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BIFURCATIONS AND FAST-SLOW DYNAMICS IN A LOW-DIMENSIONAL MODEL FOR QUASI-PERIODIC PLASMA PERTURBATIONS*

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Abstract. In a low dimensional model which describes the dynamics of the plasma pressure gradient and of the amplitude of the displacement of the magnetic field in tokamaks we study the Hopf bifurcation. We analyze the fast-slow dynamics which appears for small normal heat diffusion and we corroborate the results with those observed in experiments.

Key words: bifurcations, fast-slow dynamics, fusion plasma physics, tokamak.

1. INTRODUCTION

Oscillations of plasma's parameters (sawteeth, edge localized modes (ELMs), frequently interrupted regime of neoclassical tearing modes) are observed when some large scale plasma instabilities do not lead to an immediate termination of a discharge. The understanding of such phenomena is an important tool in controlling the whole reaction. There are many fundamental approaches to the subject, but, despite the huge theoretical and experimental effort, it seems that the phenomena are not fully understood.

In our paper we focus on a low dimensional model which describes the dynamics of the plasma pressure gradient and of the amplitude of the magnetic field displacement.

Our low-dimensional model [1] is given by

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$$\begin{cases} \dfrac{\mathrm{d}^2}{\mathrm{d}t_n^2}\zeta_n = \left(p_n' - 1\right)\cdot\zeta_n - \delta\cdot\dfrac{\mathrm{d}}{\mathrm{d}t_n}\zeta_n \\ \dfrac{\mathrm{d}}{\mathrm{d}t_n}p_n' = \eta\cdot\left(h - p_n' - \zeta_n^2\cdot p_n'\right) \end{cases}, \tag{1}$$

where ξ_n is the amplitude of the magnetic field displacement, p_n' is the plasma pressure gradient at the plasma edge, t_n is time, δ is dissipation/relaxation of the instability responsible for the ELM burst, η is diffusion and h is input power in the system. The index n means that all the quantities are normalized.

The first equation describes the evolution of the magnetic field perturbation and relaxation dynamics. The second equation describes power balance in the system including the effect of unstable modes.

The physically relevant ranges of the parameters are [1]

$$0.08 < \delta < 0.8,\ 0.006 < \eta < 0.4,\ 1 < h < 2\,.$$

The system (1) can be transformed into a system of differential equations of first order, convenient for a rigorous mathematical analysis:

$$\dot{x} = \left(z-1\right)y - \delta x\ ; \quad \dot{y} = x; \quad \dot{z} = \eta\left(h - z - y^2 z\right). \tag{2}$$

The model has some common properties with the classical Lorenz system [2], considered a benchmark for chaotic systems theory: it is dissipative and invariant under the change of coordinates $S(x,y,z) = (-x,-y,z)$. It exhibits complex dynamics, comparable with that of Lorenz's system (double scroll butterfly-like attractor, coexistence of many attractors in the phase space, bifurcations) [3]. However, it does not belong to Lorenz-like family systems [4] because the non-linear perturbation is not quadratic.

Our model is more complicated than the Lorenz system because it has two different time scales. It is a fast-slow system with two fast variables and a slow one. From a practical point of view, we are interested in studying the oscillations of the sawtooth-type, characterized by two time-scales: a long rise time, when the fast variables are almost zero and the slow variable rises almost linearly, and a very short crash time, when the fast variables suddenly increase and rapidly decrease. Many interesting papers have been recently dedicated to the study of fast-slow systems because they are related to phenomena that occur in ecosystems, medicine, changes in the climate, financial markets (see [5] and the references therein), but also in fusion plasma fusion [6].

In this paper we make a classical analysis of the system (2) for small values of the parameters (for which the sawtooth oscillations occur), starting from the classical bifurcation theory and incorporating the two time-scales.

The paper is organized as follows: in Section 2 we present the general properties of the system. Section 3 is devoted to the analysis of the main bifurcations that occur in the system and Section 4 contains the results related to its fast-slow dynamics. The conclusions are formulated in Section 5.

