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NEW NUMERICAL APPROXIMATIONS FOR SPACE-TIME FRACTIONAL BURGERS' EQUATIONS *VIA* A LEGENDRE SPECTRAL-COLLOCATION METHOD

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Abstract. Burgers' equation is a fundamental partial differential equation in fluid mechanics. This paper reports a new space-time spectral algorithm for obtaining an approximate solution for the space-time fractional Burgers' equation (FBE) based on spectral shifted Legendre collocation (SLC) method in combination with the shifted Legendre operational matrix of fractional derivatives. The fractional derivatives are described in the Caputo sense. We propose a spectral shifted Legendre collocation method in both temporal and spatial discretizations for the space-time FBE. The main characteristic behind this approach is that it reduces such problem to that of solving a system of nonlinear algebraic equations that can then be solved using Newton's iterative method. Numerical results with comparisons are given to confirm the reliability of the proposed method for FBE.

Key words: Fractional Burgers' equation; Collocation method; Shifted Legendre polynomials; Operational matrix; Caputo derivative.

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1. INTRODUCTION

Fractional differential equations (FDEs), as generalizations of classical integer order differential equations, are increasingly used to model several real phenomena emerging in engineering and science fields (see for example [1–6] and the references therein). Owing to the increasing applications, there has been an increasing interest in developing numerical methods for the solution of fractional differential equations. These methods include variational iteration method [7, 8], Adomian decomposition method [9, 10], fractional perturbation technique [11], generalized differential transform method [12], homotopy analysis method [13, 14], finite difference method [15, 16], Haar wavelet method [17], and spectral methods [18–20].

Burgers' equation occurs in various areas of applied mathematics, such as modeling of dynamics, heat conduction, and acoustic waves [21–24]. It was actually first introduced by Bateman [25] when he mentioned it as worthy of study and gave its steady solutions. The space and time FBE describes the physical processes of unidirectional propagation of weakly nonlinear acoustic waves through a gas-filled pipe [26]. The space and time FBE was firstly treated by Momani in Ref. [27] by the Adomian decomposition method. More recently, numerical solution for the space and time FBE based on variational iteration method was considered by Inc [28].

The main goal in this paper is concerned with the application of the shifted Legendre spectral collocation method to obtain the numerical solution of FBE of the form:

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} - \frac{\partial^{\beta+1} u(x,t)}{\partial x^{\beta+1}} + u(x,t)\frac{\partial u(x,t)}{\partial x} = f(x,t). \tag{1}$$

subject to the conditions

$$\begin{aligned} u(0,t) &= g_0(t), \qquad 0 < t \leq \tau, \\ u(L,t) &= g_1(t), \qquad 0 < t \leq \tau. \end{aligned} \tag{2}$$

and

$$u(x,0) = f_0(x) \qquad 0 < x < L, \tag{3}$$

where $0 < \alpha, \beta < 1$ and $f(x,t)$ is the source term. Here the fractional derivatives are defined in the Caputo sense. The main idea in the current work is to apply the shifted Legendre polynomials and the operational matrix of fractional derivative together with collocation method to discretize Eq.(1) to get a satisfactory result.

The remainder of the paper is organized as follows. In the next section, we introduce some necessary definitions and give some relevant properties of Legendre polynomials. Section 3 summarizes the application of the shifted Legendre collocation method to the solution of problems (1)-(3). As a result, a system of algebraic equations is obtained and the solution of the considered problem is given. In Section 4, some numerical results are reported to clarify the method. Finally, conclusions are given in Section 5.

2. DEFINITIONS AND FUNDAMENTALS

To begin with, we describe some necessary definitions and mathematical preliminaries of the fractional derivative theory.

Definition 2.1. The derivative of order $\alpha \geq 0$ (fractional) according to Caputo is given by

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int\limits_0^t \frac{\partial_\eta u(x,\eta)}{(t-\eta)^{1+\alpha-m}} d\eta & m-1 < \alpha < m \\ \frac{\partial^m u(x,t)}{\partial t^m} & \alpha = m \in \mathbb{N} \end{cases} \quad (4)$$

where m is the smallest integer greater than or equal to α.

