normal to the wall. The third basis vector $\mathbf{r}_w$ is normal to the wall surface and determines the w coordinate. d is the wall thickness. The metric tensor has only four components g_{uu}, g_{vv}, g_{ww}, and $g_{uv} = g_{vu} = 0$, while the contra-variant basis vectors perpendicular to the plane $(\mathbf{r}_u, \mathbf{r}_v)$ is given by $\mathbf{r}^w = \mathbf{r}/d^2$ and defines the normal vector to the wall $\mathbf{n} = d\mathbf{r}^w$. In this coordinate system, the diffusion equation

(2) looks like [21]

$$d\frac{\partial(\mathbf{r}^w\cdot\mathbf{B})}{\partial t} = \frac{1}{D}\left\{\frac{\partial}{\partial u}\left[\frac{1}{\sigma d}\left(\frac{g_{vv}}{D}\frac{\partial U}{\partial u} - \frac{g_{uv}}{D}\frac{\partial U}{\partial v}\right)\right] + \frac{\partial}{\partial v}\left[\frac{1}{\sigma d}\left(\frac{g_{uu}}{D}\frac{\partial U}{\partial v} - \frac{g_{uv}}{D}\frac{\partial U}{\partial u}\right)\right]\right\}, \tag{4}$$

with $D = \sqrt{g_{uu}g_{vv} - g_{uv}^2}$ the 2D Jacobian at the wall surface.

3.2. 2^{nd} APPROACH USING THE MAGNETIC VECTOR POTENTIAL

In the following, we are presenting an alternative way to calculate the eddy current distribution induced by the the plasma perturbation. Starting with the definition of the magnetic vector potential $\mathbf{A}$, with $\mathbf{B} = \nabla\times\mathbf{A}$, one obtains

$$\mathbf{E} = -\nabla\varphi - \frac{\partial\mathbf{A}}{\partial t}, \tag{5}$$

where $\mathbf{E}$ is the electric field intensity vector inside the conductor and φ is the scalar potential. Using Ohm's law, with σ the conductivity one obtains

$$\mathbf{J} = -\sigma\nabla\varphi - \sigma\frac{\partial\mathbf{A}}{\partial t}. \tag{6}$$

Assuming that in the conductor there are no current sources, $\nabla\cdot\mathbf{J} = 0$ in a thin wall of thickness d, one can define a surface current $\mathbf{j} = \mathbf{J}d$.

One can see that both the electric field intensity vector and the magnetic vector potential are located on the conductor surface, *i.e.*, they are surface fields. The magnetic vector potential $\mathbf{A}$ has two components: $\mathbf{A}^{pl}$ produced by external sources, and $\mathbf{A}^w$ produced by the surface currents them self $\mathbf{A} = \mathbf{A}^{pl} + \mathbf{A}^w$.

The second component $\mathbf{A}^w$ can be calculated with the Biot-Savart relation

$$\mathbf{A}^w(r) = \frac{\mu_0}{4\pi}\int_D \frac{\mathbf{J}(\mathbf{r}')dv'}{|\mathbf{r}-\mathbf{r}'|} = \frac{\mu_0}{4\pi}\int_S \frac{\mathbf{j}(\mathbf{r}')dS'}{|\mathbf{r}-\mathbf{r}'|}, \tag{7}$$

where D (or S) represents the domain of the current density, $\mathbf{r}$ is the position vector of the point where fields are calculated, and $\mathbf{r}'$ is the position vector of the source point. With Eq. (7), the equation of the current density $\mathbf{j}$ becomes

$$\nabla\cdot\left(\frac{\sigma}{d}\nabla\varphi + \frac{\sigma}{d}\frac{\mu_0}{4\pi}\int_S \frac{\partial\mathbf{j}(\mathbf{r}')/\partial t}{|\mathbf{r}-\mathbf{r}'|}dS'\right) = -\nabla\left(\frac{\sigma}{d}\frac{\partial\mathbf{A}^{pl}}{\partial t}\right). \tag{8}$$

