

## A NEW MODEL FOR CALCULATING THE MASSES OF LIGHT BARYONS RESONANCES UNDER THE DECATIC POTENTIAL

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*Abstract.* In this article we calculate the Light-baryons spectrum using a new baryon mass formula. For this purpose, we have solved the radial Schrödinger equation *via* the quasi-exact analytical ansatz method in 6-dimensional Hilbert space with the decatic potential. By applying our model, we find a good agreement for the even and odd parity resonances, the excited multiplets up to three GeV and the position of the Roper resonances of the nucleon with the spectrum of the particle data group.

*Key words:* Spectrum, Baryons, Decatic potential, Schrödinger equation, Ansatz method .

### 1. INTRODUCTION

Light baryons resonances with  $u$ ,  $d$ , and  $s$  quarks can be classified using the non-relativistic quark model. Constituent Quark Models (CQMs) have been recently widely applied to the description of baryons resonances mass [1-3]. Common to these models is the fact that the three Quark interaction can be divided in two parts: the first one, which contains the confinement interaction, is spin and flavour independent and is therefore  $SU(6)$  invariant, while the second violates the  $SU(6)$  symmetry [4, 5]. One of the most popular ways to violate the  $SU(6)$  invariance is the introduction of a spin-spin interaction [6], however in many studies a spin and isospin [1, 7], or a spin and flavour dependent interaction [1], has been considered. It is well known that the Gürsey Radicati mass formula [8] describes quite well the way  $SU(6)$  symmetry is broken, at least in the lower part of the baryon spectrum. In this paper, we applied the generalized Gürsey Radicati (GR) mass formula which is presented by Giannini and *et al.* [9] to obtain the best description of the light baryons spectrum. The model is a simple CQM where the  $SU(6)$  invariant part of the Hamiltonian is the same as in the hypercentral Constituent Quark Model (hCQM) [10, 11] and where the  $SU(6)$  symmetry is broken by a generalized GR mass formula.

This paper is organized as follows. In Sec. 2 the exact solution of six-dimensional Schrödinger equation for the decatic potential *via* wave function ansatz is given. In Sec. 3, in order to describe the splitting within the  $SU(6)$

multiplets, we introduce the generalized GR mass formula in the hCQM. We then give the results obtained by fitting the generalized GR mass formula parameters to the light baryons energies and we compare the spectrum with the experimental data. The concluding remarks are given in Sec. 4.

## 2. EXACT SOLUTION OF SCHRÖDINGER EQUATION IN THE PRESENCE DECATIC POTENTIAL

One of the important tasks of quantum mechanics is to solve the Schrödinger equation with the physical potentials. The solutions of the radial Schrödinger equation are the bound-states of a particle moving in a spherically symmetric potential. In the six-dimensional space, the hypercentral Schrödinger equation for the 3q-states with a potential  $V(r)$  can be written as:

$$\left( \frac{d^2}{dr^2} + \frac{5}{r} \frac{d}{dr} - \frac{\gamma(\gamma+4)}{r^2} \right) R_{\nu\gamma}(r) + 2m[E - V(r)] R_{\nu\gamma}(r) = 0, \quad (1)$$

where  $R_{\nu\gamma}(r)$  is the hyperradial wave function,  $r$  denotes the hyperradius,  $\gamma$  is the grand angular quantum number given by  $\gamma = 2n + l_\rho + l_\lambda$ ,  $0 \leq n \leq \infty$ ;  $l_\rho$ , and  $l_\lambda$  are the angular momenta associated with the Jacobi coordinates ( $\vec{\rho}$  and  $\vec{\lambda}$ ) described by  $\vec{\rho} = \frac{1}{\sqrt{2}}(\vec{r}_1 - \vec{r}_2)$ ,  $\vec{\lambda} = \frac{1}{\sqrt{6}}(\vec{r}_1 + \vec{r}_2 - 2\vec{r}_3)$ . Instead of  $\vec{\rho}$  and  $\vec{\lambda}$ , one can introduce the hyperspherical coordinates, with the hyperradius,  $r$  defined by  $r = \sqrt{\rho^2 + \lambda^2}$  [12].  $\nu$  denotes the number of nodes of the three-quark wave function. In Eq. (1),  $m$  is the reduced mass defined as  $m = \frac{2m_\rho m_\lambda}{m_\rho + m_\lambda}$  [13]. We get the interaction potential as the decatic potential [14, 15]:

$$V(r) = ar^2 + br^4 + cr^6 + dr^8 + er^{10}. \quad (2)$$

We have plotted the behavior of decatic potential in Fig. (1). In order to solve the corresponding equation, we will use an analytical technique, namely the quasi-exact ansatz method, rather than the numerical counterparts. The analytic solutions provide us with a deeper quantitative insight into the physics of the problem. The considered potential, has been successfully used in the analysis of the charmonium systems and is related to double-well oscillators [14, 15].

