Romanian Reports in Physics, Vol. 67, No. 2, P. 329–339, 2015

DOUBLE REDUCTION OF THE GENERALIZED ZAKHAROV EQUATIONS *VIA* CONSERVATION LAWS

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Received September 19, 2014

Abstract. In this paper, we study the generalized Zakharov equations with perspective different from those done previously, namely, by determining its symmetry generators, constructing some conservation laws and then use the associations between these to attain a double reduction of the corresponding differential equations leading to two travelling wave solutions.

Key words: Generalized Zakharov equations, Lie symmetries, Conservation laws, Double reductions, Travelling wave solutions.

PACS: 11.30.-j, 02.30.Jr, 04.20.Jb

1. INTRODUCTION

Over the years, mathematicians and physicists focused their attention to obtain exact solutions of nonlinear evolution equations, as a result, several methods have been established and developed to obtain the exact solutions of nonlinear evolution equations. Some of the most important methods found in literature include the Hirota's bilinear method [1, 2], inverse scattering method[3], Bäklund transformation [4], Darboux transformation [5], truncated Painlevé expansion [6], Lie symmetries method [7-10], first integral method [11] and others [12-15]. Lie symmetry method plays a significant role in the solution process of nonlinear differential equations. It is based upon the study of the invariance under one parameter Lie group of point transformations [6-8, 16, 17], and it systematically unify and extend well known *ad hoc* techniques to construct explicit solutions for differential equations, especially for nonlinear partial differential equations (PDEs).

The conservation laws are important in the solutions and reductions of PDEs. In the last few decades, active research efforts have been made on the derivation of conservation laws for partial differential equations. Many significant methods have been developed for the construction of conservation laws, such as the Nöether's theorem [18, 19] for variational problems, multiplier approach [8, 20], symmetry action on a known conservation law [21], partial Nöether approach [22] and new conservation method [23], etc.

The theory of double reduction of a PDE (or systems of PDEs) is well-known for the association of conservation laws with Nöether symmetries [6, 8]. The association of conservation laws with Lie Bäcklund symmetries [24] and non-local symmetries [25] lead to the expansion of the theory of double reduction for PDEs with two independent variables which do not possess Nöether symmetries [26]. In this article, we apply the double reduction, to obtain new travelling solutions of the GZEs which we could not able to derive from Lie symmetry analysis.

In this work, we consider the generalized Zakharov equations (GZEs) [27] in the form

$$\begin{cases} iE_t + E_{xx} - 2\beta|E|^2E + 2EF = 0, \\ F_{tt} - F_{xx} + (|E|^2)_{xx} = 0, \end{cases} \tag{1}$$

where E is the complex envelope of the high-frequency electric field, the low-frequency field F is the plasma density measured from its equilibrium value, and the cubic term in first equation of Eqs. (1) describes nonlinear self-interaction in the high-frequency subsystem which corresponds to a self-focusing effect in plasma physics. The GZEs are a universal model of interaction between high- and low-frequency waves in one dimension. When $\beta = 0$, (1) reduced to the classical Zakharov equation of plasma physics [28], and has an important particular solution – the Langmuir solution. Obviously, the GZE is a strongly nonlinear system and it is quiet difficult to obtain its solitary wave solutions.

Recently, extensive studies have been carried out by many authors on the GZEs. In [29,30], the solitary wave solutions of the GZEs have been obtained by the well-known He's variational approach. Wang and Li [31] introduced periodic wave solutions of GZEs using the extended F-expansion method. In [32], the Painlevé property and some exact solutions of the GZEs have been obtained using the truncated Painlevé expansion. Dai and Xu [33] studied the dynamic behaviors of some exact traveling wave solutions to the GZEs.

In order to study the Lie symmetries, conservation laws and double reductions of the GZEs (1), we express the complex envelope as $E(x,t) = u(x,t) + iv(x,t)$, with real high-frequency waves $u(x,t)$ and $v(x,t)$. By substituting it into Equations

(1) and separating the imaginary and real parts, we obtain

$$\begin{cases} 2uF - 2\beta u(u^2+v^2) - v_t + u_{xx} = 0, \\ 2vF - 2\beta v(u^2+v^2) + u_t + v_{xx} = 0, \\ F_{tt} - F_{xx} + (u^2+v^2)_{xx} = 0. \end{cases} \tag{2}$$

The layout of the paper is as follows: In the next section, symmetries and conservation laws of (1) are obtained by using Lie symmetry methods and direct method, respectively. In Section 3, new travelling wave solutions of (1) derived by employing the double reduction method to (2). Finally, conclusions are summarized in Section 4.

