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GENERALIZED GAUGE TRANSFORMATION APPROACH TO CONSTRUCT DARK SOLITONS OF COUPLED NONLINEAR SCHRÖDINGER (NLS) TYPE EQUATIONS

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Abstract. We harnesses the freedom in the celebrated Gauge transformation approach to generate dark solitons of coupled nonlinear Schrödinger (NLS) type equations. The new approach which is purely algebraic could prove to be very useful, particularly in the construction of vector dark solitons in the fields of nonlinear optics, plasma physics and Bose-Einstein condensates. We have employed this algebraic method to coupled Gross-Pitaevskii (GP) and NLS equations and obtained dark solitons.

1. INTRODUCTION

Eventhough the discovery of solitons in the numerical simulation of Korteweg de Vries (KdV) equation by Zabusky and Kruskal [1] goes back by almost fifty years and several nonlinear integrable partial differential equations (pdes) like modified KdV (mKdv) [2], Sine Gordon (SG) [3], nonlinear Schrödinger (NLS) [4] and Gross-Pitaevskii (GP) [5] equations have been identified, construction of integrable dynamical systems and the associated soliton solution continues to be a challenging problem even today [6]. In the case of NLS equation, soliton type pulse propagation in the anomalous and the normal dispersion regimes are called bright and dark solitons, respectively [7–10]. The identification of bright solitons in NLS equation arising by virtue of the subtle balance between the Kerr nonlinearity and anomalous Group Velocity Dispersion (GVD) has made quite a turnaround in the fields of optical fibre communications and responsible for several experimental demonstrations [8, 9, 11–13]. In addition to bright solitons, it was also reported that optical solitons can also be generated if GVD is negative (*i.e.*, normal dispersion regime) leading to the possibility of different type of solitons called dark solitons [8, 9]. The use of optical solitons opens new prospects for non-interference transmission since solitons are highly stable with respect to perturbations caused by fibre nonuniformities and to external interference [8, 9, 13]. During the past four decades or so, it should be mentioned that the NLS equation has also been used to analyze the dark solitons. Recent studies on the dark solitons have revealed very interesting properties which may allow their stable transmission with much less spacing between solitons when compared with bright solitons. Also, the interaction effects between two dark solitons are less than the bright solitons in the presence of fibre loss. The interaction forces between two dark solitons are always repulsive, unlike the case of bright solitons where the interaction forces change according to their relative phase. The self-induced Raman effect is found to be more destructive in the case of dark solitons. Use of dark solitons for high-speed communication systems will remain an interesting subject for future research [8, 9, 14, 15]. It is worth pointing out at this juncture that several analytical techniques like Hirota bilinear method [16, 17], Darboux transformation method [18, 19], inverse scattering method [20] have been employed to generate bright and dark solitons.

An iterative approach by gauge transforming the eigenfunctions of the associated linear eigenvalue problem was developed by Ling Lie Chau *et al.* [21], to generate soliton solutions of integrable nonlinear pdes. Eventhough this method which is quite similar to Darboux transformation approach is purely algebraic in nature as it enables one to construct soliton solutions from a trivial/nontrivial seed solution, the above approach has been employed only for generating bright solitons for zero seed solution and kink solitons for non-zero (constant) seed solution of nonlinear partial differential equations. In this paper, by choosing a plane wave as the seed solution, we suitably harness the gauge transformation approach to generate dark solitons of coupled NLS type equations.

This paper is organized as follows: In section 2, we derive the algorithm to generate dark soliton solutions by playing around the freedom in the Gauge transformation method. In section 3, as an application of the above algorithm, we experiment with coupled Gross Pitaevskii (GP) and coupled NLS equation and obtain the corresponding dark soliton solutions. We then discuss the collisional dynamics of dark solitons.

2. MODIFIED GAUGE TRANSFORMATION APPROACH

We know that any vector integrable (1+1) dimensional nonlinear Schrödinger equation (in general) can be described by the celebrated AKNS type linear eigenvalue problem of the following form

$$\Phi_x = U\Phi, \quad (1)$$
$$\Phi_t = V\Phi, \quad (2)$$

where U and V represent Lax pair governed by 3×3 matrices and $\Phi = (\phi_1, \phi_2, \phi_3)^T$ is the associated eigenfunction of the linear eigenvalue problem so that the compatibility condition

$$U_t - V_x + [U,V] = 0 \quad (3)$$

generates the desirable integrable vector NLS equation. It should be mentioned that beginning with non-zero plane wave as the seed solution is crucial for generating dark solitons. Dark solitons as is well known, are dips in density profile $|\psi|^2$. And inorder to get such dips, one needs to have a non-zero background density to begin with which is ensured by the choice of non-zero plane wave as the seed solution. We now feed the nonzero plane wave solution into the Lax pair matrices U and V and obtain the associated linear system.

$$\Phi^0_x = U^0\Phi^0, \quad (4)$$
$$\Phi^0_t = V^0\Phi^0, \quad (5)$$

where

$$\Phi^0 = \Phi|_{nonzero\ planewave\ seed}. \quad (6)$$

We now gauge transform the eigenfunction Φ^0 such that $\hat{\Phi} = g\Phi^0$ where g is a gauge function represented by a 3×3 matrix while $\hat{\Phi}$ is an iterated eigenfunction. The Lax representation in terms of the iterated eigenfunction is given by

$$\hat{\Phi}_x + U_1\hat{\Phi} = 0, \quad (7)$$
$$\hat{\Phi}_t + V_1\hat{\Phi} = 0, \quad (8)$$

so that

$$U_1 = gU^0g^{-1} + g_xg^{-1}$$
$$V_1 = gV^0g^{-1} + g_tg^{-1}$$

Now, the gauge function $g(x,t)$, can be chosen in such a way that it represents the solution of the associated Riemann problem and it is meromorphic in the complex ζ plane as

$$g(x,t;\zeta) = \left[1 + \frac{\zeta_1 - \bar{\zeta}_1}{\zeta - \zeta_1}P(x,t)\right]\cdot J. \quad (9)$$

