VECTOR SOLITON SOLUTIONS IN PT-SYMMETRIC COUPLED WAVEGUIDES AND THEIR RELEVANT PROPERTIES

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Abstract. Vector soliton solutions in PT-symmetric coupled waveguides and their relevant properties are considered by analytically and numerically solving the coupled nonlinear Schrödinger equations with linear-coupling and gain-loss effects. The results show that vector one- and two-soliton solutions, and even multisoliton solutions are stable in certain regions against both initial random perturbation for amplitude and longitudinal random fluctuation along propagation direction for gain-loss. The Newton’s cradle dynamics is also investigated. Based on vector soliton solutions on a finite background, nonlinear Talbot recurrence effects excited by linearly modulated continuous waves are discussed, and the results show that the recurrence patterns of the non-linear Talbot effects can be drawn by suitably choosing a frequency modulation factor. Finally, the evolution of the vector Peregrine solution is also studied by initially exciting a small localized perturbation on a continuous-wave background.

Key words: PT-symmetry; Darboux transformation; Newton’s cradle dynamics; Talbot effects.

1. INTRODUCTION

Theoretical and experimental studies on quantum-optical analogies have seen a spectacular resurgence, in which the propagation of optical waves in waveguides and lattices has become an important platform to investigate these phenomena [1]. For example, the extension of the canonical quantum mechanics to non-Hermitian Hamiltonians with parity-time (PT) symmetry has been applied in many optical settings, such as PT-symmetric couplers [2–4] and PT-symmetric optical lattices [5–7]. Experimental observations of PT-symmetry in optics have been demonstrated by using passive elements [8], by introducing gain or loss via photorefractive two-wave mixing [9], and in temporal photonic PT-symmetric lattices [10]. Thus, optics can provide a fertile ground to implement PT related beam dynamics including non-reciprocal light propagation, power oscillations, and optical transparency.

Furthermore, PT-related beam dynamics in nonlinear regimes has been investigated extensively. In Kerr media, optical solitons, including bright solitons, gap solitons, gray or dark solitons and vortices, supported by various complex PT-symmetric
or periodic potentials have been found during the past years [5, 11–22]. Of much interest is the fact that stable bright solitons can exist in defocusing Kerr media with PT-symmetric potentials [23]. In media with competing nonlinearities, solitons in PT-symmetric potentials have been also investigated analytically [24]. Lattice solitons in optical media described by the complex Ginzburg-Landau model with PT-symmetric periodic potentials and solitons in a chain of PT-invariant dimers have been reported [25–27]. Unidirectional optical transport induced by the balanced gain-loss profile [28, 29] and nonlinearly induced PT transition in photonic systems [30] have been also investigated.

Among these studies, the dynamics of the beam propagation in a coupler composed of a double channel waveguide with PT-symmetry gained particular attention because it can exhibit some universal properties in both linear and nonlinear regimes [2–4, 8, 9, 31–34]. Generally, the model describing the beam propagation in the coupler can be governed by a system of coupled nonlinear Schrödinger (NLS) equations [35]. This generic model and other related meaningful dynamical models have been applied in many fields such as nonlinear optics and photonics, multi-component Bose-Einstein condensates, etc., see, for example, Refs. [36–51]. Its special case is the Manakov system [52], which has been extensively studied in nonlinear optics and Bose-Einstein condensates [53–60].

However, in practical couplers, the linear-coupling and gain-loss effects must be considered. Based on the coupled NLS equations, stable soliton solutions, multisoliton Newton’s cradles and the rogue waves in PT-symmetric dual-core waveguide have been recently investigated [61–66]. In this paper, we aim to study vector multisoliton solutions and vector soliton solutions on top of a finite background in PT-symmetric coupled waveguides by employing Darboux transformation, and discuss their relevant properties, including their stabilities, the Newton’s cradle dynamics, and the Talbot recurrence effects.

The paper is organized as follows. In the next section, the model and its corresponding reduction are introduced and the Lax pair is presented. Based on this Lax pair and in terms of Darboux transformation, symmetric and antisymmetric vector multisoliton solutions are presented and their stability to various perturbations is also discussed in detail. Based on the unique features of stable soliton solutions, the Newton’s cradle dynamics is studied in Sec. 3. In Sec. 4 both symmetric and antisymmetric vector soliton solutions on finite background are presented and the nonlinear Talbot recurrence effects excited by linearly modulated continuous waves are drawn. Finally, the evolution of the vector Peregrine solution is also discussed by initially exciting a small localized perturbation on a continuous wave. The results are concluded in Sec. 5.
2. GOVERNING MODEL AND ITS REDUCTION