2. LIE SYMMETRIES AND CONSERVATION LAWS OF GZEs

The symmetry group of the generalized Zakharov equations (2) is generated by the vector field of the form The modified Riemann-Liouville derivative defined by Jumarie [28]

$$\begin{aligned} \Gamma = \xi(x,t,u,v,F)\frac{\partial}{\partial x} + \tau(x,t,u,v,F)\frac{\partial}{\partial t} + \eta(x,t,u,v,F)\frac{\partial}{\partial u} \\ + \phi(x,t,u,v,F)\frac{\partial}{\partial v} + \psi(x,t,u,v,F)\frac{\partial}{\partial F}. \end{aligned} \tag{3}$$

The application of 2nd extension $\Gamma^{(2)}$ [6-8] to (2) results in an overdetermined system of linear partial differential equations. The general solution of the overdetermined system of linear partial differential equations with the aid of Wu's method [34] is given by

$$\begin{aligned} &\xi(x,t,u,v,F) = C_1, \\ &\tau(x,t,u,v,F) = C_2, \\ &\eta(x,t,u,v,F) = -(C_3t^2 + 2C_4t - C_5)v, \\ &\phi(x,t,u,v,F) = (C_3t^2 + 2C_4t - C_5)u, \\ &\psi(x,t,u,v,F) = C_3t + C_4, \end{aligned} \tag{4}$$

where $C_i, i = 1,...,5$ are five arbitrary constants. Hence the infinitesimal symmetries of (2) form the five dimensional Lie algebra spanned by the following linearly independent operators

$$\begin{aligned}
\Gamma_1 &= \frac{\partial}{\partial x},\\
\Gamma_2 &= \frac{\partial}{\partial t},\\
\Gamma_3 &= v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v},\\
\Gamma_4 &= -2tv\frac{\partial}{\partial u} + 2tu\frac{\partial}{\partial v} + \frac{\partial}{\partial F},\\
\Gamma_5 &= -t^2v\frac{\partial}{\partial u} + t^2u\frac{\partial}{\partial v} + t\frac{\partial}{\partial F}.
\end{aligned} \tag{5}$$

The operators Γ_1 and Γ_2 are related to the space and time translation, respectively. The symmetry vector field Γ_3 is related to a rotation in the $u-v$ space. The non-vanishing commutators are given by $[\Gamma_2,\Gamma_4] = -2\Gamma_3$ and $[\Gamma_2,\Gamma_5] = \Gamma_4$.

Let $x = (x_1, x_2, \cdots, x_n)$ be n independent variables and $u = (u^1, u^2, \cdots, u^m)$ be m dependent variables. Consider a system of r PDEs of k-order with x and u, given by

$$P_\alpha[u] = P_\alpha(x, u, u_{(1)}, u_{(2)}, \cdots, u_{(k)}) = 0, \quad \alpha = 1, 2, \cdots, r, \tag{6}$$

where $u_{(1)} = \{u_i^\alpha\}$, $u_{(2)} = \{u_{ij}^\alpha\}, \cdots$, and $u_i^\alpha = \frac{\partial u^\alpha}{\partial x_i}, u_{ij}^\alpha = \frac{\partial^2 u^\alpha}{\partial x_i \partial x_j}, \cdots$. We let $U = (U^1, U^2, \cdots, U^N)$ denote arbitrary functions of the independent variables x and denote partial derivatives $\partial/\partial x_i$ by subscripts i, *i.e.*, $U_i^\sigma = \partial U^\sigma/\partial x_i, U_{ij}^\sigma = \partial^2 U_{x_i x_j}^\sigma$, etc.

1) The total derivative operators D_i with respect to x_i are

$$D_i = \frac{\partial}{\partial x_i} + u_i^\alpha \frac{\partial}{\partial u_i^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + u_{ijk}^\alpha \frac{\partial}{\partial u_{jk}^\alpha} + \cdots, \tag{7}$$

where $i, j, k, \cdots = 1, 2, \cdots, n$ and $\alpha = 1, 2, \cdots, m$.

2) Multipliers for PDE system (6) are a set of functions $\{\Lambda^\alpha[U]\}$ satisfying

$$\Lambda^\alpha[U] P_\alpha[U] = D_i T^i[U] \tag{8}$$

for some functions $\{T^i[U]\}$.

