Romanian Reports in Physics, Vol. 67, No. 2, P. 359–374, 2015

DEVELOPMENT OF A MULTI CONVERGENCE-CONTROLLER PARAMETRIC PERTURBATION APPROACH FOR SOME LINEAR AND NONLINEAR INTEGRAL EQUATIONS

K. SAYEVAND[1], DUMITRU BALEANU[2], M. FARDI[3]

[1]Faculty of Mathematical Sciences, University of Malayer, Malayer, Iran
Email: *ksayehvand@malayeru.ac.ir*
[2]Department of Chemical and Materials Engineering, Faculty of Engineering, King Abdulaziz University, P. O. Box 80204, Jeddah, 21589, Saudi Arabia
Çankaya University, Faculty of Art and Sciences, Department of Mathematics and Computer Sciences, Ankara, Turkey
Institute of Space Sciences, MG-23, RO-077125, Magurele-Bucharest, Romania
E-mail: dumitru@cankaya.edu.tr
[3]Department of Mathematics, Islamic Azad University, Najafabad Branch, Najafabad, Iran
E-mail: fardi@yahoo.com

Received July 9, 2014

Abstract. In this article, some integral equations that have been appeared in the study of physical phenomena are investigated by using the unique properties of the extended homotopy analysis method. In other words, the traditional homotopy analysis method is extended to a multi-parametric approach. Also, convergence-controller parameters are introduced to approximate the solutions of such linear and nonlinear integral equations. In this framework, the convergence of the proposed strategies is investigated. Several numerical examples are presented to illustrate the accuracy and effectiveness of the proposed approaches. The obtained results and comparison with other methods provide confirmation for the validity of our numerical schemes.

Key words: Multi-parametric approach, Convergence-controller parameters, Integral equations.

1. INTRODUCTION

In some physical phenomena related with potential and electromagnetic or acoustic radiation theory, problems are modelled as differential, integral, and integro-differential equations. Since finding the solution of these equations is too complicated, recently a lot of attention has been devoted by researchers to find their analytical and numerical solution. Many researchers have been focusing on the development of more advanced and efficient approaches for integral equations as the Taylor method, the Adomian decomposition method, homotopy perturbation method and many other methods [1, 2]. In this paper the extended *homotopy analysis method* (HAM) is applied for these type of equations.

The HAM has been presented by Liao [3–7] to obtain the analytical solutions for various linear and nonlinear problems. There are many works that deal with the HAM. For instance Abbasbandy *et al.* [8] applied the Newton-homotopy analysis method to solve nonlinear algebraic equations, Fakhari *et al.* [9] applied the HAM to approximate explicit solutions of nonlinear BBMB equations, Allan [10] constructed the analytical solutions to Lorenz system by the HAM, and Bataineh *et al.* [11, 12] proposed a new reliable modification of the HAM. Moreover, M. Ganjiani *et al.* [13] constructed the analytical solutions to coupled nonlinear diffusion reaction equations by the HAM, Alomari *et al.* [14] applied the HAM to study delay differential equations, Chen and Liu [15] applied the HAM to increase the convergent region of the harmonic balance method, Liao [16] proposed the HAM to study nonlinear problems; see also Refs. [17–28].

In this paper, after a short background on homotopy analysis method, we extend the application of this strategy to the following integral equations:

1. The second kind of linear and nonlinear two-dimensional Fredholm integral equations

$$F(u(t,x)) = u(t,x) - f(t,x) - \int_a^b \int_c^d K(t,s,x,\xi)N(u(s,\xi))d\xi ds;\ (t,x) \in D,$$

where $f(t,x)$ and $K(t,s,x,\xi)$ are analytical functions on $D = L^2([a,b]\times[c,d])$ and $E = D \times D$, respectively.

2. The mixed Volterra-Fredholm integral equations

$$N(y(x)) = y(x) - f(x) - \lambda_1 \int_0^x K_1(x,t)F(y(t))dt - \lambda_2 \int_0^1 K_2(x,t)G(y(t))dt = 0,$$

where $f(x)$ and the kernels $K_1(x,t)$ and $K_2(x,t)$ are assumed to be in $L^2(\Re)$ on the interval $0 \le x,t \le 1$.

The new generalization demonstrates an accurate solution if compared with traditional HAM, and therefore it has been shown to be computationally efficient. Moreover, we prove the convergence of the proposed approaches for these type of integral equations.

