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A NUMERICAL SOLUTION OF A FRACTIONAL OSCILLATOR EQUATION IN A NON-RESISTING MEDIUM WITH NATURAL BOUNDARY CONDITIONS

TOMASZ BLASZCZYK

Czestochowa University of Technology, Institute of Mathematics,
al. Armii Krajowej 21, 42-201 Czestochowa, Poland
E-mail: tomasz.blaszczyk@im.pcz.pl

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Abstract. In this paper we propose a numerical solution of a fractional oscillator equation with natural boundary conditions. A numerical scheme is presented to solve these equations. In the final part of this paper, examples of numerical solutions of this equation are shown. We also determine the convergence order of numerical schemes.

Key words: Fractional Euler–Lagrange equation, Fractional oscillator equation, Numerical solution, Natural boundary conditions.

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1. INTRODUCTION

Harmonic oscillators arise in various fields such as classical mechanics, electronics engineering, experimental and quantum physics, etc. Classical harmonic oscillators are defined using integer order derivatives. In recent years, the phenomena that obey the equation of motion with fractional derivatives have become an important topic. In several investigations, fractional oscillators are defined by replacing the integer time derivative in oscillator equations with fractional derivatives of order α and they are solved analytically, using, for instance, Laplace transform method, or numerical schemes [1, 2, 14, 27].

In a large part of classical and quantum mechanics we deal with simple harmonic oscillators that are obtained from conservative Lagrangian or Hamiltonian functions. This approach can also be used in fractional calculus. Riewe in [25, 26] developed mechanical models for nonconservative systems in terms of fractional derivatives. Next, Klimek [16, 17] and Agrawal [3, 4] extended Riewe's results. Since then many authors have become interested in the fractional variational calculus field (see [8, 13, 19–21]). These equations are known in the literature as the fractional Euler–Lagrange equations and they contain both the left and right fractional derivatives. Unfortunately, not all fractional Euler–Lagrange equations can be solved analytically, and numerical methods are necessary to obtain their solutions

[5, 7, 9, 11, 12].

In contrast to some references above, where the authors analysed fractional oscillator equations with the Dirichlet type of boundary conditions, we consider the fractional oscillator equation with natural boundary conditions [4, 6, 18] in this paper.

2. BASIC DEFINITIONS AND FORMULATION OF THE PROBLEM

We recall some definitions of fractional operators.

Fractional Riemann-Liouville integrals of order $\alpha > 0$ in finite interval $t \in [0, b]$ are defined by [15, 23, 24]

$$I_{0+}^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau \quad (t > 0) \tag{1}$$

$$I_{b-}^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_t^b \frac{f(\tau)}{(\tau - t)^{1-\alpha}} d\tau \quad (t < b) \tag{2}$$

Using the fractional integrals from above one can define fractional derivatives. The left Riemann-Liouville fractional derivative of order $\alpha \in (0, 1)$ has the form [15, 23, 24]

$$D_{0+}^{\alpha} f(t) := D\, I_{0+}^{1-\alpha} f(t) \tag{3}$$

where D is the operator of the first order derivative.

The right Caputo fractional derivative of order $\alpha \in (0, 1)$ looks as follows [15, 23, 24]

$$^{C}D_{b-}^{\alpha} f(t) := D_{b-}^{\alpha} f(t) - \frac{(b-t)^{-\alpha}}{\Gamma(1-\alpha)} f(b) \tag{4}$$

In this work we consider the fractional oscillator equation in the finite time interval $t \in [0, b]$ in the following form

$$-^{C}D_{b-}^{\alpha}\, D_{0+}^{\alpha} f(t) + \omega^2 f(t) = A\, g(t) \tag{5}$$

In particular, when $\alpha = 1$, then Eq. (5) becomes the differential equation of the oscillator (the harmonic oscillator in a non-resisting medium) in the presence of an external driving force $A\, g(t)$

$$D^2 f(t) + \omega^2 f(t) = A\, g(t) \tag{6}$$

where ω is the frequency and A is a constant.

Eq. (5) is one of the Euler–Lagrange equations and was derived from the following functional

$$J[f] = \int_0^b \left[-\frac{1}{2} \left(D_{0+}^{\alpha} f \right)^2 + \frac{\omega^2}{2} f^2 - f\, A\, g \right] dt, \tag{7}$$

by applying the formula of integration by parts [5, 18]

$$\int_0^b g(t)\, D_{0+}^{\alpha} f(t) dt = \int_0^b f(t)\, {}^C D_{b-}^{\alpha} g(t) dt + \left[D_{0+}^{\alpha-1} f(t)\, g(t) \right]_0^b . \tag{8}$$

When we assume that the functional (7) satisfies the condition

$$f(0) = 0 \tag{9}$$

then the boundary condition is specified only at $t = 0$ so that we can develop the natural boundary condition in the following form [4]

$$D_{0+}^{\alpha} f(t) \big|_{t=b} = 0 \tag{10}$$

To obtain the optimum solution, the conditions (9) and (10) must be satisfied.