If $U^\sigma = u^\sigma(x)$ is a solution of PDE system (6), then from (8) we obtain the conservation law

$$D_i T^i[U] = 0 \tag{9}$$

of system (6), and $T^i[U]$ are called the conserved densities.

3) The standard Euler operators with respect to the differentiable function u^j

and its derivatives $u^j_i, u^j_{i_1 i_2}, \cdots$ are the operator defined by

$$E_{u^j} = \frac{\partial}{\partial u^j} - D_i \frac{\partial}{\partial u^j_i} + \cdots + (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u^j_{i_1 \cdots i_s}} + \cdots, \tag{10}$$

for each $j = 1, 2, \cdots, m$.

$\{\Lambda^\alpha[U]\}$ yield a set of multipliers for a conservation law of system (6) if and only if each Euler operator (10) annihilates the left-hand side of (8), *i.e.*,

$$E_{U^j}(\Lambda^\alpha[U] P_\alpha[U]) \equiv 0, \;\; j = 1, ..., N, \tag{11}$$

for arbitrary $U^\sigma, U^\sigma_i, U^\sigma_{ij}, \ldots$, etc..

From the determining equation (11) for multipliers, we obtain the multipliers (with the aid of GeM [35]) $\Lambda^1(t,x,u,v,F)$, $\Lambda^2(t,x,u,v,F)$ and $\Lambda^3(t,x,u,v,F)$ for the GZEs (2) which are given by

$$\begin{aligned}
\Lambda^1 &= (\frac{1}{2} C_1 t^2 + C_2 t + C_3) V, \\
\Lambda^2 &= -(\frac{1}{2} C_1 t^2 + C_2 t + C_3) U, \\
\Lambda^3 &= -\frac{1}{12} C_1 t^3 - \frac{1}{4} C_2 t^2 + \frac{1}{12} t(-3C_1 x^2 + 12 C_6 x + 12 C_4) \\
&\quad - \frac{1}{4} C_2 x^2 + C_7 x + C_5,
\end{aligned} \tag{12}$$

where $C_i, i = 1, 2, \cdots, 7$ are arbitrary constants.

Then, from (8) and (12), we have the following seven conserved vectors of (2) satisfying (9)

$$\begin{aligned}
T_1 &= [-(\frac{1}{12} t^3 + \frac{1}{4} t x^2) F_t + \frac{1}{4}(t^2 + x^2) F - \frac{1}{4} t^2 (u^2 + v^2), -\frac{1}{2} t x F \\
&\quad + (\frac{1}{12} t^3 + \frac{1}{4} t x^2) F_x - (\frac{1}{6} t^3 + \frac{1}{2} t x^2)(u u_x + v v_x) + \frac{1}{2} t x (u^2 + v^2)]; \\
T_2 &= [-\frac{1}{4}(t^2 + x^2) F_t + \frac{1}{2} t F - \frac{1}{2} t (u^2 + v^2), \frac{1}{4}(t^2 + x^2) F_x - \frac{1}{2} x F \\
&\quad - \frac{1}{2}(t^2 + x^2)(u u_x + v v_x) + t(v u_x - u v_x) + \frac{1}{2} x (u^2 + v^2)]; \\
T_3 &= [-\frac{1}{2}(u^2 + v^2), v u_x - u v_x]; \\
T_4 &= [t F_t - F, t(2(u u_x + v v_x) - F_x)]; \\
T_5 &= [F_t, 2(u u_x + v v_x) - F_x]; \\
T_6 &= [x(t F_t - F), t(-x F_x + F + 2x(u u_x + v v_x) - u^2 - v^2)]; \\
T_7 &= [x F_t, -x F_x + F + 2x(u u_x + v v_x) - u^2 - v^2].
\end{aligned} \tag{13}$$

3. DOUBLE REDUCTIONS OF GZEs *VIA* CONSERVATION LAWS

We first look for the possible associations between Lie symmetries and the conserved vectors. We now show that Γ_1, Γ_2 and Γ_3 are associated with T_5 in (13). We obtain

$$\begin{bmatrix} T_5^{*t} \\ T_5^{*x} \end{bmatrix} = \Gamma_1^{(1)} \begin{bmatrix} T_5^t \\ T_5^x \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_5^t \\ T_5^x \end{bmatrix} + (0) \begin{bmatrix} T_5^t \\ T_5^x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{14}$$

where $\Gamma_1^{(1)} = \frac{\partial}{\partial x}$.

Thus, Γ_1 is associated with T_5.