2. MAIN RESULTS

2.1. DESCRIPTION OF APPROACH

A second kind of two-dimensional Fredholm integral equation can be considered as the following

$$F(u(t,x)) = u(t,x) - f(t,x) - \int_a^b \int_c^d K(t,s,x,\xi)N(u(s,\xi))d\xi ds;\ (t,x) \in D, \quad (1)$$

where $f(t,x)$ and $K(t,s,x,\xi)$ are analytical functions on $D = L^2([a,b]\times[c,d])$ and $E = D\times D$, respectively.

We choose $u_0(t,x) = f(t,x)$ as initial approximation guess for simplicity, in order to obtain convergent series solutions to the two-dimensional Fredholm integral Eq. (1), we first construct the zeroth order deformation equation

$$(1-A(q;\varpi_1))[\varphi(t,x;q)-u_0(t,x)] = B(q;\hbar)[\varphi(t,x;q)-f(t,x) - \int_a^b\int_c^d K(t,s,x,\xi)N(\varphi(s,\xi;q))d\xi ds], \quad (2)$$

where

$$A(q;\varpi) = (1-\varpi)\sum_{j=1}^{\infty}\varpi^{j-1}q^j, \ \ |\varpi|<1, \quad (3)$$

$$B(q;\hbar) = q\hbar, \ \hbar\neq 0. \quad (4)$$

Due to Taylor's theorem, we can write

$$\varphi(t,x;q) = u_0(t,x)+\sum_{j=1}^{\infty}u_j(t,x)q^j, \quad (5)$$

where

$$u_j(t,x) = \frac{1}{j!}\frac{\partial^j\varphi(t,x;q)}{\partial q^j}|_{q=0}. \quad (6)$$

The convergence of series (5) depends upon $\hbar$ and ϖ. Assume that $\hbar$ and ϖ are properly chosen so that the power series of (5) converges at $q=1$, then we have under this assumption the solution series

$$u(t,x) = u_0(t,x)+\sum_{j=1}^{\infty}u_j(t,x). \quad (7)$$

By differentiating (2) m times with respect to q, then dividing the equation by $m!$ and setting $q=0$, the mth-order deformation equation is formulated as follows

$$u_m(t,x)-\sum_{k=1}^{m-1}(1-\varpi)\varpi^{m-k-1}u_k(t,x) = \hbar H_m(u_0(t,x),\cdots,u_{m-1}(t,x)), \quad (8)$$

where

$$H_m = u_{m-1}(t,x)-(1-\chi_m)f(t,x) - \int_a^b\int_c^d K(t,s,x,\xi)\frac{\partial^{m-1}N(\varphi(s,\xi;q))}{(m-1)!\partial q^{m-1}}\,|_{q=0}\,d\xi ds, \quad (9)$$

$$u_m(t,x) = \frac{\partial^m\varphi(t,x;q)}{m!\partial q^m}\,|_{q=0}, \quad (10)$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1; \\ 1, & m > 1. \end{cases} \tag{11}$$

The mth-order deformation equations (8) are linear in principle. The code is developed by using the symbolic computation software *MAPLE*. Then, the Nth-order approximate solution of (8) can be written as

$$U_N(t,x) = u_0(t,x) + \sum_{j=1}^{N} u_j(t,x). \tag{12}$$

If $\varpi = 0$ the mth-order deformation equation defined by (8) becomes

$$u_1(t,x) = \hbar[u_0(t,x) - f(t,x) - \int_a^b \int_c^d K(t,s,x,\xi) N(u_0(s,\xi;q)) d\xi ds], \tag{13}$$

and

$$u_m(t,x) - u_{m-1}(t,x) = \hbar[u_{m-1}(t,x) - \int_a^b \int_c^d K(t,s,x,\xi) \frac{\partial^{m-1} N(\varphi(s,\xi;q))}{(m-1)!\partial q^{m-1}} |_{q=0} d\xi ds], \tag{14}$$

Then, we have the following remarks.

Remark 2.1.1: The value $\varpi = 0$ reduces the present approach to the traditional HAM.

Remark 2.1.2: The values $\hbar = -1$ and $\varpi = 0$ reduce the present approach to the ADM.