2. GENERAL PROPERTIES OF THE SYSTEM

The system (2) is generated by the vector field $X : R^3 \to R^3$, $X(x, y, z) = ((z-1)\cdot y - \delta \cdot x,\ x,\ \eta \cdot (h - z - y^2 z))$.

Because the divergence $\mathrm{div}(X) = -(\delta + \eta + \eta \cdot y^2)$ is negative, it results that the system is dissipative, *i.e.* its flow contracts the volumes. This means that, for any positive values of the parameters δ, η and h, all orbits $\{(x(t), y(t), z(t)),\ t \geq 0\}$ asymptotically approach one of the attractors of the system.

Practically, one can observe that the values of $\xi_n(t_n) \overset{notation}{=} y(t)$ and $p_n' \overset{notation}{=} z(t)$ exhibit temporal oscillations independently of the initial conditions. However their pattern (damped, ELM, periodic, or stochastic) depends on the parameter values as shown in Fig. 1 where the experimental regions ($8 \cdot 10^{-3} \leq \delta \leq 8 \cdot 10^{-1}$, $6 \cdot 10^{-4} \leq \eta \leq 4 \cdot 10^{-2}$) are shown for three values of the input power h, namely $h = 1.2$, $h = 1.5$ and $h = 1.8$.

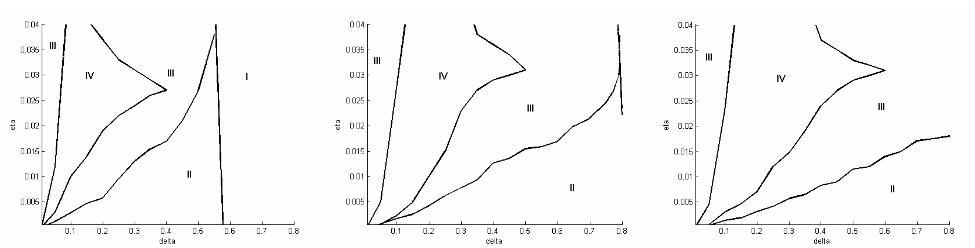


Fig. 1 – Patterns of the oscillations of the system (1) for *h* = 1.2 (left), *h* = 1.5 (center), and *h* = 1.8 (right): damped (zone I), sawtooth (zone II), periodic (zone III), stochastic (zone IV).

The figures must be interpreted in the following way:

– if (δ, η) belongs to zone I, the system exhibits damped oscillations. The values of ξ_n and η_n become almost constant after a long enough time. In this case each attractor of the system is formed by a single point.

– if (δ,η) belongs to zone II, the system is characterized by oscillations of sawtooth-type, with a long rise time and a short crash time. This is typical of ELMs. In this case the attractor of the system is a periodic orbit.

– if (δ,η) belongs to zone III, the oscillations are periodic and the rise time is comparable with the crash time. In this case the attractor of the system is also a periodic orbit.

The dynamical difference between zone II and zone III lies in the motion speed on the orbit. For the systems corresponding to (δ,η) in zone II, one can observe large variations between the small speed during the rise time and the great speed during the crash time. In systems corresponding to (δ,η) in zone III, the speed during the rise is comparable with the speed during the crash time:

– if (δ,η) belongs to zone IV the oscillations are not periodic. The attractor of the system is a complicated curve (strange attractor) and the dynamics is not regular. It is usually called stochastic (unpredictable) or chaotic.

The system (2) is invariant with respect to the map $S: R^3 \to R^3$, defined by $S(x,y,z)=(-x,-y,z)$, which is the symmetry with respect to the Oz axis.

This property has important consequences:

a) $O(-x_0,-y_0,z_0)=S(O(x_0,y_0,z_0))$.

b) If (x_0,y_0,z_0) is an equilibrium point of (2) then also $(-x_0,-y_0,z_0)$ is an equilibrium point of (2).

c) If (A) is an attractor, then $S(A)=\{S(x,y,z) \mid (x,y,z)\in A\}$ is also an attractor.

d) If the point $M=(a,b,c)$ belongs to the domain of attraction of the attractor (A), then $S(M)=(-a,-b,c)$ belongs to the domain of attraction of $S(A)$. The domains of attraction are symmetric with respect to the Oz axis.