In the above equation, ζ is the eigenvalue parameter while ζ_1 and $\bar{\zeta}_1$ represent arbitrary complex parameters and P is a 3×3 projection matrix. It should be noted that for scalar NLS type equations, $J = \sigma_3$ (Pauli's spin matrix) while for vector NLS type equation

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{10}$$

It should be mentioned that, for scalar NLS equations, the gauge function $g(x,t)$, U, V and the projection matrix P are (2× 2) in nature and ϕ is a (2 × 1) column vector. The inverse of matrix g is given by

$$g^{-1}(x,t;\zeta) = J\cdot\left[1 - \frac{\zeta_1 - \bar{\zeta}_1}{\zeta - \bar{\zeta}_1}P(x,t)\right]. \tag{11}$$

Now, choosing $\bar{\zeta}_1 = \zeta_1^*$, the projection matrix can be determined by solving the following set of partial differential equations

$$P_x = (1-P)JU^{(0)}(\bar{\zeta}_1)JP - PJU^{(0)}(\zeta_1)J(1-P), \tag{12}$$

$$P_t = (1-P)JV^{(0)}(\bar{\zeta}_1)JP - PJV^{(0)}(\zeta_1)J(1-P), \tag{13}$$

It should be mentioned that equations (2) are identical to the one proposed by Ling Lie Chau *et al.* [21] in the gauge transformation approach with a difference that P, J, $U^{(0)}$ and $V^{(0)}$ are (2×2) matrices for scalar nonlinear pdes while one has to work with (3×3) matrices for vector nonlinear pdes. One can then relate the projection matrix $P(x,t)$ with the vacuum eigenfunction $\Phi^{(0)}(x,t)$ by

$$P = J\cdot\tilde{P}\cdot J, \tag{14}$$

where

$$\tilde{P} = \frac{M^{(1)}}{\mathrm{Trace}[\mathrm{M}^{(1)}]}, \tag{15}$$

and

$$M^{(1)} = \Phi^{(0)}(x,t,\bar{\zeta}_1)\cdot\hat{m}^{(1)}\cdot\Phi^{(0)}(x,t,\zeta_1)^{-1}. \tag{16}$$

$$\Phi^{(0)}(x,t,\zeta_1) = \Phi(x,t,\zeta_1)|_{nonzero\ planewave\ seed}. \tag{17}$$

In the above equation, $\hat{m}^{(1)}$ is a 3×3 arbitrary matrix of the form

$$\hat{m}^{(1)} = \begin{pmatrix} e^{2\delta_1}\sqrt{2} & \varepsilon_1^{(1)}e^{2i(\chi_1+\xi_1)} & \varepsilon_2^{(1)}e^{2i(\chi_1+\xi_2)} \\ \varepsilon_1^{*(1)}e^{-2i(\chi_1+\xi_1)} & e^{-2\delta_1}/\sqrt{2} & 0 \\ \varepsilon_2^{*(1)}e^{-2i(\chi_1+\xi_2)} & 0 & e^{-2\delta_1}/\sqrt{2} \end{pmatrix}, \tag{18}$$

where $\chi_1, \delta_1, \xi_1, \xi_2$, are arbitrary functions of (x,t) and their choice is governed by the dispersion relation of the associated nonlinear pdes while $\varepsilon_1^{(1)}$ and $\varepsilon_2^{(1)}$ are the coupling parameters. It should be noted that the structure of $\hat{m}^{(1)}$ matrix determines the nature (bright or dark) of soliton solutions of vector NLS type equations. Hence, one can write down the dark soliton solution as,

$$\psi_1^{(1)} = \psi_1^{(0)} - 2i(\zeta_1 - \bar{\zeta_1})\tilde{P}_{12}, \quad (19)$$

$$\psi_2^{(1)} = \psi_2^{(0)} - 2i(\zeta_1 - \bar{\zeta_1})\tilde{P}_{13}. \quad (20)$$

where $\psi_i^{(0)}$(i=1,2) represents the seed solution while $\psi_i^{(1)}$(i=1,2) denotes the iterated dark soliton solutions of the corresponding coupled NLS type equation and

$$\tilde{P}_{12} = \frac{M_{12}^1}{M_{11}^1 + M_{22}^1 + M_{33}^1}; \quad \tilde{P}_{13} = \frac{M_{13}^1}{M_{11}^1 + M_{22}^1 + M_{33}^1}$$