Let us consider the domain S of the wall delimited by the closed contour Γ_0 and eventually some holes delimited by the closed contours Γ_i $i = 1,...,n_g$. On all these contours, the boundary conditions are

$$\mathbf{j}(\mathbf{r}_{\Gamma_i})\cdot\mathbf{n}_{\Gamma_i} = 0,\ i = 0,1,\ldots,n_g, \tag{9}$$

with $\mathbf{n}_\Gamma$, the normal to the contour Γ. Eq. (9) gives information on the normal component of the current density only, the tangential to the contour Γ component could take any value. Thus, the boundary conditions are of Neumann type

$$\frac{\partial\varphi}{\partial n} = -\frac{\partial A^{pl,n}}{\partial t} - \frac{\mu_0}{4\pi}\frac{\partial}{\partial t}\int_S \frac{\partial\mathbf{j}(\mathbf{r}')/\partial t\cdot\mathbf{n}(\mathbf{r})}{|\mathbf{r}-\mathbf{r}'|}dS' \tag{10}$$

For numerical calculations, Eq. (4) appears to be better suited, due to fact that if the boundary conditions are of Neumann type and there is no free term in φ (in the partial differential equation) there is no unique solution and the matrix is singular.

4. NUMERICAL SOLVING METHOD AND RESULTS

Let us resume first the solving method we have used previously, to render the new method more clear. We have considered both, wall and holes contours, as resulted by the intersection of constant coordinate lines: u=const and $v =$ const., with other words, we have considered a "theoretical" wall with holes. This simplifying assumption has been made in order to check our solving method for Eq. (4) or Eq. (8) on the most possible simple mesh *via* a Finite Difference Method (FDM). As an exemplifications of the applied FDM, a thin wall structure with an elliptical cross-section has been considered in Fig. 1. We have considered this wall section to measure six radians in the toroidal direction and six radians in the poloidal direction. This wall is located at the outer part of the plasma and is symmetric with respect to the plasma middle plane. The perturbed magnetic field generated by the rotating plasma has been taken of the form $B_\perp^{pl} = \exp[\gamma^r t]\sin(mu - nv + \omega t)$. Constant U lines obtained by solving the diffusion equation (4) with a FDM are reported in Fig. 1 (left side).

4.1. MESHLESS SOLVING METHOD

The numerical methods used by us in Ref. [21] to solve elliptical or parabolic equations were based on the Finite Difference Method (FDM) and Finite Elements Method (FEM). Both methods are based on meshing the computational domain. For many 2D domains these methods were successfully applied; for 3D domains, the tetrahedral meshing can become very expensive, especially if a certain segment of the domain requires additional expensive details. In recent years, the Radial Basis Function (RBF) collocation method has become a useful alternative to FDM and FEM for solving linear and non-linear PDEs in non-trivial geometries, since this method is meshfree and can be spectrally accurate [24]-[26]. Let us summarize the main aspects of this method in solving an elliptical PDE of the form