3. NUMERICAL SOLUTION

In this section we present a numerical scheme for a system containing Eq. (5) and conditions (9) and (10). We introduce the homogeneous grid of nodes: $0 = t_0 < t_1 < ... < t_i < t_{i+1} < ... < t_n = b$, with the constant time step $\Delta t = b/n$. Additionally we will denote the values of functions $f(t)$ and $g(t)$ at the node t_i by $f_i = f(t_i)$ and $g_i = g(t_i)$.

Using results from our previous works [10, 11] we obtain the discrete form for both fractional differential operators occurring in Eq. (5).

The discrete form of the left Riemann-Liouville fractional derivative (3) at nodes t_i for $i = 1, 2, ..., n$ is approximated by the formula [10, 11]

$$D_{0+}^{\alpha} f_i = {}^C D_{0+}^{\alpha} f_i \cong \sum_{j=0}^{i} f_j v(i,j) \tag{11}$$

and the right Caputo derivative at nodes t_i, $i = 0, 1, ..., n-1$ has the form [10, 11]

$${}^C D_{b-}^{\alpha} f_i \cong \sum_{j=i}^{n} f_j v(n-i, n-j) \tag{12}$$

where coefficients v are given by formula

$$v(i,j) = \frac{(\Delta t)^{\alpha}}{\Gamma(2-\alpha)} \begin{cases} -i^{1-\alpha} + (i-1)^{1-\alpha} & \text{for } j = 0 \\ (i-j+1)^{1-\alpha} - 2(i-j)^{1-\alpha} & \\ \quad + (i-j-1)^{1-\alpha} & \text{for } j = 1, ..., i-1 \\ 1 & \text{for } j = i \end{cases} \tag{13}$$

Using the above formulae (11) and (12) we can describe the discrete form of the composition of both operators ${}^{C}D_{b-}^{\alpha}\, D_{0+}^{\alpha} f(t)$ at nodes $t = t_i$ for $i = 1, ..., n-1$

$$ {}^{C}D_{b-}^{\alpha} D_{0+}^{\alpha} f_i \cong \sum_{k=i}^{n} \left[v(n-i, n-k) \sum_{j=0}^{k} v(k,j) f_j \right] \tag{14} $$

Finally Eq. (5) with boundary conditions (9) and (10) one can write in the discrete form as the system of $n+1$ linear equations

$$ \begin{aligned} & f_0 = 0 \\ & \sum_{k=i}^{n} \left[v(n-i, n-k) \sum_{j=0}^{k} v(k,j)\, f_j \right] - \omega^2 f_i = -A g_i, \quad \text{for } i = 1, ..., n-1 \\ & \sum_{j=0}^{n} v(n,j)\, f_j = 0 \end{aligned} \tag{15} $$

The obtained system of equations (15) can be solved numerically.