For Γ_2 and Γ_3, we obtain

$$\begin{bmatrix} T_5^{*t} \\ T_5^{*x} \end{bmatrix} = \Gamma_2^{(1)} \begin{bmatrix} T_5^t \\ T_5^x \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_5^t \\ T_5^x \end{bmatrix} + (0) \begin{bmatrix} T_5^t \\ T_5^x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{15}$$

where $\Gamma_2^{(1)} = \frac{\partial}{\partial t}$ and

$$\begin{bmatrix} T_5^{*t} \\ T_5^{*x} \end{bmatrix} = \Gamma_3^{(1)} \begin{bmatrix} T_5^t \\ T_5^x \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_7^x \\ T_7^t \end{bmatrix} + (0) \begin{bmatrix} T_5^t \\ T_5^x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{16}$$

where $\Gamma_3^{(1)} = v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v} + v_x\frac{\partial}{\partial u_x} - u_x\frac{\partial}{\partial v_x} + v_t\frac{\partial}{\partial u_t} - u_t\frac{\partial}{\partial v_t}$, respectively.

Thus, Γ_2 and Γ_3 are also associated with T_5.

Similarly, we can calculate all other possible associations. We present them in the form of Table 1.

Table 1

The associations between the symmetries and conserved vectors of Eq. (2)

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5
T_1	$\neq 0$	$\neq 0$	0	$\neq 0$	$\neq 0$
T_2	$\neq 0$	$\neq 0$	0	$\neq 0$	$\neq 0$
T_3	0	0	0	0	0
T_4	0	$\neq 0$	0	$\neq 0$	0
T_5	0	0	0	0	$\neq 0$
T_6	$\neq 0$	$\neq 0$	0	$\neq 0$	$\neq 0$
T_7	$\neq 0$	0	0	$\neq 0$	$\neq 0$

3.1. A DOUBLE REDUCTION OF (2) BY $\langle \Gamma_1, \Gamma_2, \Gamma_3 \rangle$

We consider a linear combination of Γ_1, Γ_2 and Γ_3, *i.e.*, of the form $\Gamma = c\Gamma_1 + \Gamma_2 + \Gamma_3$, and transform this generator Γ to its canonical form $Y = \frac{\partial}{\partial s}$.

From $\Gamma(r)=0, \Gamma(s)=1$, $\Gamma(p)=0$ and $\Gamma(z)=0$, we obtain

$$\frac{dx}{c}=\frac{\mathrm{d}t}{1}=\frac{\mathrm{d}u}{v}=\frac{\mathrm{d}v}{-u}=\frac{\mathrm{d}F}{0}=\frac{\mathrm{d}r}{0}=\frac{\mathrm{d}s}{1}=\frac{\mathrm{d}p}{0}=\frac{\mathrm{d}z}{0}. \tag{17}$$

The invariants of Γ from (17) are given by

$$b_1=x-ct, b_2=s-t, b_3=u^2+v^2, b_4=F, b_5=r, b_6=p, b_7=z, \tag{18}$$

where b_5, b_6, b_7 are arbitrary functions all depending on b_1, b_2, b_3 and b_4.

By choosing $b_5=b_1$, $b_2=0$, $b_6^2=b_3$ and $b_7=b_4$, we obtain the canonical coordinates

$$r=x-ct, s=t, p^2=u^2+v^2, z=F \tag{19}$$

where $p=p(r)$ and $z=z(r)$.

From (19), the inverse canonical coordinates are given by

$$t=s, x=r+cs, u^2+v^2=p^2, F=z. \tag{20}$$

The computation of A and $(A^{-1})^T$ are given by

$$A=\begin{bmatrix}0 & 1\\ 1 & c\end{bmatrix},\ (A^{-1})^T=\begin{bmatrix}-c & 1\\ 1 & 0\end{bmatrix}, \tag{21}$$

where $J=\det(A)-1$.