$$Lu(x) = f(x), \tag{11}$$

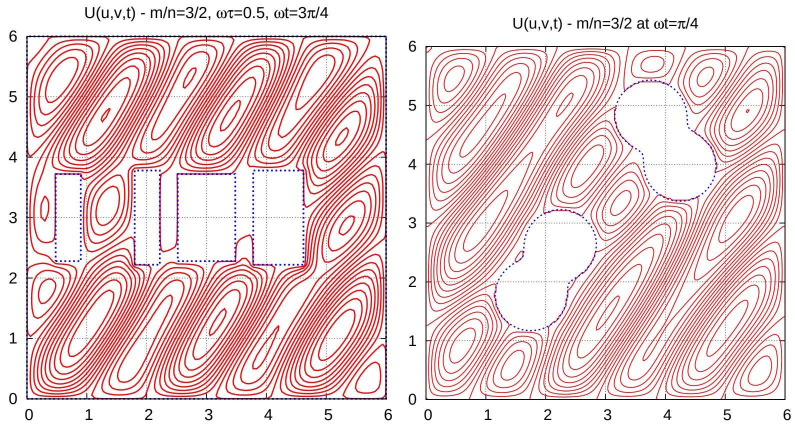

Fig. 1 – Constant U lines in a toroidal wall with elliptical cross-section and rectangular holes, in the presence of a rotating plasma obtained by using a FDM in Fig. 1 (left side) and constant U lines in the same toroidal wall with elliptical cross-section but with arbitrary holes, obtained by using a Radial Basis Function collocation meshless method in Fig. 1 (right side). The perturbed magnetic field of the plasma has been considered as $B_{\perp}^{pl} = B_0 \exp[\gamma^r t] \sin(mu - nv + \omega t)$, with B_0 =0.1 T, aspect ration R_0/r=3, wall thickness d=0.001 m and $\sigma = 10^6$ S/m. $m = 3$ and $n = 2$ represent the poloidal and toroidal wave numbers respectively. Different phases ωt, at constant $\omega\tau$ values have been considered, with τ the characteristic L/R time of the wall $\tau = \mu_0 r d\sigma$. A vanishing growth rate γ^r has been considered.

for each $x \in \Omega$ with some boundary conditions of the form

$$u(x) = g(x),\ x \in \delta\Omega_D,\ \ \frac{\partial u(x)}{\partial n} = h(x),\ x \in \delta\Omega_N, \tag{12}$$

where $\delta\Omega_D$ and $\delta\Omega_N$ denote the Dirichlet and Neumann boundary, respectively and $\delta\Omega_D \cup \delta\Omega_N = \delta\Omega$ is the boundary of the computational domain Ω. $\Omega = \{x_1, ..., x_N\} \subset \Re^n$ is a collection of points. The solution u can be approximate in terms of a radial basis function $\varphi(||x||)$ ($\varphi : \Re \to \Re$) as

$$u(x) = \sum_{k=1}^{N} \lambda_k \varphi(||x - x_k||), \tag{13}$$

which has to satisfy Eqs. (11) - (12). $\lambda_k = [\lambda_1, ..., \lambda_N]^T$ are the coefficients to be found. If L is a non-linear operator, then a multilevel Newton iteration has to be used and a linearized system has to be solved at each level. As radial basis function for our applications we have used the Gaussian

$$\varphi(x) = \mathrm{e}^{-(c||x - x_k||)^2}, \tag{14}$$

the multiquadric

$$\varphi(x) = \sqrt{1 + (c||x - x_k||)^2}, \tag{15}$$

and the polyharmonic spline

$$\varphi(x) = ||x - x_k||^l, \quad l = 1,3,5,\ldots, \tag{16}$$

with $c > 0$ a scalar parameter which may be adjusted to improve the approximation.

First, the meshless method has been applied to solve a similar problem with that presented in Fig. 1 (left side) for rectangular holes and solved using a FD method. The constant stream function lines U, obtained by using the meshless method with a Gaussian RBF are presented in Fig. 1 (right side). All considered parameters are the same as those used for the wall with rectangular holes, the single difference resides in the presence of two non-rectangular holes.

Our meshless approach on a wall with non-rectangular holes has been checked on PDEs with known analytical solutions. We have considered the following two functions:

$$f(u,v) = B_0 \exp[\gamma^r t] \sin(mu - nv + \omega t), \tag{17}$$

where we have used a Gaussian RBF and

$$f(u,v) = B_0 \exp[\gamma^r t](mu^2 + nv^2), \tag{18}$$

where now a polyharmonic spline RBF, with $l = 3$ as exponent, has been used. The correspondent numerical results are given in Fig. 2a and Fig. 2b, respectively and are in good agreement with the analytical solutions, as can be seen in Fig. 3.