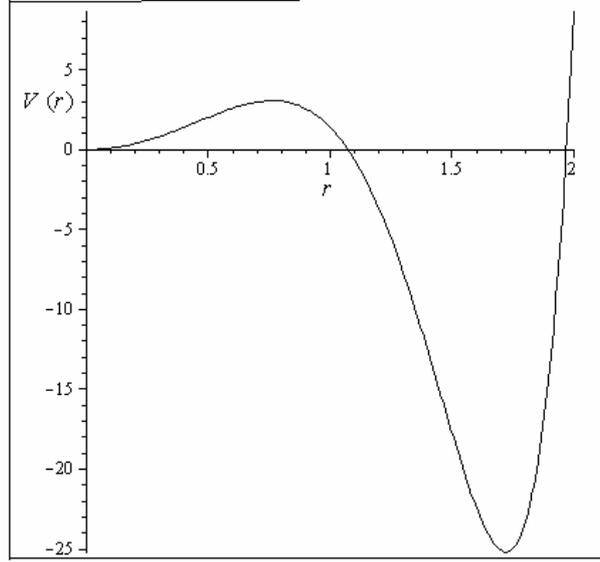


Fig. 1 – Behavior of decaying potential *versus*  $r$  with  $a = 10$ ,  $b = -7$ ,  $c = -2.5$ ,  $d = 0.9$ ,  $e = 0.01$ .

Now we want to solve the hyperradial Schrödinger equation for the three-body potential interaction Eq. (2). The transformation

$$R_{v\gamma}(r) = r^{-\frac{5}{2}} \varphi_{v\gamma}(r) \quad (3)$$

reduces Eq. (1) to the form:

$$\varphi_{v\gamma}''(r) + \left[ \varepsilon - a_1 r^2 - b_1 r^4 + c_1 r^6 - d_1 r^8 + e_1 r^{10} - \frac{(2\gamma+3)(2\gamma+5)}{4r^2} \right] \varphi_{v\gamma}(r) = 0. \quad (4)$$

The hyperradial wave function  $\varphi_{v\gamma}(r)$  is a solution of the reduced Schrödinger equation for each of the three identical particles with mass  $m$  and interacting potential Eq. (2), where

$$\varepsilon = 2mE, \quad a_1 = 2ma, \quad b_1 = 2mb, \quad c_1 = 2mc, \quad d_1 = 2md, \quad e_1 = 2me. \quad (5)$$

We use the following form for the wave function:

$$\varphi_{v\gamma}(r) = h(r) e^{g(r)}. \quad (6)$$

Now, for the functions  $f(r)$  and  $g(r)$  we make use of the ansatz [16–18]:

$$\begin{aligned}
 h(r) &= 1 & \nu &= 0 \\
 h(r) &= \prod_{i=1}^{\nu} (r - \alpha_i^{\nu}) & \nu &= 1, 2, \dots \\
 g(r) &= \frac{1}{2} \alpha r^2 - \frac{1}{4} \beta r^4 + \frac{1}{6} \tau r^6 + k \ln r.
 \end{aligned} \tag{7}$$

By calculating  $\varphi_{\nu\gamma}''(r)$  from Eq. (6) and comparing with Eq. (4), we obtain

$$\begin{aligned}
 [a_1 r^2 - b_1 r^4 + c_1 r^6 - d_1 r^8 + e_1 r^{10} + \frac{(2\gamma+3)(2\gamma+5)}{4r^2} - \varepsilon] &= \\
 = \frac{h''(r) + 2g'(r)h'(r)}{h(r)} + g''(r) + g'^2(r).
 \end{aligned} \tag{8}$$

Substitution of Eq. (7) into Eq. (8) leads to

$$\begin{aligned}
 a_1 r^2 - b_1 r^4 + c_1 r^6 - d_1 r^8 + e_1 r^{10} + \frac{(2\gamma+3)(2\gamma+5)}{4r^2} - \varepsilon &= \\
 = \tau r^{10} - (2\beta\tau)r + r^6(2\alpha\tau + \beta^2) + r^4(-2\alpha\beta + 2\tau k + 5\tau) + \\
 + r^2(\alpha^2 + 2\beta k - 3\beta) - r^2 k(k-1) + \alpha(1+2k).
 \end{aligned} \tag{9}$$