The Caputo fractional derivative D^α satisfies the following properties

$$D^\alpha C = 0, \qquad \text{(C is constant)},$$

$$D^\alpha x^n = \begin{cases} 0, & \text{for } n \in N_0 \text{ and } n < \lceil \alpha \rceil, \\ \dfrac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^{n-\alpha}, & \text{for } n \in N_0 \text{ and } n \geq \lceil \alpha \rceil. \end{cases} \quad (5)$$

Next, let us introduce some properties of the shifted Legendre polynomials. The well known Legendre polynomials are defined on the interval $[-1,1]$ and can be determined with the aid of the following recurrence formula:

$$L_{i+1}(t) = \frac{2i+1}{i+1} t L_i(t) - \frac{i}{i+1} L_{i-1}(t), \quad i = 1,2,\cdots,$$

where $L_0(t) = 1$ and $L_1(t) = t$. Let the shifted Legendre polynomials $L_i(\frac{2x}{L} - 1)$ be denoted by $L_{L,i}(x)$, $x \in (0,L)$. Then $L_{L,i}(x)$ can be obtained as follows:

$$L_{L,i+1}(x) = \frac{(2i+1)(2x-L)}{(i+1)L} L_{L,i}(x) - \frac{i}{i+1} L_{L,i-1}(x), \quad i = 1,2,\cdots, \quad (6)$$

where $L_{L,0}(x) = 1$ and $L_{L,1}(x) = \frac{2x}{L} - 1$. The analytic form of the shifted Legendre polynomials $L_{L,i}(x)$ of degree i is given by

$$L_{L,i}(x) = \sum_{k=0}^{i} (-1)^{i+k} \frac{(i+k)!\, x^k}{(i-k)!\, (k!)^2\, L^k}. \quad (7)$$

The orthogonality condition is

$$\int_0^L L_{L,j}(x) L_{L,k}(x) dx = \begin{cases} \dfrac{h}{2j+1}, & j = k, \\ 0, & j \neq k. \end{cases} \quad (8)$$

A function $u(x)$, square integrable in $[0,L]$, may be expressed in terms of shifted Legendre polynomials as

$$u(x) = \sum_{j=0}^{\infty} c_j L_{L,j}(x),$$

where the coefficients c_j are given by

$$c_j = \frac{2j+1}{L} \int_0^L u(x) L_{L,j}(x) dx, \quad j = 0,1,2,\cdots. \tag{9}$$

In practice, only the first $(N+1)$-terms shifted Legendre polynomials are considered. Hence we can write

$$u_M(x) \simeq \sum_{j=0}^{M} c_j L_{L,j}(x) = C^T \Phi_{L,M}(x), \tag{10}$$

where the shifted Legendre coefficient vector C and the shifted Legendre vector $\Phi_{L,M}(x)$ are given by

$$C^T = [a_0, a_1, \cdots, a_M],$$

$$\Phi_{L,M}(x) = [L_{L,0}(x), L_{L,1}(x), \cdots, L_{L,M}(x)]^T. \tag{11}$$

Similarly a function $u(x,t)$ of two independent variables defined for $0 < x < L$ and $0 < t \leq \tau$ may be expanded in terms of the double shifted Legendre polynomials as

$$u_{N,M}(x,t) = \sum_{i=0}^{N} \sum_{j=0}^{M} a_{ij} L_{\tau,i}(t) L_{L,j}(x) = \Phi_{\tau,N}^T(t) \mathbf{A} \Phi_{L,M}(x), \tag{12}$$

where the shifted Legendre vectors $\Phi_{\tau,N}(t)$ and $\Phi_{L,M}(x)$ are defined similarly to Eq. (11); also the shifted Legendre coefficient matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0M} \\ a_{10} & a_{11} & \cdots & a_{1M} \\ \vdots & \vdots & \cdots & \vdots \\ a_{N0} & a_{N1} & \cdots & a_{NM} \end{pmatrix},$$

where

$$a_{ij} = \left(\frac{2i+1}{\tau}\right)\left(\frac{2j+1}{L}\right) \int_0^\tau \int_0^L u(x,t) L_{\tau,i}(t) L_{L,j}(x) dx dt, \\ i = 0,1,\cdots,N, \; j = 0,1,\cdots,M. \tag{13}$$