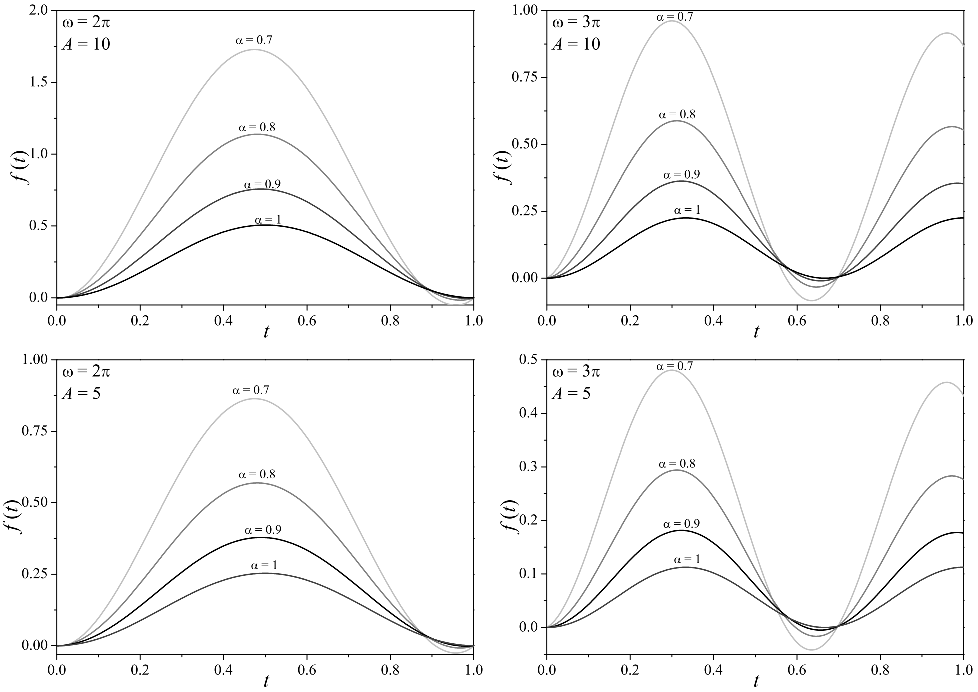

Fig. 1 – Examples of numerical solutions of Eq. (16) for $\alpha \in (0,1]$.

4. EXAMPLES

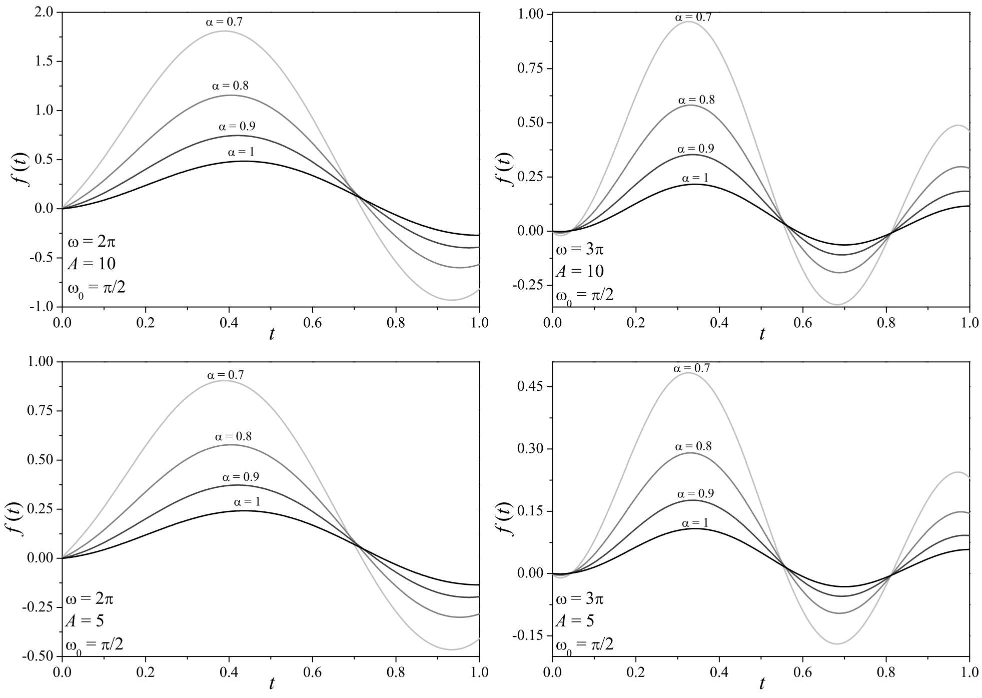

Fig. 2 – Examples of numerical solutions of Eq. (16) for $\alpha \in (0,1]$.

In this section, we consider two examples of forcing function $g(t)$ to show numerical solutions of Eq. (5) with boundary conditions (9) and (10). We used the LUP decomposition method [22] to solve the system of equations (15). In all examples we have assumed $b = 1$. We present several examples of calculations for different values of parameters α, ω, A and forms of function $g(t)$. For all presented graphs of functions (see Figs. 1, 2, and 3), in the calculations we assume that the time domain $t \in [0,1]$ has been divided into $n = 1024$ subintervals.

4.1. STEADY FORCING FUNCTION, $g(t) = 1$.

As the first numerical example, we consider the following fractional oscillator equation

$$-{}^{C}D^{\alpha}_{1-}\, D^{\alpha}_{0+}\, f(t) + \omega^2 f(t) - A = 0 \tag{16}$$

with boundary conditions (9) and (10). For computational purposes, we take $\omega \in \{2\pi, 3\pi\}$ and $A \in \{5, 10\}$. The numerical solutions for $\alpha \in \{0.7; 0.8, 0.9\}$ are dis-