The propagation of optical beams in a nonlinear coupler composed of a double parallel waveguides can be described by two coupled NLS equations for field variables $\psi_1$ and $\psi_2$ in the dimensionless form [63]

\begin{align}
-i \frac{\partial \psi_1}{\partial z} &= -\frac{\partial^2 \psi_1}{\partial x^2} + \left( \chi_1 |\psi_1|^2 + \chi |\psi_2|^2 \right) \psi_1 + i \gamma_1 \psi_1 - \psi_2, \\
i \frac{\partial \psi_2}{\partial z} &= -\frac{\partial^2 \psi_2}{\partial x^2} + \left( \chi |\psi_1|^2 + \chi_1 |\psi_2|^2 \right) \psi_2 - i \gamma_2 \psi_2 - \psi_1, 
\end{align}

(1), (2)

where $z$ and $x$ are the dimensionless propagation and transverse coordinates in physical units. Here the linear-coupling constant is scaled to be 1, the nonlinear coefficients $\chi_1$ and $\chi$ correspond to the self-phase modulation and cross-phase modulation, and the coefficients $\gamma_1 > 0$ and $\gamma_2 > 0$ represent the gain and loss, respectively. Specifically, when $\gamma_1 = \gamma_2$, the Eqs. (1) and (2) can describe the beam propagation in a PT-symmetric coupled waveguide.

Equations (1) and (2) can appear in optics in several different contexts. In the absence of the linear-coupling effect, it can be used to describe the propagation of two coupled pulses with different frequencies and incoherently pulses in nonlinear fibers, and two orthogonally polarized pulses in birefringence fibers, in which $x$ stands for the time and $z$ for the propagation coordinate in the frame moving with the pulse group velocity [67, 68]. Also, such model can appear in Bose-Einstein condensates to describe the interaction of multi-component condensates [53–55].

In order to solve Eqs. (1) and (2), we assume that $\gamma_1 = \gamma_2 \equiv \gamma$ and restrict our consideration to $\gamma \leq 1$, which describes the PT-balance between gain in Eq. (1) and dissipation in Eq. (2). Following Refs. [62, 63], we look for symmetric and antisymmetric solutions of Eqs. (1) and (2) as

\begin{equation}
\psi_2(x,z) = (i \gamma \pm \sqrt{1 - \gamma^2}) \psi_1(x,z) = u^\pm(x,z),
\end{equation}

(3)

where the “+” sign is for the symmetric and the “−” is for the antisymmetric solution, respectively. Substituting Eq. (3) into Eqs. (1) and (2), one can find the function $u^\pm$ obeys the following single equation

\begin{equation}
i \frac{\partial u^\pm}{\partial z} = -\frac{\partial^2 u^\pm}{\partial x^2} + \left( \chi_1 + \chi \right) |u^\pm|^2 u^\pm \mp \sqrt{1 - \gamma^2} u^\pm.
\end{equation}

(4)

Note that the coefficient $\sqrt{1 - \gamma^2}$ in Eq. (4) includes both the PT-balanced gain/loss and linear-coupling effect, which is normalized to unity. It can be shown that Eq. (4) is an integrable system. Indeed, in terms of the Ablowitz-Kaup-Newell-Segur technique, the Lax pair for Eq. (4) can be constructed as

\begin{equation}
\Psi_x = U \Psi, \Psi_z = V \Psi,
\end{equation}

(5)
where \( \Psi = (\Psi_1, \Psi_2)^T \), \( U = \lambda J + mP \) and \( V = (i2\lambda^2 \pm i\sqrt{1-\gamma^2}/2)J + 2im\lambda P + im^2Q + imR \) with

\[
J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & u^\pm \\ -u^{\pm*} & 0 \end{pmatrix},
\]

\[
Q = \begin{pmatrix} |u^\pm|^2 & 0 \\ 0 & -|u^\pm|^2 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & u^\pm_x \\ u^{\pm*_x} & 0 \end{pmatrix},
\]

and \( m = \sqrt{-(\chi_1 + \chi)/2} \), provided that the condition \( \chi_1 + \chi < 0 \) is satisfied. Here \( \lambda \) is the complex spectral parameter, \( \ast \) denotes the complex conjugate, and the superscript \( T \) denotes the matrix transpose. Accordingly, Eq. (4) can be recovered from the compatibility condition \( U_z - V_x + [U, V] = 0 \).

3. VECTOR MULTISOLITON SOLUTIONS AND NEWTON’S CRADLES

In this section, we briefly introduce the procedure for getting the soliton solutions by employing Darboux transformation, and present the multisoliton solution for Eq. (4). Thus one can acquire the symmetric and antisymmetric vector soliton solutions of Eqs. (1) and (2) in terms of the relation (3). Based on the vector soliton solutions, we can discuss properties of the PT-symmetric and antisymmetric solutions for Eqs. (1) and (2).