Theorem 2.1. *Let $\Phi_{\tau,N}(t)$ be the shifted Legendre vector and $\alpha > 0$, then the Caputo fractional derivative of order $\alpha > 0$ of $\Phi_{L,M}(x)$ can be expressed as*

$$D^\alpha \Phi_{L,M}(x) \simeq \boldsymbol{D}^{(\alpha)} \Phi_{L,M}(x), \tag{14}$$

where $\boldsymbol{D}^{(\alpha)}$ is the $(M+1) \times (M+1)$ Legendre operational matrix of the fractional *derivative of order α and is defined as follows:*

$$
\boldsymbol{D}^{(\alpha)} = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & 0 & \cdots & 0 \\
\Theta_{\lceil\alpha\rceil,0} & \Theta_{\lceil\alpha\rceil,1} & \Theta_{\lceil\alpha\rceil,2} & \cdots & \Theta_{\lceil\alpha\rceil,M} \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
\Theta_{i,0} & \Theta_{i,1} & \Theta_{i,2} & \cdots & \Theta_{i,M} \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
\Theta_{M,0} & \Theta_{M,1} & \Theta_{M,2} & \cdots & \Theta_{M,M}
\end{pmatrix}, \tag{15}
$$

where

$$
\Theta_{i,j} = \sum_{k=\lceil\alpha\rceil}^{i} \frac{(-1)^{i+k}\,(2j+1)\,(i+k)!\,(k-j-\alpha+1)_j}{L^{\alpha}\,(i-k)!\,k!\,\Gamma(k-\alpha+1)\,(k-\alpha+1)_{j+1}}, \tag{16}
$$

(see [29, 30] for proof).

3. LEGENDRE SPECTRAL COLLOCATION METHOD

Since the Legendre spectral collocation method approximates the initial boundary problems in physical space and it is a global method, it is very easy to implement and adapt it to various problems, including variable coefficient and nonlinear problems (see, for instance [31]-[35]). In this section, a new algorithm for solving time fractional Burgers' equation is proposed based on Legendre-Gauss-Lobatto spectral collocation approximation together with the Legendre operational matrix for fractional derivative.

To solve problem (1)-(3), we approximate $u(x,t)$ by the shifted Legendre polynomials as

$$
u_{N,M}(x,t) = \Phi^{T}_{\tau,N}(t)\mathbf{A}\Phi_{L,M}(x), \tag{17}
$$

where $\mathbf{A}$ is a $(N+1)\times(M+1)$ unknown matrix. Using Eq. (14) and (17), we can write

$$
\begin{aligned}
\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} &= \Phi_{\tau,N}^{T}(t)\mathbf{D}^{(\alpha)^{T}}\mathbf{A}\Phi_{L,M}(x),\\
\frac{\partial^{\beta+1} u(x,t)}{\partial t^{\beta+1}} &= \Phi_{\tau,N}^{T}(t)\mathbf{A}\mathbf{D}^{(\beta+1)}\Phi_{L,M}(x),\\
\frac{\partial u(x,t)}{\partial x} &= \Phi_{\tau,N}^{T}(t)\mathbf{A}\mathbf{D}^{(1)}\Phi_{L,M}(x),\\
u(x,0) &= \Phi_{\tau,N}^{T}(0)\mathbf{A}\Phi_{L,M}(x),\\
u(0,t) &= \Phi_{\tau,N}^{T}(t)\mathbf{A}\Phi_{L,M}(0),\\
u(L,t) &= \Phi_{\tau,N}^{T}(t)\mathbf{A}\Phi_{L,M}(L).
\end{aligned}
\tag{18}
$$