The modified gauge transformation approach can be extended to generate multidark soliton solutions. For example, the general form of the N-th dark soliton solution can be written as

$$\psi_1^{(N)} = \psi_1^{(N-1)} - 2i(\zeta_N - \bar{\zeta_N})\frac{\tilde{P}_{12}}{R}, \quad (21)$$

$$\psi_2^{(N)} = \psi_2^{(N-1)} - 2i(\zeta_N - \bar{\zeta_N})\frac{\tilde{P}_{13}}{R}, \quad (22)$$

where $\tilde{P}_{12}$ and $\tilde{P}_{13}$ are given by

$$\begin{aligned}
\tilde{P}_{12}^{N-1} = & - [M_{12}^{(N-1)}((\tau + \gamma M_{11}^{(N-1)})M_{11}^{(N)} + \gamma(M_{12}^{(N-1)} \\
& M_{21}^{(N)}\gamma^*/\tau^2 + M_{13}^{(N-1)}M_{31}^{(N)})) + M_{32}^{(N-1)} \\
& ((\tau + \gamma M_{11}^{(N-1)})M_{13}^{(N)} + \gamma(M_{12}^{(N-1)}M_{23}^{(N)} \\
+ & M_{13}^{(N-1)}M_{33}^{(N)}))\gamma^*/\tau^2 + ((\tau + \gamma M_{11}^{(N-1)})M_{12}^{(N)} \\
+ & \gamma(M_{12}^{(N-1)}M_{22}^{(N)} + M_{13}^{(N-1)}M_{32}^{(N)})) \\
& (\tau + M_{22}^{(N-1)}\gamma *)/\tau^2], \\
\tilde{P}_{13}^{N-1} = & - [M_{13}^{(N-1)}((\tau + \gamma M_{11}^{(N-1)})M_{11}^{(N)} + \gamma(M_{12}^{(N-1)} \\
& M_{21}^{(N)} + M_{13}^{(N-1)}M_{31}^{(N)}))\gamma^*)/\tau^2 + M_{23}^{(N-1)} \\
& ((\tau + \gamma M_{11}^{(N-1)})M_{12}^{(N)} + \gamma(M_{12}^{(N-1)}M_{22}^{(N)} \\
+ & M_{13}^{(N-1)}M_{32}^{(N)}))\gamma^*/\tau^2 + ((\tau + \gamma M_{11}^{(N-1)})M_{13}^{(N)} \\
+ & \gamma(M_{12}^{(N-1)}M_{23}^{(N)} + M_{13}^{(N-1)}M_{33}^{(N)})) \\
& (\tau + M_{33}^{(N-1)}\gamma^*)/\tau^2],
\end{aligned}$$

and

$$\tau = M_{11}^{(N-1)} + M_{22}^{(N-1)} + M_{33}^{(N-1)}, \qquad \gamma = \frac{\zeta_1 - \bar{\zeta_1}}{\zeta_2 - \zeta_1},$$
$$R = \tilde{P}_{11}^{N-1} + \tilde{P}_{22}^{N-1} + \tilde{P}_{33}^{N-1}, \qquad \gamma^* = -\frac{\zeta_1 - \bar{\zeta_1}}{\zeta_2 - \bar{\zeta_1}},$$

with

$$\begin{aligned}
\tilde{P}_{11}^{N-1} &= M_{21}^{(N-1)}((\tau + \gamma M_{11}^{(N-1)})M_{12}^{(N)} + \gamma(M_{12}^{(N-1)}M_{22}^{(N)} \\
&+ M_{13}^{(N-1)}M_{32}^{(N)}))\gamma^*/\tau^2 + M_{31}^{(N-1)}((\tau + \gamma M_{11}^{(N-1)}) \\
&\quad M_{23}^{(N)} + M_{13}^{(N-1)}M_{33}^{(N)}))\gamma^*/\tau^2 M_{13}^{(N)} + \gamma(M_{12}^{(N-1)} \\
&\quad +((\tau + \gamma M_{11}^{(N-1)})M_{11}^{(N)} + \gamma(M_{12}^{(N-1)}M_{21}^{(N)} + \\
&\quad M_{13}^{(1)}M_{31}^{(N)}))(\tau + M_{11}^{(N-1)}\gamma^*)/\tau^2, \\
\tilde{P}_{22}^{N-1} &= M_{12}^{(N-1)}(\gamma M_{11}^{(N)}M_{21}^{(N-1)} + M_{21}^{(N)}(\tau + \gamma M_{22}^{(N-1)}) \\
&+ \gamma M_{23}^{(N-1)}M_{31}^{(N)})\gamma^*/\tau^2 + M_{32}^{(N-1)}(\gamma M_{13}^{(N)}M_{21}^{(N-1)} \\
&+ (\tau + \gamma M_{22}^{(N-1)})M_{23}^{(N)} + \gamma M_{23}^{(N-1)}M_{33}^{(N)})\gamma^*/\tau^2 \\
&+ (\gamma M_{12}^{(N)}M_{21}^{(N-1)} + (\tau + \gamma M_{22}^{(N-1)})M_{22}^{(N)} + \gamma M_{23}^{(N-1)} \\
&\quad M_{32}^{(N)})(\tau + M_{22}^{(N-1)}\gamma^*))/\tau^2 \\
\tilde{P}_{33}^{N-1} &= M_{13}^{(N-1)}(\gamma M_{11}^{(N)}M_{31}^{(N-1)} + \gamma M_{21}^{(N)}M_{32}^{(N-1)} + M_{31}^{(N)} \\
&\quad (\tau + \gamma M_{33}^{(N-1)}))\gamma^*/\tau^2 + M_{23}^{(N-1)}(\gamma M_{12}^{(N)}M_{31}^{(N-1)} \\
&+ \gamma M_{22}^{(N)}M_{32}^{(N-1)} + M_{32}^{(N)}(\tau + \gamma M_{33}^{(N-1)}))\gamma^*/\tau^2 \\
&+ (\gamma M_{13}^{(N)}M_{31}^{(N-1)} + \gamma M_{23}^{(N)}M_{32}^{(N-1)} + (\tau \\
&+ \gamma M_{33}^{(N-1)})M_{33}^{(N)})(\tau + M_{33}^{(N-1)}\gamma^*)/\tau^2,
\end{aligned}$$