The symmetry group is $Z^2=\{Id_{R^3}, S\}$ and we can use some specific techniques to study the bifurcations [7, pp. 276–288].

An important consequence of the symmetry is that there are twin bifurcations of the equilibrium points.

3. BIFURCATIONS OF THE EQUILIBRIUM POINTS

Technical computations show that:

Proposition 1. a) *For* $h<1$ *the system (2) has a unique equilibrium* $P_1=(0,0,h)$ *which is a hyperbolic sink.*

b) *For* $h>1$ *the system (2) has three equilibria:* $P_1=(0,\,0,\,h)$ *which is a saddle point and two S-conjugate equilibria* $P_{2,3}=\left(0,\ \pm\sqrt{h-1},\ 1\right)$.

Using the Routh-Hurwitz criterion one can prove that $P_{2,3}$ are stable if

$$\delta h(\delta+\eta h)-2(h-1)>0. \tag{3}$$

In this situation the points $P_{2,3}$ are only local attractors. Numerical simulations show the existence of other attractors (limit cycles or strange attractors, depending on the values of $\delta,\,\eta,\,h$) in the phase space.

This mathematical result is interesting from a physical point of view. If the normalized input power h is small ($h<1$), the system is stabilized in time. The magnetic field approaches the unperturbed configuration (because $y(t)\to 0$) and the normalized plasma pressure gradient at the plasma edge becomes almost constant (because $z(t)\to h$).

If the normalized input power h is large enough ($h>1$), which is commonly used in experiments, the displacement of the magnetic field, and the plasma pressure gradient at the plasma edge generally oscillate, excepting the situation when the initial conditions of the system are close to $\xi_n(0)=\pm\sqrt{h-1},\ p_n'=1$ and $\xi_n'(0)=0$.

3.1. PITCHFORK BIFURCATION

The pitchfork bifurcation occurs when the equilibrium point P_1 looses its stability.

Proposition 2. *For* $h=1$ *the system (2) undergoes a supercritical pitchfork bifurcation at* $P_1=(0,\,0,\,h)$. *For* $h<1$ *there is a unique equilibrium point* $P_1=(0,0,h)$ *which is stable and for* $h>1$ *there are three equilibrium points:* $P_1=(0,0,h)$ *(which is unstable) and* $P_{2,3}=\left(0,\ \pm\sqrt{h-1},\ 1\right)$, *which are stable near* $h=1$.

In the bifurcation diagram (Fig. 2) the solid and dashed curves represent the stable and unstable equilibrium points, respectively. The pitchfork bifurcation, which occurs at $h=1$, is labelled "*A*". The values of the bifurcation parameter h are plotted on the horizontal.