2.2. CONVERGENCE THEOREMS

Theorem 2.2.1 If the solution series

$$u(t,x) = u_0(t,x) + \sum_{j=1}^{\infty} u_j(t,x), \tag{15}$$

is convergent, then we have the following statement

$$\sum_{m=1}^{\infty} H_m = 0. \tag{16}$$

Proof. Since the solution series

$$u(t,x) = u_0(t,x) + \sum_{j=1}^{\infty} u_j(t,x), \tag{17}$$

is convergent, we have

$$\lim_{m \to \infty} u_m(t,x) = 0. \tag{18}$$

The left-hand side of (8) satisfies

$$\begin{aligned}\sum_{m=1}^{\infty}\big[u_m(t,x)-\sum_{k=1}^{m-1}(1-\varpi)\varpi^{m-k-1}u_k(t,x)\big]&\\ &=u_1(t,x)\\ &+u_2(t,x)-(1-\varpi)u_1(t,x)\\ &+u_3(t,x)-(1-\varpi)\varpi u_1(t,x)-(1-\varpi)u_2(t,x)\\ &\vdots\\ &=(1-(1-\varpi)\sum_{j=0}^{\infty}\varpi^j)\sum_{j=0}^{\infty}u_j(t,x)=0.\end{aligned} \tag{19}$$

Then, from (8) and (19) we have

$$\sum_{m=1}^{\infty}H_m=0. \tag{20}$$

Theorem 2.2.2 Assume that the operator $N[u(t,x)]$ is contraction and the solution series

$$u(t,x)=u_0(t,x)+\sum_{j=1}^{\infty}u_j(t,x), \tag{21}$$

is convergent, then it must be the solution of two-dimensional Fredholm integral equation.

Proof. Let

$$\varepsilon(t,x;q)=\varphi(t,x;q)-f(t,x)-\int_a^b\int_c^d K(t,s,x,\xi)N(\varphi(s,\xi;q))d\xi ds. \tag{22}$$

Using Taylor's series around $q=0$ for $\varepsilon(t,x;1)$, we have

$$\begin{aligned}\varepsilon(t,x;1)=&\sum_{m=0}^{\infty}\frac{1}{m!}\frac{\partial^m\varphi(t,x;q)}{\partial q^m}|_{q=0}-f(t,x)\\ &-\int_a^b\int_c^d K(t,s,x,\xi)\sum_{m=0}^{\infty}\frac{1}{m!}\frac{\partial^m N(\varphi(s,\xi;q))}{\partial q^m}|_{q=0}d\xi ds.\end{aligned} \tag{23}$$

If the solution series

$$u(t,x)=u_0(t,x)+\sum_{j=1}^{\infty}u_j(t,x), \tag{24}$$

is convergent, then the series

$$\sum_{m=0}^{\infty} \frac{1}{m!} \frac{\partial^m N(\varphi(t,x;q))}{\partial q^m}|_{q=0}, \tag{25}$$

will converge to $N[u(t,x)]$ (see [29]).
Now, by using Theorem 1 we have

$$\varepsilon(t,x;1) = u(t,x) - f(t,x) - \int_a^b \int_c^d K(t,s,x,\xi) N(u(s,\xi)) d\xi ds = 0. \tag{26}$$

This ends the proof.

3. RESULTS AND DISCUSSION

The Nth-order approximation of the solution $u(t,x)$ can be expressed as

$$U_N(t,x) = u_0(t,x) + \sum_{j=1}^{N} u_j(t,x), \tag{27}$$

which is mathematically dependent upon the convergence-control parameters $\hbar$ and ϖ.

In our work for optimal values of $\hbar$ and ϖ, we use a technique that has been shown to produce a fast converging approximation. In principle, the technique seeks to minimize the exact residual error (ERE) of (1) at the Nth-order approximation. The ERE is given by

$$\widehat{E}_M(\hbar,\varpi) = \int_a^b \int_c^d (F(U_N(s,\xi)))^2 d\xi ds, \tag{28}$$

In practice, however, the evaluation of $\widehat{E}_M(\hbar,\varpi)$ tends to be time-consuming. A simpler alternative consists of calculating the averaged residual error (ARE). We use here the ARE defined by

$$E_M^n(\hbar,\varpi) = \frac{(b-a)(d-c)}{n^2} \sum_{j=0}^{n} \sum_{i=0}^{n} (F(U_N(t_i,x_j)))^2, \tag{29}$$

where

$$t_i = \frac{(b-a)i}{n},\ i = 1,2,...,n,\ \ x_j = \frac{(d-c)j}{n},\ j = 1,2,...,n. \tag{30}$$