$$\begin{aligned}
M_{11}^{(N)} &= e^{-\theta_N}\sqrt{2}; \quad M_{12}^{(N)} = e^{-i\xi_N + \chi_1}\varepsilon_1^{(N)}; \quad M_{13}^{(N)} = e^{-i\xi_N + \chi_1}\varepsilon_2^{(j)}; \\
M_{21}^{(N)} &= e^{i\xi_N + \chi_1}\varepsilon_1^{*(N)}; \quad M_{22}^{(N)} = e^{\theta_N}/\sqrt{2}; \quad M_{23}^{(N)} = 0; \\
M_{31}^{(N)} &= e^{i\xi_N + \chi_1}\varepsilon_2^{*(N)}; \quad M_{32}^{(N)} = 0; \quad M_{33}^{(N)} = e^{\theta_j}/\sqrt{2},
\end{aligned}$$

To generate dark solitons for the scalar NLS type equations, one can feed the same nonzero plane wave solution and follow the above procedure except that one has to choose the $\hat{m}^{(1)}$ matrix of the following form

$$\hat{m}^{(1)} = \begin{pmatrix} m_1 & 1/n_1 \\ n_1 & 1/m_1 \end{pmatrix} \tag{23}$$

$$m_1 = \frac{k + i\lambda}{|c_i|} \ \ and \ \ n_1 = i. \tag{24}$$

Then, equation (19) yields the dark soliton solution for the corresponding integrable scalar NLS type equation.

3. APPLICATIONS

(i) Coupled GP (CGP) equation:

The coupled GP equation representing a binary BEC comprising of the hyperfine states of the same atomic species say, Rubidium (^{87}Rb) in a transient harmonic trap can be written down (in dimensionless form) as

$$i\frac{\partial \psi_j}{\partial t} + \frac{\partial^2 \psi_j}{\partial x^2} + \Big[a(t)\sum_{k=1}^{2} b_{jk}|\psi_k|^2 + v(x,t)\Big]\psi_j, \tag{25}$$

In the above equation, $v(x,t) = \lambda(t)^2 x^2$ represents the transient harmonic trap and $a(t)$ represents the temporal scattering length (a(t) should be negative for defocussing (attractive) and positive for focussing (repulsive) cases respectively) and ψ_j, $j = 1,2$ describes the order parameter of the condensates. The above integrable coupled GP equation has also been investigated [23, 24] and the dynamics of the vector BECs has been explored by constructing bright and dark solitons.

The above equation (25) admits the following eigenvalue problem,

$$\begin{aligned} \phi_x &= U\phi \\ \phi_t &= V\phi \end{aligned} \tag{26}$$

where, $\phi = (\phi_1, \phi_2, \phi_3)^T$ and

$$U = \begin{pmatrix} i\zeta(t) & Q_1 & Q_2 \\ -Q_1^* & -i\zeta(t) & 0 \\ -Q_2^* & 0 & -i\zeta(t) \end{pmatrix}, \tag{27}$$

$$V = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix}, \tag{28}$$

where,

$$\begin{aligned}
V_{11} &= -i\zeta(t)^2 + i\Gamma(t)x\zeta(t) + \frac{i}{2}\gamma(t)A(t)Q_1Q_1^* + \frac{i}{2}\gamma(t)A(t)Q_2Q_2^* \\
V_{12} &= [\Gamma(t)x - \zeta(t)]Q_1 + \frac{1}{2}Q_{1x} \\
V_{13} &= [\Gamma(t)x - \zeta(t)]Q_2 + \frac{1}{2}Q_{2x} \\
V_{21} &= -[\Gamma(t)x - \zeta(t)]Q_1^* + \frac{1}{2}Q_{1x}^* \\
V_{22} &= i\zeta(t)^2 - i\Gamma(t)x\zeta(t) - \frac{i}{2}\gamma(t)A(t)Q_1Q_1^* \\
V_{23} &= -\frac{i}{2}Q_2Q_1^* \\
V_{31} &= -[\Gamma(t)x - \zeta(t)]Q_2^* + \frac{1}{2}Q_{2x}^* \\
V_{32} &= -\frac{i}{2}Q_1Q_2^* \\
V_{33} &= i\zeta(t)^2 - i\Gamma(t)x\zeta(t) - \frac{i}{2}\gamma(t)A(t)Q_2Q_2^*
\end{aligned}$$

with

$$Q_1 = \frac{1}{\sqrt{A(t)}}\psi_1(x,t)e^{i(-\Gamma(t)x^2)},$$
$$Q_2 = \frac{1}{\sqrt{A(t)}}\psi_2(x,t)e^{i(-\Gamma(t)x^2)}. \tag{29}$$

In the above nonisospectral eigenvalue problem, spectral parameter $\zeta(t)$ obeys the following equation

$$\zeta(t) = \mu e^{-(\int \Gamma(t)dt)} \tag{30}$$

where μ is a hidden complex constant and $\Gamma(t)$ is an arbitrary function of time and

$$\Gamma(t) = \frac{d}{dt}logA(t). \tag{31}$$

$$a(t) = \frac{1}{A(t)}, \tag{32}$$

$$\lambda(t)^2 = \Gamma(t)^2 - \Gamma'(t) \tag{33}$$

It is known [23, 24] that the coupled GP equation is completely integrable only if the transient trap $\lambda(t)$ and the scattering length $a(t)$ are connected by the following