By equating the corresponding powers of  $r$  on both sides of Eq. (9), we obtain

$$\begin{aligned}
 \tau &= \sqrt{e_1}, \quad \beta = \frac{d_1}{2\sqrt{e_1}}, \quad \alpha = \frac{d_1^2 - 4c_1 e_1}{8e_1 \sqrt{e_1}}, \quad b = \frac{8e_1 \sqrt{e_1} (5+2k) - d_1 (d_1^2 - 4e_1 c_1)}{8e_1^2}, \\
 k &= \gamma + \frac{3}{2}, \quad k = \gamma + \frac{5}{2}, \quad a = \frac{d_1^4 - 8e_1 c_1 d_1^2 + 16c_1^2 e_1^2 + 64d_1 e_1^2 (k + \frac{3}{2})}{64e_1^3}, \\
 \varepsilon &= -\alpha(1+2k).
 \end{aligned} \tag{10}$$

From Eq. (10), we have  $k = \gamma + \frac{5}{2}$ , which ensures the existence of well-behaved solutions at the origin and infinity. The energy eigenvalue for the mode  $\nu = 0$  and grand angular momentum  $\gamma$  from Eqs. (5) and (10) is given as

$$E_{0\gamma} = \frac{-2}{(2m)^{\frac{1}{2}}} \frac{d^2 - 4ce}{8e\sqrt{e}} (\gamma + 3). \tag{11}$$

### 3. CALCULATING THE MASSES OF LIGHT BARYON RESONANCES

Although the description of the light baryons spectrum by the hCQM [19] is fairly good and comparable to the results of other approaches, the approach is not satisfactory in some cases for the splitting of various  $SU(6)$  multiplets. The following results [7, 20, 21] show that both spin and isospin dependent terms in the quark Hamiltonian are important. Description of the splitting within the  $SU(6)$  baryon multiplets is presented by the Gürsey Radicati mass formula [8]:

$$M = M_0 + CC_2[SU_S(2)] + DC_1[U_Y(1)] + E[C_2[SU_I(2)] - \frac{1}{4}(C_1[U_Y(1)])^2], \quad (12)$$

where  $M_0$  is the average energy value of the  $SU(6)$  multiplet,  $C_2[SU_S(2)]$  and  $C_2[SU_I(2)]$  represents the  $SU(2)$  (quadratic) Casimir operators for spin and isospin, respectively, and  $C_1[U_Y(1)]$  denotes the Casimir for the  $U(1)$  subgroup generated by the hypercharge  $Y$ . This mass formula has been successful in the description of the ground-state baryon masses, however, as stated by the authors themselves, it is not the most general mass formula that can be written on the basis of a broken  $SU(6)$  symmetry. In order to generalize Eq. (12), Giannini *et al.* considered a dynamical spin-flavor symmetry  $SU_{SF}(6)$  [9] and described the  $SU_{SF}(6)$  symmetry breaking mechanism by generalizing Eq. (12) as

$$M = M_0 + AC_2[SU_{SF}(6)] + BC_2[SU_F(3)] + CC_2[SU_S(2)] + DC_1[U_Y(1)] + E[C_2[SU_I(2)] - \frac{1}{4}(C_1[U_Y(1)])^2]. \quad (13)$$

In Eq. (13) the spin term represents spin-spin interactions, the flavor term stands for the flavor dependence of the interactions, and the  $SU_{SF}(6)$  term depends on the permutation symmetry of the wave functions and represents "signature-dependent" interactions. The signature-dependent (or exchange) interactions were extensively investigated years ago within the framework of Regge theory [22]. As our analysis is restricted to nonstrange baryon resonances only, the  $D$  term can be absorbed into an overall constant and the only relevant parameters are  $A$ ,  $C$  and the combination  $(3B + E)$ . Therefore, we have chosen the  $D$  parameter to be zero. The last two terms represent the isospin and hypercharge dependence of the masses. The generalized Gürsey-Radicati mass formula of Eq. (13) can be used to describe the light baryons spectrum, provided that two conditions are fulfilled. The first condition is the feasibility of using the same splitting coefficients for different  $SU(6)$  multiplets. This seems actually to be the case, as shown by the algebraic approach to the baryon spectrum [1]. The second condition is given by the feasibility of getting reliable values for the unperturbed mass values  $M_0$  [9]. For this goal we regarded the  $SU(6)$  invariant part of the hCQM, which provides a good description of the baryons spectrum and used the Gürsey-Radicati inspired  $SU(6)$  breaking interaction to describe the splitting within each  $SU(6)$  multiplet.