At the Nth-order of approximation, the ARE contains two unknown convergence-control parameters, whose "optimal" values are determined by solving the nonlinear algebraic equations

$$\frac{\partial E_M^n}{\partial \hbar} = 0,\ \frac{\partial E_M^n}{\partial \varpi} = 0. \tag{31}$$

Table 1

Absolute errors of the proposed approach and ADM (Example 3.1).

	convergence-control parameters $(\hbar, \varpi)$		
$t = x$	$(-1.521, -0.148)$ *with seven terms*	$(-1.456, 0)$ *with seven terms*	ADM *with seven terms*
1	$1.531E-4$	$3.996E-4$	$2.503E-3$
$\frac{1}{2}$	$7.651E-5$	$1.998E-4$	$1.252E-3$
$\frac{1}{2^2}$	$3.826E-5$	$9.999E-5$	$6.258E-4$
$\frac{1}{2^3}$	$1.913E-5$	$4.995E-5$	$3.129E-4$
$\frac{1}{2^4}$	$9.567E-6$	$2.497E-5$	$1.565E-4$
$\frac{1}{2^5}$	$4.783E-6$	$1.249E-5$	$7.823E-5$
$\frac{1}{2^6}$	$2.392E-6$	$6.244E-6$	$3.911E-5$

In this section, two two-dimensional integral equations are employed to illustrate the validity of the present approach described in Section 2. The convergence, accuracy, and efficiency of this approach are investigated by comparing it with the THAM and the ADM.

Example 3.1 First, consider the following nonlinear two-dimensional integral equation

$$u(t,x) = x\sin(\pi t) - \frac{x}{6} + \int_0^1 \int_0^1 (x + \cos(\pi s))u^2(s,\xi)d\xi ds, \tag{32}$$

with the exact solution

$$u(t,x) = x\sin(\pi t). \tag{33}$$

For $\hbar \neq 0$ and $\varpi = 0$, our approach gives the "optimal" value of the convergence-control parameter $\hbar \neq 0$ by solving the equation $\frac{dE_4^{20}}{dh} = 0$, which leads to $\hbar = -1.456$ with the corresponding minimum ARE $E_4^{20} = 5.483E-7$. For $\hbar \neq 0$ and $\varpi \neq 0$, we obtain the "optimal" values of $\hbar = -1.521$ and $\varpi = -0.148$ by solving the algebraic equations $\frac{dE_4^{20}}{dh} = 0$ and $\frac{dE_4^{20}}{d\varpi} = 0$, which leave us with the corresponding minimum ARE $E_4^{20} = 4.458E-22$.

For comparison of the solution series given by the present approach with the exact solution, we report the absolute error which is defined by

$$|e_N(t,x)| = |u(t,x) - U_N(t,x)|.$$

In Table 1, we compared the present approach with the ADM. The approximate solutions given by the present approach are more accurate than the solution given by the ADM, as shown in Table 1. **Example 3.2** Consider the following linear two-dimensional integral equation

$$u(t,x) = xe^{-t} + (\frac{e^{-2}}{4} - \frac{1}{4})x + \frac{e^{-2}}{6} - \frac{1}{6} + \int_0^1 \int_0^1 (x+\xi)e^{-(2t-s)}u(s,\xi)d\xi ds, \tag{34}$$

Table 2

Absolute errors of the proposed approach and ADM (Example 3.2).

	convergence-control parameters $(\hbar, \varpi)$		
$t=x$	$(-0.756,-0.755)$ *with nine terms*	$(-1.581,0)$ *with nine terms*	ADM *with nine terms*
1	$3.712E-6$	$1.109E-3$	$1.699E-2$
$\frac{1}{2}$	$2.520E-6$	$9.600E-4$	$1.161E-2$
$\frac{1}{2^2}$	$1.929E-6$	$8.953E-4$	$8.914E-3$
$\frac{1}{2^3}$	$1.625E-6$	$8.629E-4$	$7.567E-3$
$\frac{1}{2^4}$	$1.481E-6$	$8.467E-4$	$6.894E-3$
$\frac{1}{2^5}$	$1.401E-6$	$8.386E-4$	$6.557E-3$
$\frac{1}{2^6}$	$1.366E-6$	$8.345E-4$	$6.389E-3$

with the exact solution

$$u(t,x) = xe^{-t}. \tag{35}$$

For $\hbar \neq 0$ and $\varpi = 0$, the present approach reduces to traditional HAM and E_8^{20} has the minimum $1.393E-7$ at the "optimal" value $\hbar = -1.581$. For $\hbar \neq 0$ and $\varpi \neq 0$, the optimal convergence occurs at $\hbar = -1.155$ and $\varpi = -0.306$ and has an ARE of $E_8^{20} = 9.892E-22$.
The approximate solutions given by the present approach are more accurate than the solutions given by the ADM and the THAM, as shown in Table 2.