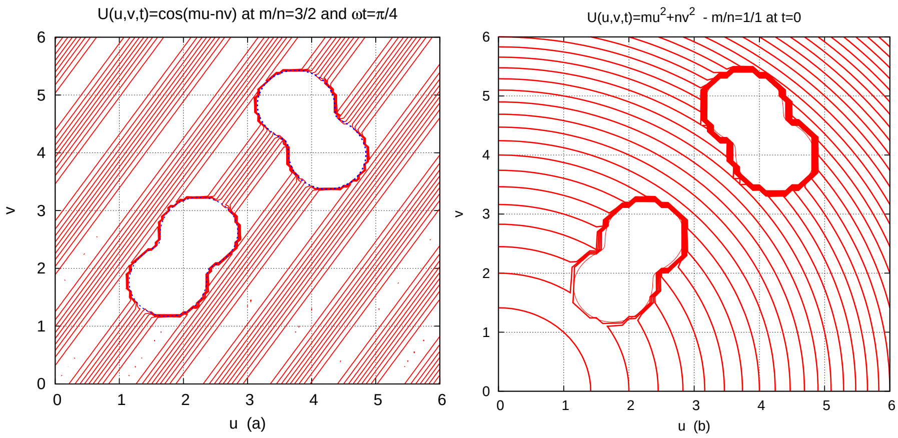

Fig. 2 – Constant U lines in a toroidal wall with elliptical cross-section and arbitrary holes obtained using a RBF meshless method for a PDE with known analytical solution given by Eq. (17) in Fig. 2a and Eq. (18) in Fig. 2b. B_0 =0.1 T, aspect ration R_0/r=3, wall thickness d=0.001 m and $\sigma = 10^6$ S/m. $m = 3$ and $n = 2$ represent the poloidal and toroidal wave numbers respectively. A phase ωt, at constant $\omega\tau$ value has been considered, with τ the characteristic L/R time of the wall $\tau = \mu_0 r d\sigma$. A vanishing growth rate γ^r has been considered.

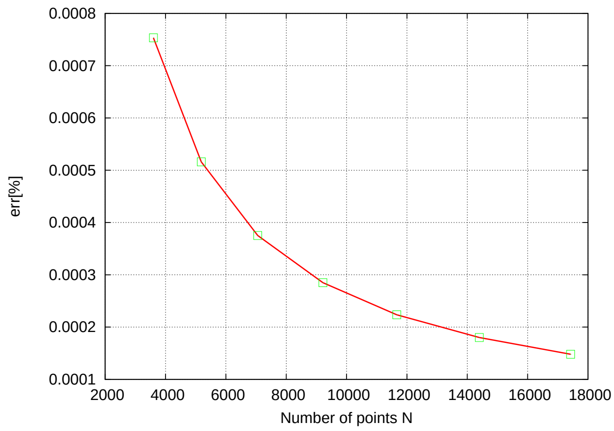

Fig. 3 – Relative error [%] dependence on the number N of considered points.

Finally, if a real tokamak wall has to be considered and both, wall contour and holes could not be described by analytical functions like in our meshless examples, we have developed a routine to interpolate any experimental curve *via* Chebyshev polynomials [27], because Chebyshev polynomials appear to be the best suited for open curves, like real tokamak wall contours are.

5. CONCLUSIONS

In this paper, a performant meshless method, newly introduced in the computational praxis, to calculate the response of the 3-D thin conducting shell to the magnetic perturbations is presented. Due to its high accuracy of calculations, the method is complementary to the widely-used approach, based on wall representation by conducting triangles with a uniform current density. While for a cylindrical or elliptical wall cross-section as well as for hole contours represented by constant coordinate lines, the metric coefficients could be described analytically, the Chebyshev polynomials interpolation have been tested and used for other wall and holes geometries.

Our method is applicable for RWM studies and at the same time is consistent with the physics requirements of the Wall Touching Kink Mode (WTKM) [28] and disruption simulations - the object of our next stability investigation in tokamaks.