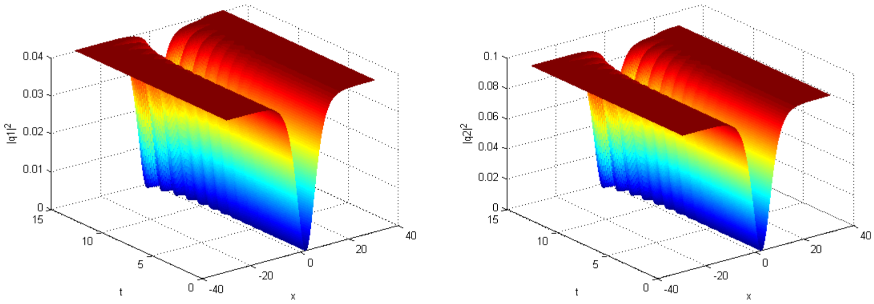

Fig. 1 – The density profiles of the dark solitons in the presence of periodic modulated potential for the choice of parameters $\Gamma(t) = \cos(\omega t + \delta)\exp(\sigma t)$, $a(t) = \exp(\frac{\exp(\sigma t(\sigma\cos(\delta+t\omega)+\omega\sin(\delta+t\omega)))}{\omega^2+\sigma^2})$, $\omega = 6$, $\delta = 0$, $\sigma = 0.4$, $\varepsilon_1^{(1)} = 0.3$, $\alpha_{10} = 0.1$, $\beta_{10} = 0.3$, $\chi_1 = 0.1$, $\delta_1 = 0.2$, $a_1 = -a_2 = 1, c_1 = c_2 = 1$

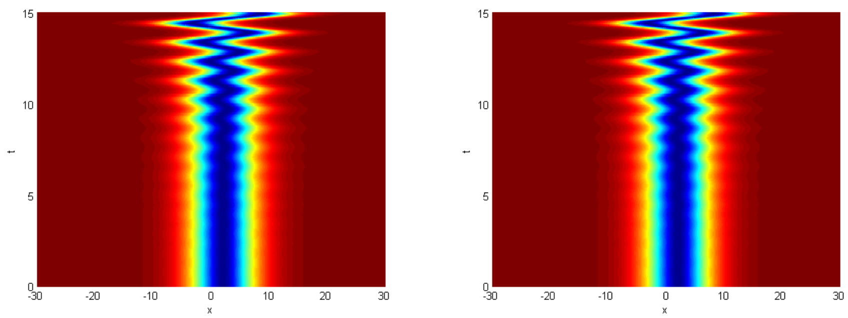

Fig. 2 – Contour plots of Fig.1 exhibiting the beating effect in the propagation dark solitons

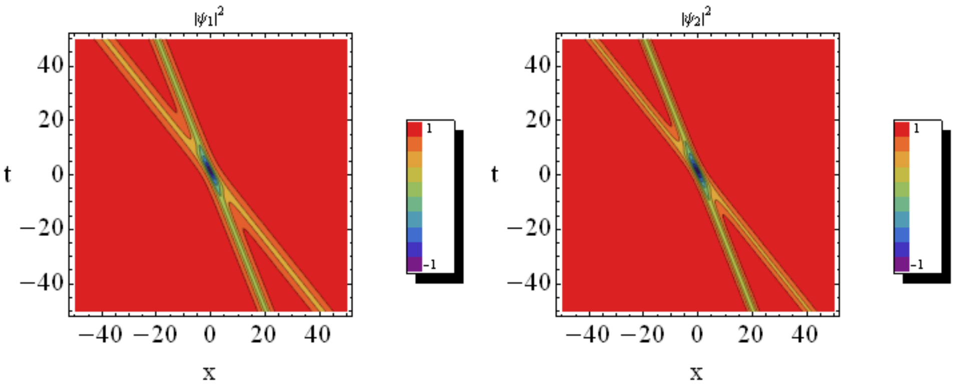

Fig. 3 – Collisional dynamics of dark solitons for the choice of parameters $a(t)=0.1$,$\Gamma(t)=0.1\times10^{-}2t$, $\alpha_1=0.1$, $\alpha_2=0.25$, $\beta_1=0.3$, $\beta_2=0.25$, $\delta_1=0.1$, $\delta_2=0.2$, $\chi_1=0.3$, $\chi_2=0.4$, $\varepsilon_1^{(1)}=0.85i$, $\varepsilon_1^{(2)}=0.5$ such that $|\varepsilon_1^{(j)}|^2+|\varepsilon_2^{(j)}|^2=1,(j=1,2)$,$a_1=-a_2=1,c_1=c_2=1$

equation (obtained by substituting eqn.(31) and eqn.(32) into eqn.(33))

$$-\frac{1}{2a(t)}\frac{d^2a(t)}{dt^2}+\frac{1}{a^2(t)}\Big(\frac{da(t)}{dt}\Big)^2+2\lambda^2=0, \tag{34}$$