4. BIPARAMETRIC HOMOTOPY METHOD: A FRESH VIEW ON MIXED VOLTERRA-FREDHOLM INTEGRAL EQUATIONS

4.1. ANALYSIS OF METHOD

Lets define the (jointly continuous) map $Y(x;q) \to y(x)$, where the embedding parameter $q \in [0,1]$ is such that, as q increases from 0 to 1, $Y(x;q)$ varies from the initial guess to the exact solutions $y(x)$. To ensure this, we construct a general zero-order deformation equation of the governing equation

$$\begin{aligned}(1-q)[Y(x;q) - y_0(x) - q\varpi[K_1^*(F(Y(x;q)) - F(y_0(x)))\\ + K_2^*(G(Y(x;q)) - G(y_0(x)))]] = q\hbar N(Y(x;q)),\end{aligned} \tag{36}$$

where $\hbar$ and ϖ are two convergence-controller parameters which help ensure the convergence of the solution series; the operator $N(Y(x;q))$ is defined by the governing equation (1) and

$$K_1^*(F(Y(x;q)) - F(y_0(x))) = \int_0^x K_1(x,t)[F(Y(t;q)) - F(y_0(t))]dt, \tag{37}$$

$$K_2^*(G(Y(x;q)) - G(y_0(x))) = \int_0^1 K_2(x,t)[G(Y(t;q)) - G(y_0(t))]dt. \quad (38)$$

Clearly, when $q = 0$ the zero-order deformation equation (36) gives

$$Y(x;0) = y_0(x). \quad (39)$$

When $q = 1$ it becomes

$$Y(x;1) = y(x). \quad (40)$$

Here, $y_0(x)$ is the initial guess. As q increases from 0 to 1, $Y(x;q)$ varies from the initial guess $y_0(x)$ to the exact solution $y(x)$. With the help of Taylor's theorem, (39) and (3), we have

$$Y(x;q) = y_0(x) + \sum_{m=1}^{\infty} y_m(x), \quad (41)$$

where

$$y_m(x) = D_m(Y(x;q)). \quad (42)$$

It should be noted that the convergence of series (41) depends upon $\hbar$ and w, assuming that $\hbar$ and w are selected such that series (41) is convergent at $q = 1$; then due to (41) we can write

$$y(x) = y_0(x) + \sum_{m=1}^{\infty} y_m(x). \quad (43)$$

This expression provides us with a relationship between the initial guess $y_0(x)$ and the exact solution $y(x)$ by means of the terms $y_m(x),\ m = 1,2,3,\cdots$, which are determined by the so-called high-order deformation equations described below.

Let us define the vectors

$$\overrightarrow{y_m} = \{y_0(x), y_1(x), \cdots, y_m(x)\}. \quad (44)$$

According to definition (6), the governing equation of $y_m(x)$ can be derived from the zero-order deformation equation (36). Differentiating the zero-order deformation equation (36) m times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equation,

$$\begin{aligned} &y_m(x) - \chi_m y_{m-1}(x) = \\ &\qquad \hbar R_m(\overrightarrow{y_{m-1}}, x) + \chi_m \varpi H_{m-1}(\overrightarrow{y_{m-2}}, x) + Z_m \varpi H_{m-2}(\overrightarrow{y_{m-3}}, x), \end{aligned} \quad (45)$$

where

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1, \end{cases} \quad (46)$$

$$Z_m = \begin{cases} 0, & m \leq 2, \\ -1, & m > 2, \end{cases} \tag{47}$$

$$R_m(\overrightarrow{y_{m-1}}, x) = D_{m-1}(N[Y(x;q)]). \tag{48}$$

and

$$H_m(\overrightarrow{y_{m-1}}, x) = K_1^* D_{m-1}(F(Y(x;q))) + K_2^* D_{m-1}(G(Y(x;q))). \tag{49}$$

The mth-order deformation equation is linear in principle. The code is developed by using the symbolic computation software *MAPLE*. Then, the Nth-order approximate solution can be written as

$$y(x) \approx Y_N(x, \hbar, \varpi) = y_0(x) + \sum_{m=1}^{N} y_m(x). \tag{50}$$