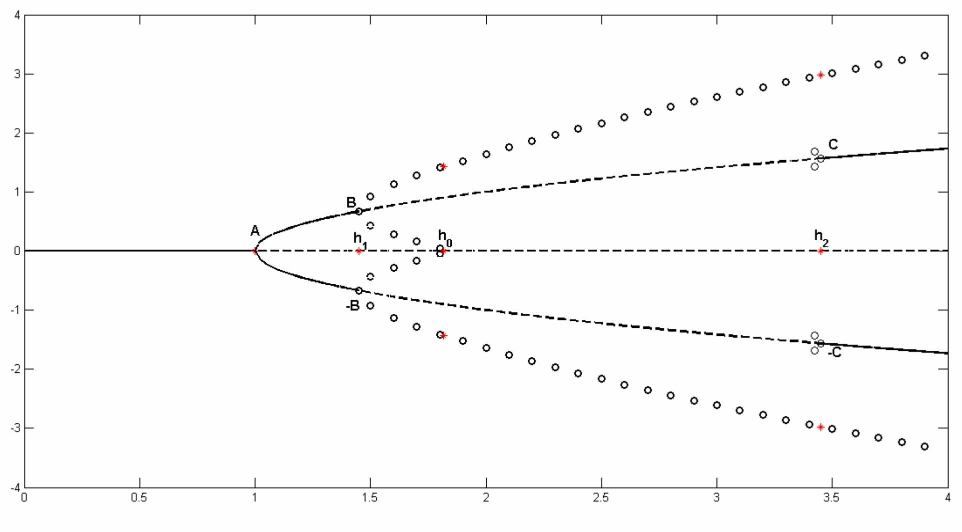


Fig. 2 – Bifurcation diagram of system (2) for $\delta = 0.2, \quad \eta = 2.$

In Fig. 2 one can observe that, for fixed values of δ and η, the twin equilibrium points $P_{2,3}$ are stable near $h = 1$ (the segment AB), then they become unstable for larger values of h (the segment BC) and finally they gain their stability when h increases.

3.2. HOPF BIFURCATION

The Hopf Theorem [8, pp. 150–156] assumes the dependence of the system on one parameter, but we have three of them. Thus we have to assume two parameters to be fixed and one to be varied. The structure of the system (2) allows us not to be specific about which two are considered fixed and which one is assumed to be variable. Also, we need not choose between the twin equilibrium points because they have the same properties. We consider P_2 without loss of generality.

Proposition 3. *If* $0 < \delta < \sqrt{2}$ *and* $\eta < \dfrac{(2-\delta^2)^2}{8\delta}$, *the system (2) undergoes (twin) Hopf bifurcations of the equilibrium points* $P_{2,3} = \left(0, \pm\sqrt{h-1}, 1\right)$ *in*

$$h_1 = \frac{(2-\delta^2) - \sqrt{(2-\delta^2)^2 - 8\delta\eta}}{2\delta\eta} \quad \textit{and} \quad h_2 = \frac{(2-\delta^2) + \sqrt{(2-\delta^2)^2 - 8\delta\eta}}{2\delta\eta}.$$

Remark: If $\delta\eta h_1^2 > 2$, the Hopf bifurcation is above h_1, and if $\delta\eta h_2^2 < 2$ the Hopf bifurcation is below h_2.

In Fig. 2 the Hopf bifurcations are labeled $B, -B$ and $C, -C$, respectively.

The Hopf bifurcations, labeled "B" and "$-B$", occur simultaneously on the two branches of the equilibrium points at $h_1 = 1.4488$. The other two, labeled "C" and "$-C$" occur at $h_2 = 3.4512$. In Fig. 2 are also presented the maximal and the minimal values of y on the periodic orbits formed through the Hopf bifurcations (small circles). We observe that the size of the stable periodic orbits generated by the Hopf bifurcation at $h_1 = 1.4488$ increases when h increases, the orbits approach until they collide. At $h_0 \approx 1.8183$ a stable double scroll homoclinic orbit of P_1 is formed (Fig. 3a).

Figure 2 also shows that double scroll periodic orbits are formed for $h > h_0$. They have twin symmetry (which is due, of course, to the symmetry of the system (2)).

When h increases, the size of the double scroll periodic orbit also increases and the orbit moves away from the equilibrium point P_1, but also from P_2 and P_3.

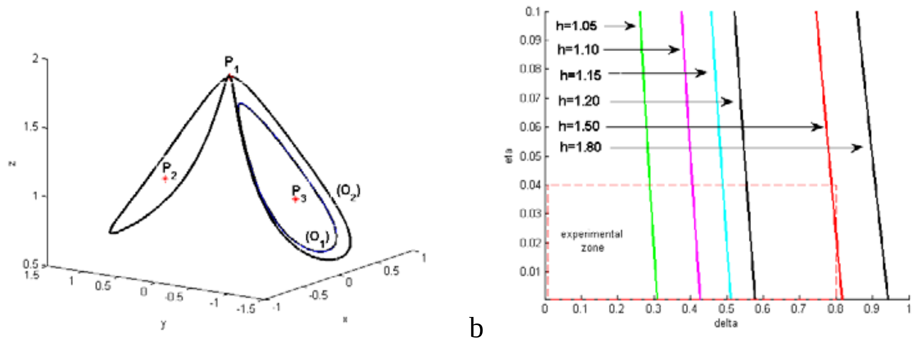


Fig. 3 – a) Periodic orbits (O_2) of the system (2) for δ = 0.2, η = 2, h = 1.8183 collide in the equilibrium point P_1. The orbit (O_1) is obtained for δ = 0.2, η = 2, h = 1.78;
b) Hopf curves for various values of h.