REFERENCES

1. D. Pfirsch and H. Tasso, Nucl. Fusion **11**, 259 (1971).
2. R. Fitzpatrick, Phys. Plasmas **1**, 2931 (1994).
3. M.S. Chu, M.S. Chance, A.H. Glasser and M. Okabayashi, Nucl. Fusion **43**, 441 (2003).

4. Y. Liu, A. Bondeson, Y. Gribov, and A. Polevoi, Nucl. Fusion **44**, 232 (2004).
5. L. Degtyarev, A. Martynov, S. Medvedev, F. Troyon, L. Villard, and R. Gruber, Comput. Phys. Commun. **103**, 10 (1997).
6. F. Villone, G. Rubinacci, Y. Q. Liu, and Y. Gribov, Europhys. Conf. Abstr. **29C**, P-5.049 (2005).
7. J. Bialek, A. Boozer, M. E. Mauel, and A. Navratil, Phys. Plasmas **8**, 2170 (2001).
8. P. Merkel and M. Sempf, 21st IAEA Fusion Energy Conference 2006, Chengdu, China (International Atomic Energy Agency, Vienna, 2006), paper TH/P3-8.
9. P. Merkel, C. Nührenberg, and E. Strumberger, Europhys. Conf. Abstr. **28G**, P-1.208 2004.
10. E. Strumberger, P. Merkel, M. Sempf, and S. Günter, Phys. Plasmas **15**, 056110 (2008).
11. L.A. Tseitlin, Soviet Physics-Technical Physics, **10**, (1970).
12. H. Ammari, A. Buffa, and J.-C. Nedelec, SIAM J. Appl. Math. **60** (2000).
13. A. Kameari, Journ. Comp. Physisc **42**, 124 (1981).
14. R. Albanese. G. Rubinacci, M.Canali, S. Stangherlin A.B.B. Ricerca, S.p.A., Sesto S. Giovanni, A. Musolino and M. Raugi, IEEE Transaction on Magnetics, **32**, (1996).
15. I. Senda, T. Shoji, T. Tsunematsu, T. Nishino, and H. Fujieda, Nucl. Fusion **37** (1997).
16. A.M. Miri, N.A. Riegel and C. Meinecke, Int. J. Numer. Model. **11** , 307 (1998).
17. G.O. Ludwig, E. Del Bosco and J.G. Ferreira, Nucl. Fusion **45**, 675 (2005).
18. A.H. Boozer, Phys. Plasmas **19**, (2012).
19. C.V. Atanasiu, A. Moraru and L.E. Zakharov, *Influence of a Nonuniform Resistive Wall on the RWM Stability in a Tokamak*, American Physical Society Plasma 51st Annual Meeting, Atlanta, USA, 2-6 November 2009.
20. C.V. Atanasiu, A. Moraru and L.E. Zakharov, *MHD Modeling in Diverted Tokamak Configurations*, 7th ATEE-2011 International Symposium, May 12-14, 2011, Bucharest, Romania.
21. C.V. Atanasiu, L.E. Zakharov, Phys. Plasmas, **20**, 092506 (2013).
22. C.V. Atanasiu, A.H. Boozer, L.E. Zakharov, and A.A. Subbotin, Phys. Plasmas **6**, (1999).
23. C.V. Atanasiu, S. Günter, K. Lackner, and L.E. Zakharov, Phys. Plasmas **12**, (2004).
24. E.J. Kansa, Computational Applied Mathemtics, **19**, 147-161 (1990).
25. R. Schaback and Y.C. Hon, Journal of Applied Mathematics and Computation, **119**, 177-186 (2001).
26. B. Fornberg and E. Letho, Journal Comp. Phys. **230**, 2270-2285 (2011).
27. M. Abramovitz and I.A. Stegun, *Handbook of Mathematical Functions* (Dover Publications Inc., New York, 1994).
28. L.E. Zakharov, S.A. Galkin, S.N. Gerasimov, Phys. Plasmas, **19**, (2012).