The present method contains the convergence-controller parameters $\hbar$ and ϖ, which provide us with a simple way to ensure the convergence of series solution [30]. In the present paper the convergence-controller parameters $\hbar$ and ϖ were chosen optimally from the so-called averaged residual error (ARE)

$$E_N^n(\hbar, \varpi) = \frac{1}{n+1} \sum_{j=0}^{n} (N(Y_N(\frac{j}{n}, \hbar, \varpi))^2, \tag{51}$$

which is evaluated at the Nth-order approximation. The best values of $\hbar$ and ϖ are given by two nonlinear algebraic equations

$$\frac{\partial E_N^n}{\partial \hbar} = 0, \ \frac{\partial E_N^n}{\partial \varpi} = 0. \tag{52}$$

4.2. ANALYSIS OF CONVERGENCE

In this part, we also prove the convergence of the solution for mixed Volterra-Fredholm integral equations.

Lemma 4.2.1 Assume that the operators $F[y(x)]$ and $G[y(x)]$ are contraction and the series

$$y_0(x) + \sum_{m=1}^{\infty} y_m(x), \tag{53}$$

converges to $y(x)$, then

$$\sum_{m=0}^{\infty} [\chi_m H_{m-1}(\overrightarrow{y_{m-2}}, x) + Z_m H_{m-2}(\overrightarrow{y_{m-3}}, x)] = 0. \tag{54}$$

Proof. If the series

$$y_0(x) + \sum_{m=1}^{\infty} y_m(x), \tag{55}$$

converges to $y(x)$, then the series

$$\sum_{m=0}^{\infty} H_m(\overrightarrow{y_{m-1}}, x), \tag{56}$$

will converge to $K_1^* F(y(x)) + K_2^* G(y(x))$ (see [29]).
Now, we have

$$\begin{aligned}
&\sum_{m=0}^{\infty} [\chi_m H_{m-1}(\overrightarrow{y_{m-2}}, x) + Z_m H_{m-2}(\overrightarrow{y_{m-3}}, x)] \\
&= \sum_{m=2}^{\infty} H_{m-1}(\overrightarrow{y_{m-2}}, x) - \sum_{m=3}^{\infty} H_{m-2}(\overrightarrow{y_{m-3}}, x) \\
&[K_1^* F(y(x)) + K_2^* G(y(x))] - [K_1^* F(y(x)) + K_2^* G(y(x))] = 0.
\end{aligned} \tag{57}$$

This ends the proof.
Theorem 4.2.1 Let the operators $F[y(x)]$ and $G[y(x)]$ be contraction operators. If the series $y_0(x) + \sum_{m=1}^{\infty} y_m(x)$ is convergent, then it must be the exact solution of equation (1).
Proof. Since the series

$$y_0(x) + \sum_{m=1}^{\infty} y_m(x), \tag{58}$$

is convergent, we have

$$\lim_{m\to\infty} y_m(x) = 0. \tag{59}$$

Using the left-hand side of high-order deformation equation, we have

$$\sum_{m=1}^{\infty} \left[y_m(x) - \chi_m y_{m-1}(x) \right] = 0. \tag{60}$$

Then, by using Lemma we have

$$\begin{aligned}
\hbar \sum_{m=1}^{\infty} R_m(\overrightarrow{y_{m-1}}, x) + \varpi \sum_{m=1}^{\infty} [\chi_m H_{m-1}(\overrightarrow{y_{m-2}}, x) - Z_m H_{m-2}(\overrightarrow{y_{m-3}}, x)] \\
= \hbar \sum_{m=1}^{\infty} R_m(\overrightarrow{y_{m-1}}, x) = 0.
\end{aligned} \tag{61}$$

Since $\hbar \neq 0$ then the above equation gives

$$\sum_{m=1}^{\infty} R_m(\overrightarrow{y_{m-1}}, x) = 0. \tag{62}$$

Now, it holds that

$$\begin{aligned} &\sum_{m=1}^{\infty} D_{m-1}(N[Y(x;q)]) \\ &= \sum_{m=1}^{\infty} [D_{m-1}(Y(x;q)) - f(x)(1-\chi_m) \\ &- \lambda_1 K_1^* D_{m-1}(F(Y(x;q))) - \lambda_2 K_2^* D_{m-1}(G(y(t)))] = 0. \end{aligned} \tag{63}$$

In the above equations we have used the fact that the series

$$\sum_{m=1}^{\infty} D_{m-1}(F(Y(x;q))), \tag{64}$$

and

$$\sum_{m=1}^{\infty} D_{m-1}(G(Y(x;q))), \tag{65}$$

converge to $F(y(x))$ and $G(y(x))$, respectively. Now, we have

$$y(x) - f(x) - \lambda_1 K_1^* F(y(x)) - \lambda_2 K_2^* G(y(x)) = 0. \tag{66}$$

This ends the proof.