By using Eq. (17) and (18), the Eqs.(1)-(3) can be represented in the following matrix form,

$$
\begin{aligned}
&\Phi_{\tau,N}^{T}(t)\mathbf{D}^{(\alpha)^{T}}\mathbf{A}\Phi_{L,M}(x) - \Phi_{\tau,N}^{T}(t)\mathbf{A}\mathbf{D}^{(\beta+1)}\Phi_{L,M}(x)\\
&+(\Phi_{\tau,N}^{T}(t)\mathbf{A}\Phi_{L,M}(x))(\Phi_{\tau,N}^{T}(t)\mathbf{A}\mathbf{D}^{(1)}\Phi_{L,M}(x)) = f(x,t),\\
&\Phi_{\tau,N}^{T}(0)\mathbf{A}\Phi_{L,M}(x) = f_0(x),\\
&\Phi_{\tau,N}^{T}(t)\mathbf{A}\Phi_{L,M}(0) = g_0(t),\\
&\Phi_{\tau,N}^{T}(t)\mathbf{A}\Phi_{L,M}(L) = g_1(t).
\end{aligned}
\tag{19}
$$

A collocation scheme is defined in Eq. (19) by evaluating the result at the points (x_i,t_j). For suitable collocation points we use the shifted Legendre-Gauss-Lobatto nodes x_i $(0 \le i \le M)$ and the shifted Legendre roots t_j $(0 \le j \le N-1)$ of $L_{L,N}(t)$; whereof $L_{L,N}(t_j) = 0$ $(0 \le j \le N-1)$. The previous system can be rewritten in form:

$$
\begin{aligned}
&\Phi_{\tau,N}^{T}(t_j)\mathbf{D}^{(\alpha)^{T}}\mathbf{A}\Phi_{L,M}(x_i) - \Phi_{\tau,N}^{T}(t_j)\mathbf{A}\mathbf{D}^{(\beta+1)}\Phi_{L,M}(x_i)\\
&\quad+(\Phi_{\tau,N}^{T}(t_j)\mathbf{A}\Phi_{L,M}(x_i))(\Phi_{\tau,N}^{T}(t_j)\mathbf{A}\mathbf{D}^{(1)}\Phi_{L,M}(x_i)) = f(x_i,t_j),\\
&\qquad\qquad 0 \le i \le M-1, \quad (1 \le j \le N-1),
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
&\Phi_{\tau,N}^{T}(0)\mathbf{A}\Phi_{L,M}(x_i) = f_0(x_i), && 0 \le i \le M,\\
&\Phi_{\tau,N}^{T}(t_j)\mathbf{A}\Phi_{L,M}(0) = g_0(t_j), && 0 \le j \le N-1,\\
&\Phi_{\tau,N}^{T}(t_j)\mathbf{A}\Phi_{L,M}(L) = g_1(t_j), && 0 \le j \le N-1.
\end{aligned}
\tag{21}
$$

This constitute a system of $(N+1)\times(M+1)$ nonlinear algebraic equations in the required double shifted Legendre expansion coefficients $a_{ij},\ i = 0,1,\cdots,M, \quad j = 0,1,\cdots,N$, which can be solved by using any standard iteration technique, like Newton's iteration method. Consequently, the $u_{N,M}(x,t)$ given in Eq. (17) can be calculated.

Table 1

Comparison of MAE of the LSC method and method in [36] at $\alpha = 0.1$ and $\beta = 0.999$.

$N = M$	LSCM	$\frac{1}{\Delta t}$	method in [36]
4	$2.10914.10^{-4}$	5	$4.97588.10^{-3}$
8	$2.19115.10^{-5}$	10	$3.49223.10^{-3}$
12	$5.08634.10^{-6}$	20	$2.37772.10^{-3}$
16	$1.17627.10^{-6}$	40	$1.81607.10^{-3}$
20	$7.80581.10^{-7}$	80	$1.43319.10^{-3}$

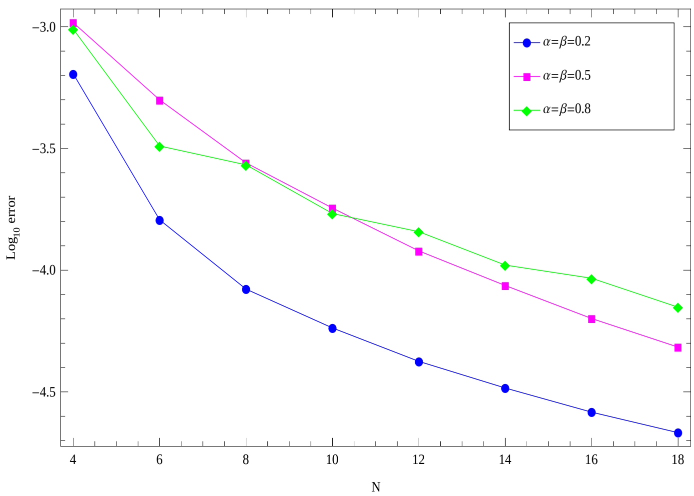

Fig. 1 – Convergence at $\alpha = \beta = 0.2, 0.5, 0.8$.