To generate dark vector solitons of the above GP eq.(25), we choose the following non zero plane wave solution as the seed

$$q_i=c_ie^{[i(a_ix-(a_i^2/2+\sigma_lc_l^2)t)]} \tag{35}$$

to obtain the vacuum eigenfunction

$$\Phi^{(0)}=\begin{pmatrix}\phi_{11}^{(0)} & 0 & 0\\ 0 & \phi_{22}^{(0)} & 0\\ 0 & 0 & \phi_{33}^{(0)}\end{pmatrix}exp(i(\mu x+(\zeta\mu-\frac{1}{2}\zeta^2+\frac{1}{2}\mu^2+\sigma_lc_l^2)t)) \tag{36}$$

where

$$\begin{aligned}\phi_{11}^{(0)} &= e^{-2i\zeta(t)x-6i\zeta(t)^2t},\\ \phi_{22}^{(0)} &= e^{i\zeta(t)x+3i\zeta(t)^2t},\\ \phi_{33}^{(0)} &= e^{i\zeta(t)x+3i\zeta(t)^2t},\end{aligned}$$

Employing the gauge transformation method, one obtains the dark soliton solution for the coupled GP equation (25) (for the defocussing case keeping the temporal scattering length a(t)=-a(t)) of the following form

$$\psi_1^{(1)} = \sqrt{\frac{1}{a(t)}} 2c_i\varepsilon_1^{(1)}\beta_1(t)tanh(\theta_1)e^{i(-\xi_1+f(t)\frac{x^2}{2})}, \tag{37}$$

$$\psi_2^{(1)} = \sqrt{\frac{1}{a(t)}} 2c_i\varepsilon_2^{(1)}\beta_1(t)tanh(\theta_1)e^{i(-\xi_1+f(t)\frac{x^2}{2})}, \tag{38}$$

where

$$\theta_1 = 2\beta_1 x + 4\int \alpha_1\beta_1 dt - 2\delta_1, \tag{39}$$

$$\xi_1 = 2\alpha_1 x + 2\int (\alpha_1^2 - \beta_1^2)dt + \Lambda - 2\chi_1, \tag{40}$$

and

$$\Lambda = e^{[i(a_i x - (a_i^2/2 + \sigma_l c_i^2)t)]} \tag{41}$$

with $\alpha_1 = \alpha_{10}e^{[\int \Gamma(t)dt]}$, $\beta_1 = \beta_{10}e^{[\int \Gamma(t)dt]}$ while δ_1 and χ_1 are arbitrary parameters and $\varepsilon_1^{(1)}$,$\varepsilon_2^{(1)}$ are coupling parameters connected by the relation $|\varepsilon_1^{(j)}|^2 + |\varepsilon_2^{(j)}|^2 = 1, (j = 1,2)$. The dark solitons given by eqs. (37) and (38) are identical to the one reported in [24]. The density profile of dark solitons given by eqs. (37) and (38) is shown in Fig. 1 while its contour plots shown in Fig.2 display the beating effect of the dark solitons during time evolution. It should be mentioned that this beating effect arises due to the temporal nature of the harmonic trap [24, 25]. Eventhough the nature of the solitons depends on the scattering length $a(t)$ and trap frequency $\lambda(t)$ (or $\Gamma(t)$) in accordance with the integrabilily condition given by equation (34), the trap frequency $\Gamma(t)$ predominates over the scattering length. As we have chosen the trap frequency $\Gamma(t)$ to be a periodic wave with exponentially varying amplitude ($\sigma > 0$), the amplitude of the soliton visibly gets higher for large "t" as it is evident from Figs. 1 and 2. This beating effect of dark solitons is consistent with the experimental results discussed in Ref. [26]. Our paper also gives a simple experimental protocol to observe "beating effect" in the collisional dynamics of dark solitons without the impact of external thermal cloud. The collisional dynamics of two solitons is shown in Fig. 3. One can easily extend this modified gauge transformation approach to construct multi dark soliton solutions and study their collisional dynamics.

It should also be mentioned that the present integrable model does not have the luxury of observing parametric resonance excitations in dark solitons (BECs) by virtue of the constraint imposed by the integrability condition given by eq.(34) as one may not be able to choose the frequency of the trap $\Gamma(t)$ and $a(t)$ desirably.

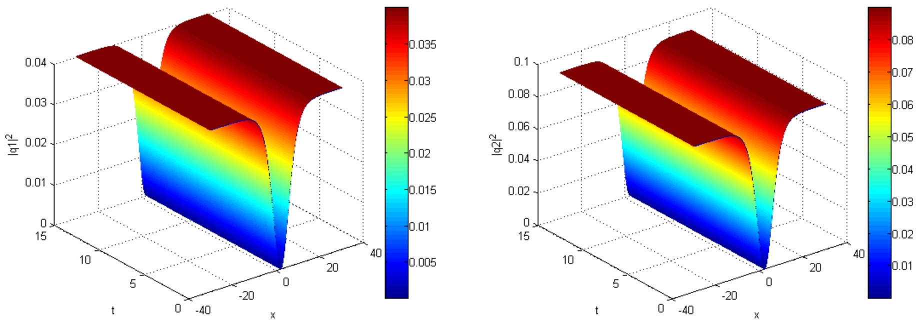

Fig. 4 – Density profiles of dark solitons of coupled NLS equation for the choice $\varepsilon_1^{(1)} = 0.3$, $\alpha_{10} = 0.3, \beta_{10} = 0.5, \chi_1 = 0.3, \delta_1 = 0.5$,$a_1 = -a_2 = 1, c_1 = c_2 = 1$