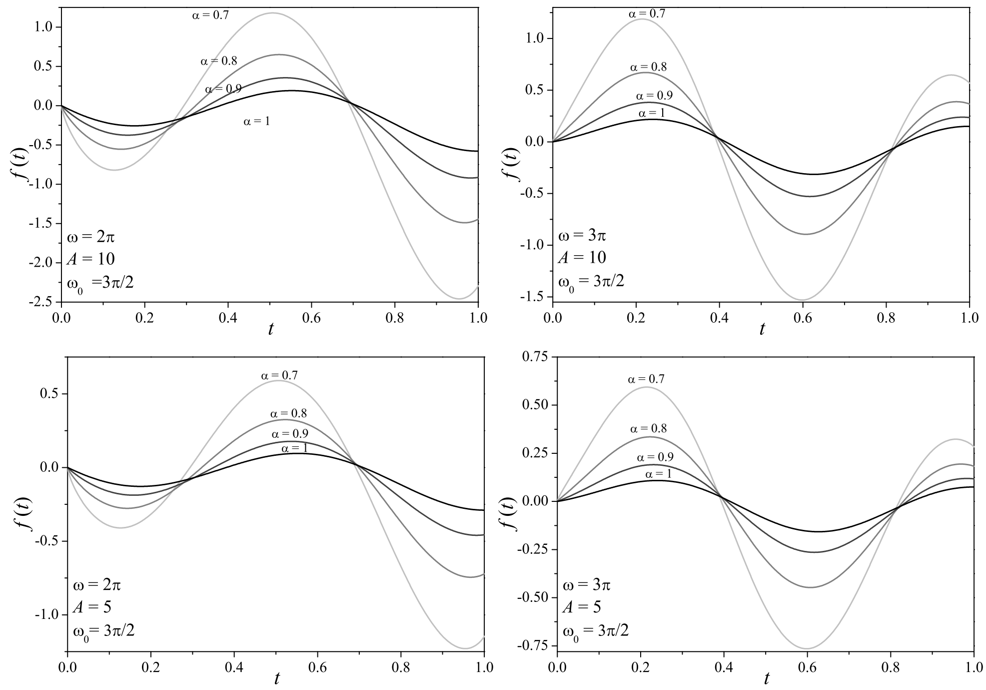

Fig. 3 – Examples of numerical solutions of Eq. (16) for $\alpha \in (0,1]$.

played in Fig. 1.

The case $\alpha = 1$ represents the harmonic oscillator with integer derivatives. We can see how changes of the parameter α influence the frequency of oscillations in comparison to the classical oscillator equation ($\alpha = 1$).

4.2. PERIODIC FORCING FUNCTION, $g(t) = \cos(\omega_0 t)$.

As the second numerical example, we consider the following fractional oscillator equation

$$-{}^{C}D^{\alpha}_{1^-}\, D^{\alpha}_{0^+} f(t) + \omega^2 f(t) - A\cos(\omega_0 t) = 0 \tag{17}$$

with boundary conditions (9) and (10). Figures 2 and 3 present solutions for $\omega \in \{2\pi, 3\pi\}$, $A \in \{5, 10\}$, variable values of parameter α, and different forms of function $q(t)$. We show the influence of parameters α, λ and forms of function $q(t)$ on the solution.

To demonstrate the stability and convergence of our numerical scheme, we assume that the numerical solutions for $\Delta t = 1024$ is the best approximation of the results, in compliance with paper [28]. And then, we used this result to estimate the error in the solution for a given step size and to determine the convergence order of the numerical scheme. We used the following formula

$$p = \log_2 \left(\frac{Err\,(h)}{Err\,(h/2)} \right), \tag{18}$$

where Err represents maximum absolute error.

The results are shown in Table 1.

Table 1

Errors Err and convergent order p for computation of Eq. (16) with parameters $\omega = 1$ and $F = 1$.

α	$h = 1/n$	Err	p
0.7	1/16	0.26887294	-
	1/32	0.14340200	0.91
	1/64	0.07280034	0.98
	1/128	0.03492306	1.06
0.8	1/16	0.19949658	-
	1/32	0.10635234	0.91
	1/64	0.05405180	0.98
	1/128	0.02601033	1.05
0.9	1/16	0.13952208	-
	1/32	0.07359016	0.92
	1/64	0.03728320	0.98
	1/128	0.01795833	1.05

Analysing the values in the table, one can observe that the rate of convergence p does not depend on the fractional order α. We observe that as the step size is reduced by half, the errors are also reduced by half. Therefore, the numerical scheme is first-order convergent.

5. CONCLUSIONS

In this paper the fractional oscillator equation in a non-resisting medium with natural boundary conditions is considered. The discrete form of this equation was presented as a system of linear algebraic equations. The obtained system of equations was solved numerically. The equation was solved for derivatives of different orders α, different values of parameters ω, A and different forms of function $g(t)$. In order to ensure stability of the computation, the convergent order is evaluated numerically. We observe that the numerical method has first-order convergence.

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