Therefore, the light baryons masses are obtained by three quark masses and the eigenenergies ( $E_{v\gamma}$ ) of the radial Schrödinger equation with the expectation values of  $H_{GR}$  as follows:

$$M = 3m + E_{v\gamma} + \langle H_{GR} \rangle. \quad (14)$$

It must be noticed that, in order to simplify the solving procedure, the constituent quarks masses are assumed to be the same for up and down quark flavors ( $mu = md$ ). In the previous section, we determined the eigenenergies ( $E_{v\gamma}$ ) by a quasi-exact solution of the radial Schrödinger equation for the hypercentral potential Eq. (2). The expectation values of  $H_{GR}(\langle H_{GR} \rangle)$ , are completely identified by the expectation values of the Casimir operators [23]:

$$\begin{aligned} \langle C_2[SU_{SF}(6)] \rangle &= \begin{cases} \frac{45}{4} \text{ for [56]} \\ \frac{33}{4} \text{ for [70]} \\ \frac{21}{4} \text{ for [20]} \end{cases} \\ \langle C_2[SU_F(3)] \rangle &= \begin{cases} 3 \text{ for [8]} \\ 6 \text{ for [10]} \\ 0 \text{ for [1]} \end{cases} \\ \langle C_2[SU_I(2)] \rangle &= I(I+1) \\ \langle C_1[U_Y(1)] \rangle &= Y \\ \langle C_2[SU_S(2)] \rangle &= S(S+1). \end{aligned} \quad (15)$$

For calculating the light baryons mass *via* Eq. (14), we need to find the unknown parameters. For this purpose, we choose a limited number of well-known light resonances and express their mass differences using  $H_{GR}$  and the Casimir operator expectation values:

$$\begin{aligned} N(1650)S11 - N(1535)S11 &= 3C \\ \Delta(1232)P33 - N(938)P11 &= 9B + 3C + 3E \\ N(1535)S11 - N(1440)S11 &= (E10 - E01) + 12A. \end{aligned} \quad (16)$$

We calculated the  $C$  parameter from Eq. (16) and determined  $m$ ,  $d$ ,  $c$ ,  $e$  (in Eq. (14)) and the three coefficients  $A$ ,  $B$  and  $E$  of Eq. (16) in a simultaneous fit to of 3 and 4 star resonances of Table 2. The fitted parameters are reported in Table 1.

We have reported our results in Table 2. The first column in Table 2 identifies the considered baryons. The second column is given for the baryons resonances where 3 and 4 star resonances of Table 2 have been assigned as octet and decuplet states. Reported baryons masses from Ref. [24] is shown in the third column. In fourth column, states of the baryons are given. The used notation of the state column is  $(^{2S+1} \dim(SU(3)), [\dim(SU(6)), L^P])$  where  $\dim(SU(n))$  is the dimension of the  $SU(n)$  representation,  $S$  and  $L$  are the total spin and orbital angular momentum of the quark system, respectively.  $J$  and  $P$  are the spin and parity of the resonance. For an example in the first row of Table 2,  $^2_{8_{1/2}}[56, 0^+]$  numbers 2, 8,  $\frac{1}{2}$ , 56, 0 indicate  $(2s+1)$  the spin of the quark system, dimension of the  $SU(3)$  representation, spin of the resonance, dimension of the  $SU(6)$  representation, orbital angular momentum of the quark system, respectively. Also the (+) sign identifies the parity of the resonance related to N baryon. We have shown our results for baryon masses in the fifth column. A comparison with Ref. [9] is reported in column 6. The relative error is presented in the seventh column. The percentage of relative error for our calculations is between 0.04 and 6% (column 7, in Table 2). Comparison between our results and the experimental masses [24] shows that the light baryons spectra are, in general, fairly well reproduced.

#### 4. CONCLUSION

In this article, we have calculated the mass spectra of light baryons resonances on the basis of a simple Constituent Quark Model and the generalized Gürsey Radicati mass formula [9]. In the model employed by us, the energy splittings within the  $SU(6)$  multiplets are considered as perturbations added to the  $SU(6)$  invariant levels, which are given by our suggested hypercentral potential, Eq. (2). About the addition of other even terms, as we see, the results have been improved in some cases. Nevertheless, we obviously see that in some other cases the single harmonic term is superior. In fact, this is an outcome of the fact that the existence of nonconfining term(s) is necessary in the potential model. On the other hand, the results may indicate that the harmonic term is not singly sufficient. For reproducing the spectrum of light baryons resonances, we calculated the energy eigenvalues by a quasi-exact analytical solution of the hyperradial Schrödinger equation for the decaic potential. Then, we fitted the generalized GR mass formula parameters to the light baryons energies and calculated the baryons mass according to Eq. (14). The overall good description of the spectrum obtained here shows that our model can be also used to give a fair description of the energies of the excited multiplets up to three GeV and not only for the ground-state octets and decuplets. Moreover, our model reproduces the position of the Roper resonances of the nucleon and negative-parity resonance. We have also reported the masses of the baryons for some resonance states of N and  $\Delta$  baryons which are in agreement with the experimental data.