(ii) Coupled Nonlinear Schrödinger (CNLS) equation

Under the dependent variable transformation

$$\psi_1 = \frac{1}{\sqrt{a(t)}l(t)}\Phi_1(X,T)exp(i[\Gamma(t)x^2]) \tag{42}$$

$$\psi_2 = \frac{1}{\sqrt{a(t)}l(t)}\Phi_2(X,T)exp(i[\Gamma(t)x^2]) \tag{43}$$

where Φ_i is an arbitrary function with spatial and temporal variables chosen as $X = \frac{x}{l(t)}$ and $T = l^{-2}$ with $\Gamma(t), \lambda(t)$ given by

$$\Gamma_t + 4\Gamma^2 + \lambda^2 = 0, l_t - 4\Gamma l = 0, \tag{44}$$

eqn. (25) can be reduced to the standard coupled nonlinear Schödinger equation in the normal GVD region of the following form

$$i\Phi_{1T} - \Phi_{1XX} + \nu(|\Phi_1|^2 + |\Phi_2|^2)\Phi_1 = 0, \tag{45}$$

$$i\Phi_{2T} - \Phi_{2XX} + \nu(|\Phi_1|^2 + |\Phi_2|^2)\Phi_2 = 0. \tag{46}$$

The above coupled nonlinear equation is nothing but the celebrated integrable model proposed by Manakov [22]. The dark soliton solution of the CNLS equations employing the gauge transformation approach is of the following form (defocussing-defocussing Manakov model for $\nu = -1$)

$$\psi_1^{(1)} = -c_1\varepsilon_1^{(1)}\beta_1 tanh(\theta_1)e^{i(-\xi_1)}, \tag{47}$$

$$\psi_2^{(1)} = -c_2\varepsilon_2^{(1)}\beta_1 tanh(\theta_1)e^{i(-\xi_1)}, \tag{48}$$

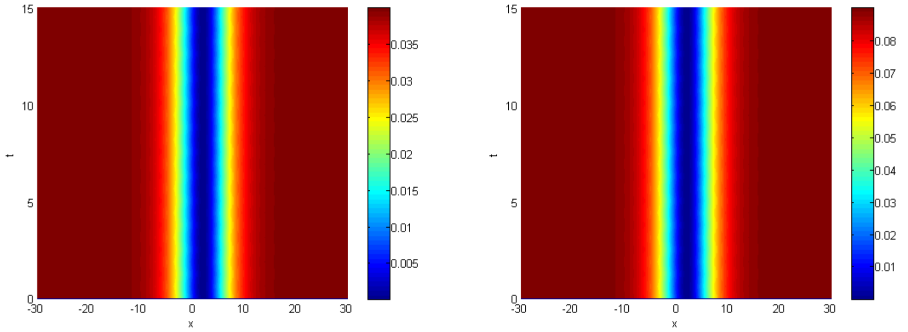

Fig. 5 – Contour plots of Fig. 4 showing the trajectory of dark solitons

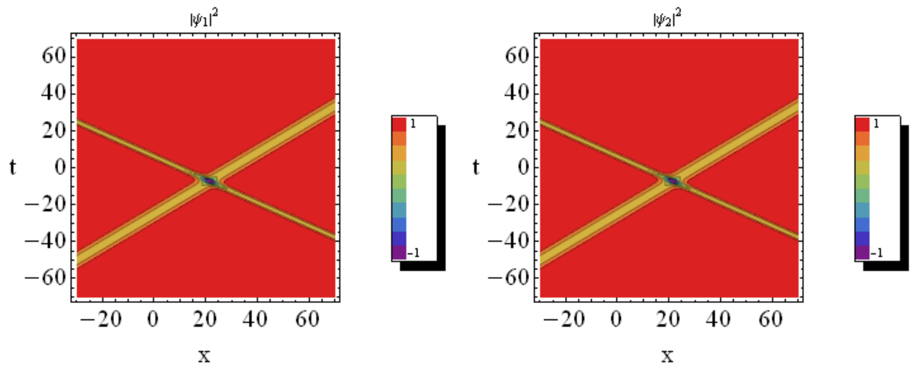

Fig. 6 – Collisional dynamics of dark solitons of the coupled NLS equation for the parametric choice $\alpha_1 = 0.1$, $\alpha_2 = 0.25$, $\beta_1 = 0.3$, $\beta_2 = 0.2$, $\delta_1 = 0.1$, $\delta_2 = 0.2$, $\chi_1 = 0.3$, $\chi_2 = 0.4$, $\varepsilon_1^{(1)} = 0.85i$, $\varepsilon_1^{(2)} = 0.5$ such that $|\varepsilon_1^{(j)}|^2 + |\varepsilon_2^{(j)}|^2 = 1/\nu, (j = 1, 2)$, $a_1 = -a_2 = 1, c_1 = c_2 = 1$

where

$$\begin{aligned}\theta_1 &= 2\beta_1 x + 8\alpha_1\beta_1 t - 2\delta_1,\\ \xi_1 &= 2\alpha_1 x + 4(\alpha_1^2 - \beta_1^2)t + \Lambda - 2\chi_1,\end{aligned}$$

and

$$\Lambda = e^{[i(a_i x - (a_i^2/2 + \sigma_l c_i^2)t)]} \tag{49}$$

subject to $|\varepsilon_1^{(j)}|^2 + |\varepsilon_2^{(j)}|^2 = \frac{1}{\nu}, (j = 1,2)$ where $\varepsilon_{1,2}$ represent coupling parameters. $\alpha_j = Re[\mu]$ and $\beta_j = Im[\mu]$ while $j = 1,2$ are the real constants and μ is the hidden spectral parameter. The dark soliton given by eqn.(3) is identical to the one reported in ref [27]. The density profile of dark solitons and its trajectory are shown in Figs. 4 and 5 respectively. The collisional dynamics of dark solitons of the Manakov model is displayed in Fig. 6.