Without loss of generality, we suppose $u=p\cos(r), v=p\sin(r)$ for (20) and (22), the partial derivatives of u, v and F from (22) are given by

$$\begin{aligned}&u_x=p_r\cos(r)-p\sin(r), v_x=p_r\sin(r)+p\cos(r),\\ &F_t=-cz_r, F_x=z_r, F_{tt}=c^2z_{rr}, F_{xx}=z_{rr}.\end{aligned} \tag{22}$$

Using the following formula, we can get the reduced conserved form,

$$\begin{bmatrix}T_5^r\\ T_5^s\end{bmatrix}=J(A^{-1})^T\begin{bmatrix}T_5^t\\ T_5^x\end{bmatrix}=\begin{bmatrix}(1-c^2)z_r-2pp_r\\ cz_r\end{bmatrix}, \tag{23}$$

where the reduced conserved form also satisfies

$$D_rT_5^r=0. \tag{24}$$

The second step of double reduction can be given as

$$(1-c^2)z_r-2pp_r=C, \tag{25}$$

where C is an integration constant. Differentiating (26) implicitly with respect to r results in

$$(1-c^2)z_{rr}-2(p_r^2+pp_r)=0. \tag{26}$$

From multiplier $(0,0,1)$ corresponding to (T_5^r, T_5^s) and Eq. (2), we get

$$(c^2-1)z_{rr}+2(p_r^2+pp_{rr})=0. \tag{27}$$

Substituting (27) into (26) and then taking out a common factor of p yields the second order linear ordinary differential equation

$$p_{rr} - p_r = 0. \tag{28}$$

We obtain the general solution of Eq. (28) as follows

$$p(r) = \delta e^r + \lambda, \tag{29}$$

where δ, λ are two arbitrary constants. Then, substituting (29) into Eq. (27), obtain

$$z(r) = \frac{2}{1-c^2}\delta(\frac{1}{2}\delta e^{2r} + \lambda e^r) + \mu r + \nu, \tag{30}$$

where μ, ν are two arbitrary constants.

For simplicity, let $c = 2, \lambda = \mu = 0, \nu = -1$, from the canonical coordinates (19) and Eqs. (29), (30), hold the solution of Eq. (2) as follows

$$\begin{aligned} u(x,t) &= \delta e^{x-2t}\cos(x-2t), \\ v(x,t) &= \delta e^{x-2t}\sin(x-2t), \\ F(x,t) &= -\frac{1}{3}\delta^2 e^{2(x-2t)} - 1. \end{aligned} \tag{31}$$

From (31), we obtain a new solitary wave solution of the GZEs (1) as follows

$$\begin{aligned} E(x,t) &= \delta e^{x-2t+I(x-2t)}, \\ F(x,t) &= -\frac{1}{3}\delta^2 e^{2(x-2t)} - 1. \end{aligned} \tag{32}$$

3.2. A DOUBLE REDUCTION OF (2) BY $\langle \Gamma_1, \Gamma_2 \rangle$

From Table 1, we know that Γ_1 and Γ_2 are associated with T_5.

We consider a linear combination of Γ_1 and Γ_2, *i.e.*, of the form $\Gamma = c\Gamma_1 + \Gamma_2$, and transform this generator Γ to its canonical form $Y = \frac{\partial}{\partial s}$.

From $\Gamma(r) = 0, \Gamma(s) = 1$, $\Gamma(p) = 0$, $\Gamma(q) = 0$ and $\Gamma(z) = 0$, we obtain

$$\frac{dx}{c} = \frac{\mathrm{d}t}{1} = \frac{\mathrm{d}u}{0} = \frac{\mathrm{d}v}{0} = \frac{\mathrm{d}F}{0} = \frac{\mathrm{d}r}{0} = \frac{\mathrm{d}s}{1} = \frac{\mathrm{d}p}{0} = \frac{\mathrm{d}q}{0} = \frac{\mathrm{d}z}{0}. \tag{33}$$

The invariants of Γ from (33) are given by

$$b_1 = x - ct, b_2 = s - t, b_3 = u, b_4 = v, b_5 = F, b_6 = r, b_7 = p, b_8 = q, b_9 = z, \tag{34}$$

where b_6, b_7, b_8 and b_9 are arbitrary functions all dependent on b_1, b_2, b_3, b_4 and b_5.

By choosing $b_6 = b_1$, $b_2 = 0$, $b_7 = b_3$, $b_8 = b_4$ and $b_9 = b_5$, we obtain the canonical coordinates

$$r = x - ct, s = t, p = u, q = v, z = F, \tag{35}$$

where $p = p(r), q = q(r)$ and $z = z(r)$.

From (34), the inverse canonical coordinates are given by

$$t = s, x = r + cs, u = p, v = q, F = z. \tag{36}$$

The computation of A and $(A^{-1})^T$ are given by

$$A = \begin{bmatrix} 0 & 1 \\ 1 & c \end{bmatrix}, \ (A^{-1})^T = \begin{bmatrix} -c & 1 \\ 1 & 0 \end{bmatrix} \tag{37}$$

where $J = -1$.