From Proposition 3 it results that the relation that must be fulfilled by δ, η and h in order to have a Hopf bifurcation is $\delta h(\delta + \eta h) - 2(h-1) = 0$. It is the limiting equality case of inequality (3). This shows that the equilibrium points $P_{2,3}$

lose or gain stability if and only if a solution branch passes through a Hopf point, *i.e.* in parameter space the Hopf surface coincides with the surface of marginal stability. The Hopf surface has the explicit form

$$S_H : \delta h(\delta + \eta h) - 2(h-1) = 0 \ . \tag{4}$$

For fixed values of h we obtain Hopf curves in the parameters' plane (δ, η). In Fig. 3b are presented the Hopf curves obtained for various values of h. The Hopf curves are moving to the right when h increases and they do not intersect the experimental zone for $h > 1.8$.

The Hopf curves have an important meaning for the dynamics of the system. Practically, they separate zone I (damped oscillations) from zone II, III (periodic oscillations) in Fig. 1. They also have influence on the slow-fast dynamics of the system.

4. FAST-SLOW DYNAMICS

In the case $0 < \eta << 1$ the system (2) is a fast-slow system with two fast variables, x and y, and a slow one, z. This dynamics is important for the study of our model because the experimental interval of η is $0.006 < \eta < 0.4$.

Fast-slow systems are characterized by two different time scales, fast and slow time. The dynamics consists of fast motions in x, y directions and slow motions in z direction. This structure yields to nonlinear phenomena as relaxation oscillations which are also observed in many physical, chemical, biological [9] or medical problems [10].

In Fig. 4 are presented typical relaxation oscillations of system (2), for $\delta = 0.6, \quad \eta = 0.009, \quad h = 1.5.$ The fast variables x and y remain nearly zero for most of the time (the rise time) except for a very short time (the crash time) when they suddenly increase and rapidly decrease. These oscillations are also known as sawtooth-type oscillations, due to the specific form of the graph of the slow variable. Practically, a point spends long time (the rise time) to climb the vertical line and has a rapid descending motion (rise time) on the loop of its orbit.

The mechanism of sawtooth-type oscillation can be understood from the study of the fast and slow subsystems. The fast and slow subsystems of (2), respectively, are

$$\dot{x} = (z-1)y - \delta x, \ \dot{y} = x, \ \dot{z} = 0 \ \text{ and } \ \dot{x} = 0, \quad \dot{y} = 0, \quad \dot{z} = \eta\left(h - z - y^2 z\right).$$

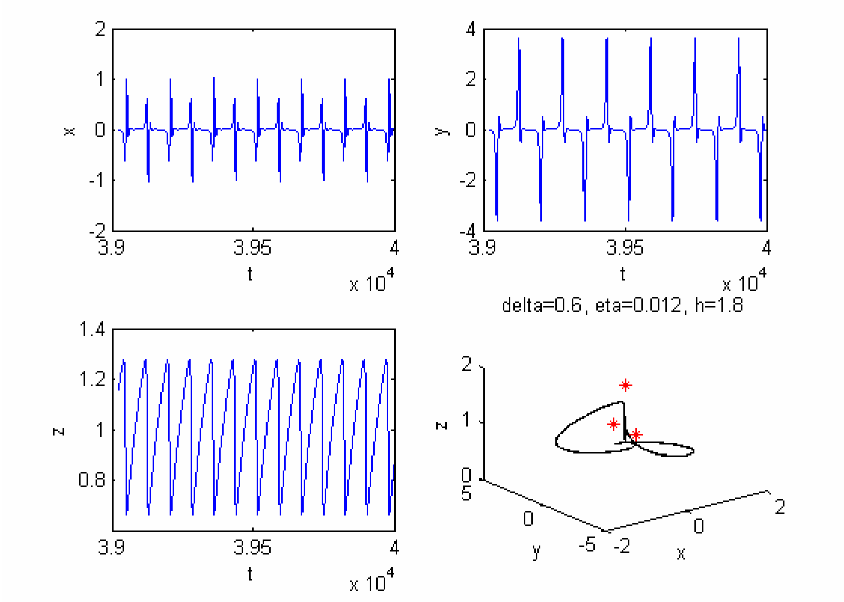


Fig. 4 – Sawtooth orbit $\delta = 0.6,\ \eta = 0.009,\ h = 1.5$.