4. CONCLUSION

In this paper, we have formulated a simple algebraic approach by harnessing the freedom in the celebrated gauge transformation approach to construct dark solitons of coupled NLS type equations. As an application, we have constructed the dark solitons of coupled GP and coupled NLS equations and studied their properties. It should be emphasized that the present approach is purely algebraic and enables one to generate multi dark soliton solutions from a trivial/nontrivial seed solution and it is quite superior to other analytical techniques like Darboux transformation method [28], Hirota method [27, 29] and IST [30]. As far the limitations of this approach are concerned, it should be emphasized that this method generates only bright and dark solitons for zero and non zero seeds respectively. We have not yet exploited it to generate breathers, Ma solitons, rogue waves etc.,

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REFERENCES

1. N.J.Zabusky, M.D.Kruskal, Phys. Rev. Lett. **15**, 240 (1965).
2. R. M. Miura, C. S. Gardner and M. D. Kruskal, J. Math. Phys. **9**, 1204 (1968).
3. Ryogo Hirota, J. Phys. Soc. Jpn. **33**, 1459 (1972).
4. V. E. Zhakharov and E. I. Schulman, Physica D **4**, 270 (1982).
5. V. S. Bagnato *et al.*, Rom. Rep. Phys. **67**, 5 (2015);
   R. Radha and P. S. Vinayagam, Rom. Rep. Phys. **67**, 89 (2015).

6. A. T. Avelar, D. Bazeia, and W. B. Cardoso, Phys. Rev. E **79**, 025602(R) (2009);
D J Frantzeskakis, J. Phys. A: Math. Theor. **43** 213001 (2010);
L A Toikka and K-A Suominen, Phys. Rev. A **87**, 043601 (2013).
7. A Hasegawa and Y Kodama, *Solitons in Optical Communications* (Oxford University press, Oxford, 1995).
8. A Hasegawa and F D Tappert, Appl.Phys. Lett. **23**, 142 (1973).
9. G. P Agrawal, *Nonlinear Fibre Optics* (Academic, New York, 2012).
10. K. J. Blow and N. J. Doran, Phys. Lett. A **107**, 55 (1985).
11. R. Radha, P. S. Vinayagam and K. Porsezian, Phys. Rev. E. **88**, 032903 (2013).
12. M. F. Saleh and F. Biancalana, Phys. Rev. A **87**, 043807 (2013).
13. J. R. Taylor, *Optical Solitons-Theory and Experiment* (Cambridge University press, New York 1992).
14. D. Krokel, N. J. Halas, G. Giuliani and D. Grischkowsky, Phys. Rev. Lett. **60**, 29 (1988).
15. A. M. Weiner, J. P. Heritage, R. J. Hawkins, R. N. Thurston, E. M. Kirschner, D. E. Leaird and W. J. Tomlinson, Phys. Rev. Lett. **61**, 2445 (1988).
16. Ryogo Hirota and Junkichi Satsuma, Prog. Theor. Phys. Supplement. **59**, 64 (1976).
17. R. K. Bullough and P. J. Caudrey *Solitons* (Springer, Berlin, 1980) p. 157.
18. V. B. Matveev, Phys. Lett. A **166**, 205 (1992).
19. V. B. Matveev and M. Salle, *Darboux transformations and Solitons* (Springer-Verlag, Berlin, 1991).
20. C. S. Gardner, J. M. Greene, M. D. Kruskal and R. M. Miura, Phys. Rev. Lett. **19**, 1095 (1967).
21. L.-L. Chau, J.C. Shaw, H.C. Yen, J. Math. Phys. **32(7)**, 1737 (1991).
22. S V Manakov, Sov. Phys. JETP. **38(2)**, 248 (1974).
23. V. Ramesh Kumar, R. Radha and M. Wadati, Phys.Lett.A **374**, 3685 (2010).
24. S. Rajendran, P. Muruganandam and M. Lakshmanan, J. Phys. B: At. Mol. Opt. Phys. **42**, 145307 (2009).
25. Vladimir N. Serkin and Akira Hasegawa, Phys. Rev. Lett. **85**, 4502 (2000).
26. C. Backer, S. Stellmer, Soltan-Panahi, S. Dorscher, M. Baumert, E. M. Richter, J. Kronjager, K. Bongs, Sengstock and Klaus, Nature Phys. **4**, 496 (2008);
A. Weller, J. P. Ronzheimer,C. Gross, D. J. Frantzeskakis, G. Theocharis, P. G. Kevrekidis, J. Esteve and M. K. Oberthaler, Phys Rev L, **101**, 130401 (2008);
S. Stellmer, C. Becker, P. Soltan-Panahi, E. M. Richter, S. Dorscher, M. Baumert, J. Kronjager, K. Bongs and K. Sengstock, Phys Rev L **101**, 120406 (2008).
27. R. Radhakrishnan and M. Lakshmanan, J. Phys. A:Math. Gen. **28**, 2683 (1995).
28. Liming Ling, Li-Chen Zhao and Boling Guo, arXiv:1309.1037v1[nlin.SI].
29. A. Mahalingam and K. Porsezian, Phys. Rev. E. **64**, 046608 (2001).
30. M. J. Ablowitz, G. Biondini and B. Prinari, J. Math. Phys. **47**, 063508 (2006).