5. NUMERICAL RESULTS

Hereunder, we present two examples with analytical solutions to show efficiency of the present method given in Subsection 4.1 for solving the mixed Volterra-Fredholm integral equations. Also approximate solutions are compared with their exact solutions. According to Theorem in Subsection 4.2, we need only to focus on properly choosing the parameters $\hbar$ and ϖ so that the series (43) is convergent.

Example 5.1 Consider the mixed Volterra-Fredholm integral equation of second kind

$$\begin{aligned} y(x) = &-\frac{29}{12} - \frac{6}{5}x + 3x^2 - \frac{1}{3}x^4 + \frac{1}{30}x^6 \\ &+ \int_0^1 (t+x)(y(t)+y^2(t))dt + \int_0^x (t-x)y^2(t)dt, \end{aligned} \tag{67}$$

which has the exact solution $y(x) = x^2 - 2$.

For this example, we choose $y_0(x) = -\frac{29}{12} - \frac{6}{5}x + 3x^2 - \frac{1}{3}x^4 + \frac{1}{30}x^6$ as initial approximation guess. We consider $N = 5$ and $n = 20$. For $\hbar \neq 0$ and $\varpi \neq 0$, the

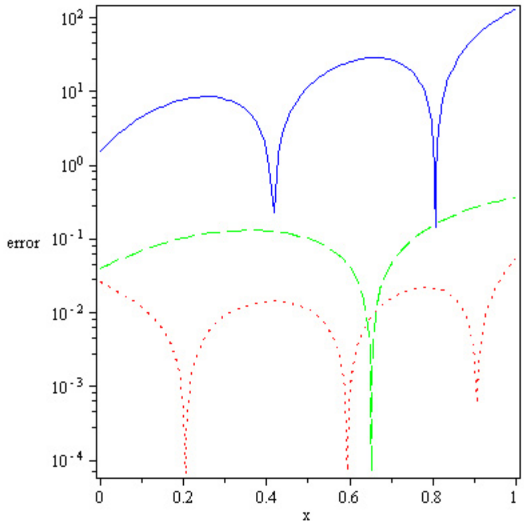

Fig. 1 – Dotted line: The error with $\hbar = -0.129E-1$ and $\varpi = 0.109$, dashed line: The error with $\hbar = -0.244E-1$ and $\varpi = 0$, solid line: The error with $\hbar = -1$ and $\varpi = 0$.

optimal convergence occurs at $\hbar = -0.129E-1$ and $\varpi = 0.109$ and has a ARE of $E_5^{20} = 3.291E-4$. For $\hbar \neq 0$ and $\varpi = 0$, the present method reduces to standard HAM and E_5^{20} has the minimum $3.95E-2$ at the "optimal" value $\hbar = -0.244E-1$, see Fig. 1.

Remark 5.1: If $\hbar = -1$, $\varpi = 0$ and $y_0(x) = f(x)$, the present method will be converted to ADM.

Example 5.2 Consider the mixed Volterra-Fredholm integral equation of second kind

$$y(x) = \frac{1}{2}e^{-x} - \frac{2}{e}x + \frac{5}{e} - 2 + x + \frac{1}{2}e^{-3x} + \int_0^1 t(t-x)y(t)dt + \int_0^x e^{-t-x}y(t)dt, \quad (68)$$

which has the exact solution $y(x) = e^{-x}$. For this example, we choose $y_0(x) = 0$ as initial approximation guess. We consider $N = 6$ and $n = 20$ and by using the technique developed in Section 2 we find the best values $\hbar$ and ϖ i.e.

$$\hbar = -0.142, \varpi = 1.821, \quad (69)$$

which give the corresponding minimum ARE $E_6^{20} = 5.851E-4$.

The value $\varpi = 0$ reduces the present method to the standard HAM. For $\varpi = 0$, the optimal convergence occurs at $\hbar = -1.437$ and has a ARE of $E_6^{20} = 8.453E-3$. The Error function $|Y_N(x,\hbar,\varpi) - y(x)|$ with $N = 6$ has been plotted for different values of $\hbar$ and ϖ in Fig. 2.