The **critical manifold** (non-uniformly hyperbolic) of the fast subsystem is

$$M = \{(0,\ y, 1),\ y \in R\} \cup \{(0,\ 0,\ z),\ z \in R\}.$$

The **slow motion** of $(0,\ 0,\ z_0)$, when $z_0 < 1 - \delta^2/4$, is given by

$$x(t) = 0,\ y(t) = 0, z(t) = h + (z_0 - h)\mathrm{e}^{-\eta t}.$$

In this situation $z(t)$ slowly increases toward h. When $1 - \delta^2/4 < z(t)$ we approach the bifurcation point $(0,\ 0,\ 1)$ on the critical manifold and the motion is governed by the fast subsystem.

The **fast motion of** $(0,\ 0,\ z_0)$, when $1 - \delta^2/4 < z_0 < 1$, is given by

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}, y(t) = \frac{x' + \delta x}{z_0 - 1},\ z(t) = h + (z_0 - h)\mathrm{e}^{-\eta t},$$

where: $\lambda_1 = \dfrac{-\delta - \sqrt{\delta^2 + 4(z_0 - 1)}}{2} < 0$ and $\lambda_2 = \dfrac{-\delta + \sqrt{\delta^2 + 4(z_0 - 1)}}{2} > 0$ and C_1, C_2 are constants depending on z_0.

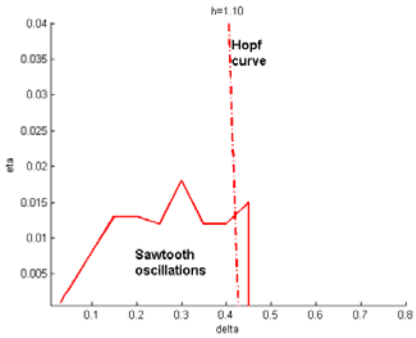
We observe that the values of $|x(t)|$ and $|y(t)|$ increase exponentially (rapidly) when t increases. When $|y(t)|$ is large enough, the values of $\dot{z}$ in system (2) become negative, z decreases and become smaller than $1-\delta^2/4$. From this moment the cycle is repeated: the orbit becomes close to the Oz axis, then it slowly moves up along Oz until $|y(t)|$ is large enough, then it rapidly moves down until $z(t)<1-\delta^2/4$.

The oscillations of $x(t), y(t), z(t)$ are generated by the existence of an attractor in the phase space of the system (2).

For the sawtooth analysis of an orbit we look only at the variation of z and we compare the rise time and the crash time, *i.e.* the time necessary to go up from the minimum values of z to the next maximum value (increasing part of the graph of z) and the time necessary to go down from the maximum values of z to the next minimum value (decreasing part of the graph of z).

An orbit is considered of sawtooth-type if $\dfrac{\text{Rise time}}{\text{Crash time}} > 4$.

Zones of the parameter plane (δ,η) where sawtooth oscillations occur are presented in Fig. 5 (for various values of h).



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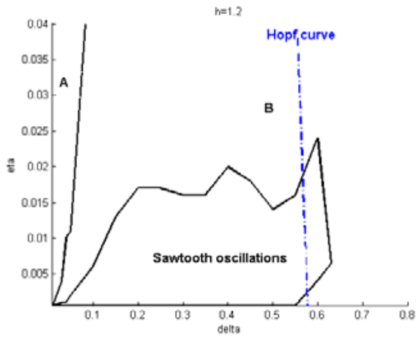