The partial derivatives of u, v and F from (34) are given by

$$\begin{aligned} &u_x = p_r, v_x = q_r, F_x = z_r, u_t = -cp_r, v_t = -cq_r, F_t = -cz_r, \\ &u_{xx} = p_{rr}, v_{xx} = q_{rr}, F_{xx} = z_{rr}, F_{tt} = c^2 z_{rr}. \end{aligned} \tag{38}$$

Using the following formula, we can get the reduced conserved form,

$$\begin{bmatrix} T_5^r \\ T_5^s \end{bmatrix} = J(A^{-1})^T \begin{bmatrix} T_5^t \\ T_5^x \end{bmatrix} = \begin{bmatrix} (1-c^2)z_r - 2(pp_r + qq_r) \\ cz_r \end{bmatrix}, \tag{39}$$

where the reduced conserved form also satisfies (24).

The second step of double reduction can be given as

$$(1-c^2)z_r - (p^2+q^2)_r = C, \tag{40}$$

where C is an integration constant. Integrating (40) with respect to r, and take C and other integration constant to zero, then obtain

$$z = \frac{1}{1-c^2}(p^2+q^2). \tag{41}$$

For simplicity, we suppose the following transformation for (41)

$$\begin{aligned} p &= \sqrt{(1-c^2)z}\sin(r), \\ q &= \sqrt{(1-c^2)z}\cos(r), \end{aligned} \tag{42}$$

then substituting (42) into (2), obtain $c = -2$ and the second order linear ordinary differential equation

$$(2+6\beta)\sqrt{z}z + \sqrt{z} + (\sqrt{z})_{rr} = 0. \tag{43}$$

Under the transformation $w = \sqrt{z}$, Eq. (43) becomes

$$(2+6\beta)w^3 + w + w_{rr} = 0. \tag{44}$$

Solving (44), get a solution

$$w = C_1 \alpha J((\sqrt{2+3\beta}r + C_2)\alpha, C_1\gamma), \tag{45}$$

where J is the JacobiSN function, C_1, C_2 are two arbitrary constants, and $\alpha =$

$\frac{1}{\sqrt{2+3\beta-C_2^2(1+3\beta)}}$ and $\gamma=\frac{\sqrt{-(2+3\beta)(1+3\beta)}}{2+3\beta}$. Thus,

$$\begin{aligned} p &= C_1\delta J((\sqrt{2+3\beta}r+C_2)\alpha, C_1\gamma)\sin(r), \\ q &= C_1\delta J((\sqrt{2+3\beta}r+C_2)\alpha, C_1\gamma)\cos(r), \\ z &= C_1^2\alpha^2 J^2((\sqrt{2+3\beta}r+C_2)\alpha, C_1\gamma), \end{aligned} \tag{46}$$

where $r=x+ct$ and $\delta=\sqrt{\frac{3}{C_2^2(1+3\beta)-2-3\beta}}$.

From (46), we obtain a new periodic wave solution of the GZEs (1) as follows

$$\begin{aligned} E(x,t) &= C_1\delta J((\sqrt{2+3\beta}(x+ct)+C_2)\alpha, C_1\gamma)[\sin(x+ct)+i\cos(x+ct)], \\ F(x,t) &= C_1^2\alpha^2 J^2((\sqrt{2+3\beta}(x+ct)+C_2)\alpha, C_1\gamma). \end{aligned} \tag{47}$$

4. CONCLUSIONS

In this paper we firstly employed the Lie symmetry method and direct method to obtain the symmetries and conservation laws of the GZEs. The accuracy of the solutions has been verified by substituting them back in to GZEs. Secondly, two new travelling wave solutions have been obtained by exploiting the double reduction method, which has been applied after establishing the conserved vectors association with the Lie symmetries. It is important to note that these two solutions could not be obtainable through Lie symmetry method. Further these solutions can be used as benchmarks against the numerical solutions. It is worth to mention that this work provides a mechanism to obtain new solutions though double reduction. By using Table 1, one can use the associated Lie generators (or combinations of Lie generators) and conservation laws to get new solution and could explore further new physical properties of GZEs. Besides, present work can be further perused from the point of view of Hamiltonian structures and numerical simulations.

Acknowledgements. This work is supported by the Research Project of China Scholarship Council (No.201208155076), the Natural Science Foundation of Inner Mongolia (No.2013MS 0118), the College Science Research Project of Inner Mongolia (No.NJZZ12182, No.NJZY13268) and the program for Innovative Research Team of Hohhot University for Nationalities (No.CXTD1402).

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