Example 5.3 Consider the mixed Volterra-Fredholm integral equation of second

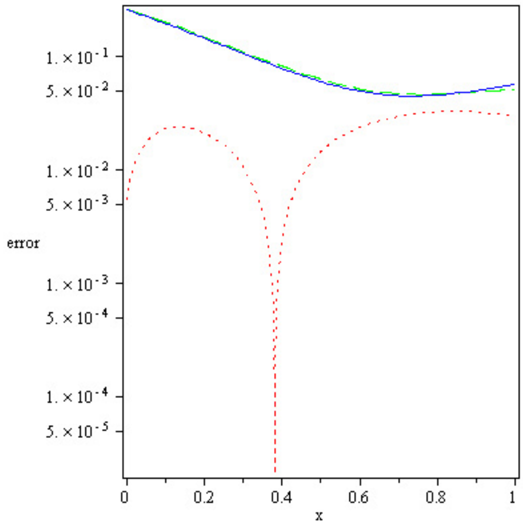

Fig. 2 – Dotted line: The error with $\hbar = -0.142$ and $\varpi = 1.821$, dashed line: The error with $\hbar = -1.437$ and $\varpi = 0$, solid line: The error with $\hbar = -1$ and $\varpi = 0$.

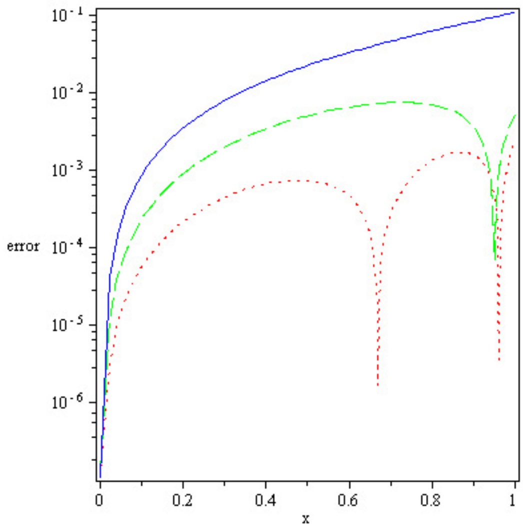

Fig. 3 – Dotted line: The error with $\hbar = -0.895$ and $\varpi = -2.163$, dashed line: The error with $\hbar = -0.417$ and $\varpi = 0$, solid line: The error with $\hbar = -1$ and $\varpi = 0$.

kind

$$y(x) = \frac{5}{6}x^2 - \frac{1}{4}x^5 + \int_0^1 tx^2y^2(t)dt + \int_0^x txy(t)dt, \tag{70}$$

which has the exact solution $y(x) = x^2$. For this example, we choose $y_0(x) = \frac{5}{6}x^2 -$

$\frac{1}{4}x^5$ as initial approximation guess. We consider $N = 6$ and $n = 20$ and by using the technique developed in Section 4 we find the best values $\hbar$ and ϖ i.e.

$$\hbar = -0.895, \varpi = -2.163, \tag{71}$$

which give the corresponding minimum ARE $E_6^{20} = 7.375E - 7$.
For $\varpi = 0$, the optimal convergence occurs at $\hbar = -0.417$ and has a ARE of $E_6^{20} = 1.493E - 5$. The Error function $|Y_N(x, \hbar, \varpi) - y(x)|$ with $N = 6$ has been plotted for different value of $\hbar$ and ϖ in Fig. 3.

6. CONCLUDING REMARKS

In this paper, we have studied the application of an extended homotopy analysis strategy for solving two-dimensional Fredholm integral equations and mixed Volterra-Fredholm integral equations. Based on the obtained results, we have the following conclusions:

1. This approach contains two convergence-controller parameters which provide us with a simple way to adjust and control the convergence region and the convergence rate of the obtained series solution.

2. The obtained results reveal once again the very fast convergence of the present approach, which does not need higher-orders of approximation.

3. All the given examples reveal that the multi-parametric homotopy yields a very effective and convenient approach to the approximate solutions of two-dimensional Fredholm integral equations.

4. Two appropriate auxiliary parameters provide a simple way to ensure the convergence of series solution.

5. The ADM and HAM cannot give better results than the present approach. In fact, the ADM and HAM are only the special cases of the present approach.

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