Table 1

The fitted values of the parameters of the Eq. (14), obtained with resonances mass differences and global fit to the experimental resonance masses [24]

Parameter	$A$	$B$	$C$	$E$	$m$	$d$	$c$	$e$
Value	-12.3 MeV	0.233 MeV	38.3	59 MeV	275 MeV	-0.709	0.264	0.402

Table 2

Mass spectrum of baryons resonances (in MeV) calculated with the mass formula Eq. (14). The column  $M_{Our Calc}$  contains our calculations with the parameters of Table 1 and column 7 indicate the percentage of relative error for our calculations

Baryon	Status	Mass(exp)[24]	State	$M_{Our Calc}$	$M_{[9]}$	Percent of relative error
N(938) P11	****	938	$^2 8_{1/2}[56, 0^+]$	938.9	938.0	0.095%
N(1440) P11	****	1420-1470	$^2 8_{1/2}[56, 0^+]$	1454.7	1448.7	2.44% - 1.04%
N(1520) D13	****	1515-1525	$^2 8_{3/2}[70, 1^-]$	1523.9	1543.7	0.58% - 0.07%
N(1535) S11	****	1525-1545	$^2 8_{1/2}[70, 1^-]$	1523.9	1543.7	0.07% - 1.36%
N(1650) S11	****	1645-1670	$^4 8_{1/2}[70, 1^-]$	1671	1658.6	1.58% - 0.05%
N(1675) D15	****	1670-1680	$^4 8_{5/2}[70, 1^-]$	1671	1658.6	0.05% - 0.53%
N(1680) F15	***	1680-1690	$^2 8_{5/2}[56, 2^+]$	1712.6	1651.4	1.94% - 1.33%
N(1700) D13	***	1650-1750	$^4 8_{3/2}[70, 1^-]$	1671	1658.6	1.27% - 4.5%
N(1710) P11	***	1680-1740	$^2 8_{1/2}[70, 0^+]$	1748	1795.4	4.04% - 0.45%
N(1720) P13	****	1700-1750	$^2 8_{3/2}[56, 2^+]$	1712.6	1651.4	0.74% - 2.13%
N(2190) G17	****	2100-2200	$^2 8_{7/2}[70, 3^-]$	2200.9	-	4.8% - 0.04%
N(2220) H19	****	2200-2300	$^2 8_{9/2}[56, 4^+]$	2228.4	-	1.2% - 3.11%
N(2250) G19	****	2200-2350	$^4 8_{9/2}[70, 3^-]$	2315.8	-	5.2% - 1.45%
N(2600) I1, 11	***	2550-2750	$^2 8_{11/2}[70, 5^-]$	2587.7	-	1.47% - 5.9%
$\Delta$ (1232) P33	****	1231-1233	$^4 10_{3/2}[56, 0^+]$	1231.5	1232.0	0.04% - 0.1%
$\Delta$ (1600) P33	***	1550-1700	$^4 10_{3/2}[56, 0^+]$	1618.4	1683.0	4.41% - 4.8%
$\Delta$ (1700) D33	****	1670-1750	$^2 10_{3/2}[70, 1^-]$	1733.8	1722.8	3.8% - 0.92%
$\Delta$ (1905) F35	****	1865-1915	$^4 10_{5/2}[56, 2^+]$	1876.3	1945.4	0.6% - 2.02%
$\Delta$ (1910) P31	****	1870-1920	$^4 10_{1/2}[56, 2^+]$	1876.3	1945.4	0.33% - 2.27%
$\Delta$ (1920) P33	***	1900-1970	$^4 10_{3/2}[56, 0^+]$	2005.2	2089.4	5.5% - 1.7%
$\Delta$ (1950) F37	****	1915-1950	$^4 10_{7/2}[56, 2^+]$	1876.3	1945.4	2.02% - 3.77%
$\Delta$ (2420) H3, 11	****	2300-2500	$^4 10_{11/2}[56, 4^+]$	2392.1	-	4% - 4.3%

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