4. NUMERICAL SIMULATION AND COMPARISON

This section presents numerical illustrations to demonstrate the accuracy of the proposed method for solving the problem (1)-(3). We will report the accuracy and efficiency of the new method based on maximum absolute error (MAE) defined as

$$MAE = \max\{|u(x,t) - u_{N,M}(x,t)|, \quad 0 < x < L, \quad 0 < t < \tau\}.$$

Consider the following fractional Burgers' equation [36, 37],

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} - \frac{\partial^{\beta+1} u(x,t)}{\partial x^{\beta+1}} + u(x,t)\frac{\partial u(x,t)}{\partial x} = f(x,t). \tag{22}$$

with initial condition

$$u(x,0) = x^2(1-x)^2, \qquad 0 < x < L, \tag{23}$$

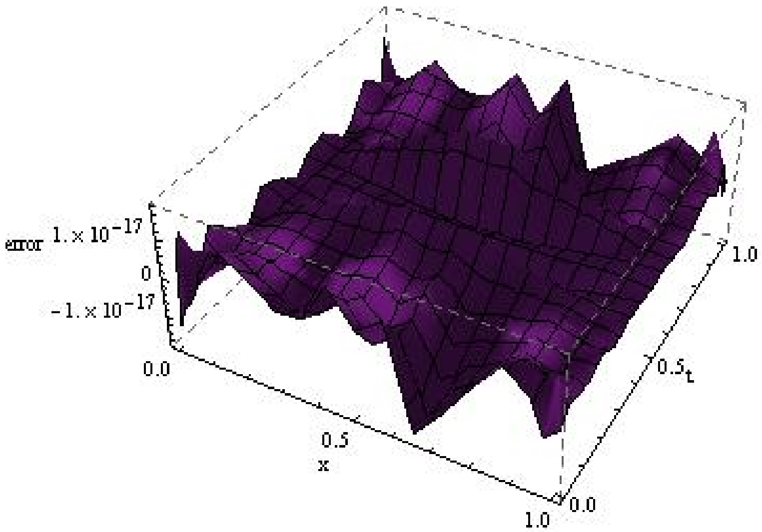

Fig. 2 – The error function at $N = M = 12$ and $\alpha = \beta = 1$.

and boundary conditions

$$u(0,t) = u(1,t) = 0, \qquad 0 < t \le \tau. \tag{24}$$

The function $f(x,t)$ may be chosen such that the exact solution of (22) is $u(x,t) = (4t^2 - 4t + 1)x^2(1-x)^2$. We applied the proposed method and made a comparison of our method with the method in [36]. The obtained results are shown in Table 1. Figure 1 shows the logarithmic graphs of MAEs $(log_{10}Error)$ at $\alpha = \beta = 0.2,\ ,0.5,\ 0.8$. Also, Fig. 2 shows the error function $u(x,t) - u_{12,12}(x,t)$ for $\alpha = \beta = 1$. We observe that the suggested algorithm provides accurate and stable numerical results. This numerical experiment demonstrates the utility of the proposed method.

5. CONCLUSIONS AND FUTURE WORK

We have presented a new space-time spectral algorithm based on shifted Legendre spectral technique combined with the associated operational matrix of Caputo fractional derivative. This algorithm was employed for solving space-time FBE. According to the numerical results given in the previous section, it has been concluded that the proposed method may be extended to solve several types of nonlinear partial FDEs, such as nonlinear space-time fractional Klein-Gordon equation, time-fractional Schrödinger equation, fractional coupled Klein-Gordon-Schrödinger equation, and space-time fractional coupled Burgers' equation.

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