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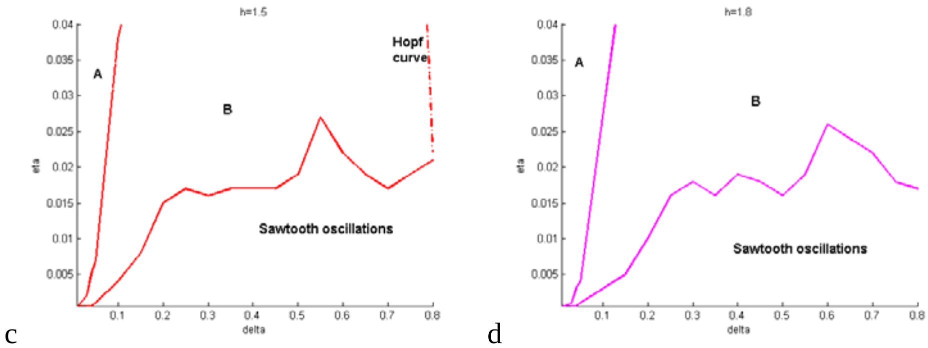

Fig. 5 – Sawtooth zones for $h = 1.10$ (a), $h = 1.20$ (b), $h = 1.50$ (c) and $h = 1.80$ (d).

The sawtooth zone is limited by the Hopf bifurcation curve, which moves to the right when h increases. Practically, it leaves the experimental zone at $h \approx 1.52$.

In Fig. 6 one can observe that the area of the sawtooth zone represents less than 40 % from the area of the experimental zone on the parameter plane (δ, η) and the larger value is obtained for $h \approx 1.5$. It is also interesting to see that ELM oscillations (which are sawtooth-type oscillations) are obtained for small diffusion ($\eta < 0.025$, less than 50% of the maximal possible value). In the same time δ, the dissipation/relaxation of the instability, must be considerable in order to observe sawtooth oscillations. This is in agreement with the experimental observation [11].

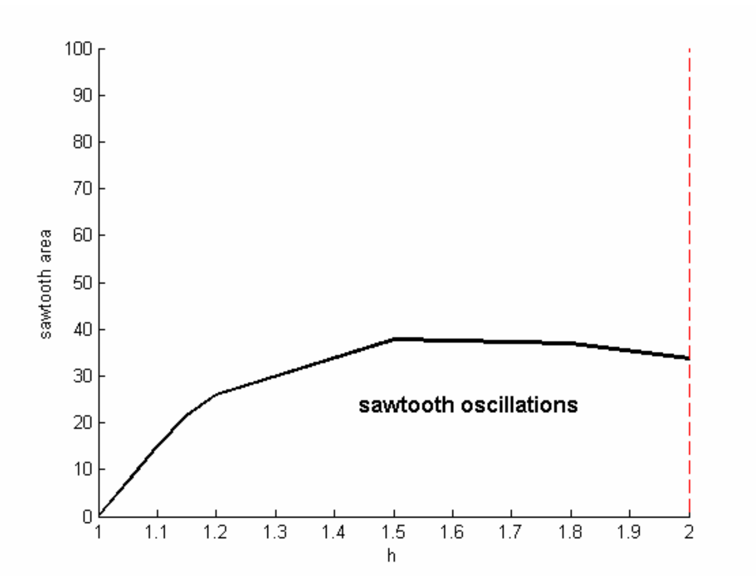

Fig. 6 – Area of sawtooth zones for $h \in [1, 2]$.

5. CONCLUSIONS

The system we studied in this paper is important both from a mathematical and a practical point of view. It was introduced in order to explain the quasi-periodic plasma dynamics observed in fusion experiments in Tokamaks. The system has some similar properties with the Lorenz system (dissipativity and symmetry with respect to the O*z* axis) and it is a fast-slow system.

In this paper we analyzed the stability of the equilibrium points and we studied the some bifurcations of the system (pitchfork and Hopf bifurcations). We theoretically explained the fast-slow dynamics of the system and we applied the results in specific cases which are important from a practical point of view. A saturation of the sawtooth zone (ELM zone) was observed when the input power is $h \approx 1.5$. This phenomenon is related to the position of Hopf bifurcation curve.

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