Based on the Lax pair (5) and introducing the following transformation

\[
\tilde{\Psi} = (\lambda I - S)\Psi, \quad S = H\Lambda H^{-1}, \quad \Lambda = \text{diag}(\lambda_1, \lambda_2),
\]

where \( H \) is a nonsingular matrix that satisfies equation \( H_x = JHA + mPH \), and letting

\[
\tilde{\Psi}_x = U_1\tilde{\Psi}, \quad U_1 = \lambda J + mP_1, \quad P_1 = \begin{pmatrix} 0 & u^\pm_1 \\ -u^{\pm*_1} & 0 \end{pmatrix},
\]

one obtains the Darboux transformation in the form [69]

\[
P_1 = P + \frac{1}{m} [J, S].
\]

Also, It is easy to verify that, if \( \Psi = (\Psi_1, \Psi_2)^T \) is an eigenfunction of Eq. (5) with eigenvalue \( \lambda = \lambda_1 \), then \( (\Psi^*_2, \Psi^*_1)^T \) is also an eigenfunction, but its eigenvalue is \( -\lambda_1^* \). Thus we can take \( H \) and \( \Lambda \) in the form

\[
H = \begin{pmatrix} \Psi_1 & -\Psi_2^* \\ \Psi_2 & \Psi_1^* \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_1^* \end{pmatrix},
\]

to obtain

\[
S = -\lambda^* \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (\lambda_1 + \lambda_1^*) \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix},
\]
where the matrix element of $S$ are given by $s_{11} = |\Psi_1|^2/\Psi^T\Psi^*$, $s_{12} = \Psi_1 \Psi^*_2/\Psi^T\Psi^*$, $s_{21} = \Psi^*_1 \Psi_2/\Psi^T\Psi^*$, $s_{22} = |\Psi_2|^2/\Psi^T\Psi^*$ with $\Psi^T\Psi^* = |\Psi_1|^2 + |\Psi_2|^2$.

Finally, the Darboux transformation for Eq. (4) follows from relation (6)

$$u_1^\pm = u^\pm + \frac{2(\lambda_1 + \lambda_1^*)}{m} s_{12}. \tag{7}$$

Thus by solving the Eq. (5) which is a first-order linear differential equation, one can generate a new solution $u_1^\pm$ of the Eq. (4) from Eq. (7) once a seed solution $u^\pm$ is known. In particular, a one-soliton solution can be generated if the seed is a trivial zero state. Next, taking $u_1^\pm$ as the new seed solution, one can derive the corresponding two-soliton solution from Eq. (7). This can be continued as a recursion procedure generating multisoliton solutions.

Accordingly, one-soliton solution for Eq. (4) can be found in the form

$$u_1^\pm (x, z) = \sqrt{2A_s} \cdot \frac{\sech (A_s \theta)}{\sqrt{\lambda_1 + \lambda_1^*}} e^{i\Phi_1^\pm}, \tag{8}$$

where $\theta = x - 2\omega_s z - x_0$ and $\Phi^\pm = \omega_s x + (A_2^2 - \omega_s^2 \pm \sqrt{1 - \gamma^2}) z + \phi_0$ with $A_s$, $\omega_s$, $x_0$, and $\phi_0$ being arbitrary constants. Specifically, when $\omega_s = 0$, $\phi_0 = 0$, and $x_0 = 0$, the solution (8) is the same as that in Ref. [63].

Similarly, we can obtain the exact two-soliton solution of Eq. (4) as follows

$$u_2^\pm (x, z) = \frac{\sqrt{2}}{\sqrt{\lambda_1 + \lambda_1^*}} G^\pm F^\pm, \tag{9}$$

where $F^\pm = a_1 \cosh \theta_1 e^{i\Phi_1^\pm} + a_2 \cosh \theta_2 e^{i\Phi_2^\pm} + ia_3 (\sinh \theta_1 e^{i\Phi_1^\pm} - \sinh \theta_2 e^{i\Phi_2^\pm})$, $G^\pm = b_1 \cosh (\theta_1 + \theta_2) + b_2 \cosh (\theta_1 - \theta_2) + b_3 \cos (\Phi_2^\pm - \Phi_1^\pm)$, $\theta_1 = (x - 2\omega_1 z - x_0), \theta_2 = (x - 2\omega_2 z - x_0), \Phi_1^\pm = (A_1^2 - \omega_1^2 \pm \sqrt{1 - \gamma^2}) z + \phi_0$ and $a_1 = 2A_2 [A_2^2 - A_1^2 + (\omega_2 - \omega_1)^2], a_2 = 2A_1 [A_1^2 - A_2^2 + (\omega_2 - \omega_1)^2], a_3 = 4A_1 A_2 (\omega_2 - \omega_1), b_1 = (A_1 - A_2)^2 + (\omega_1 - \omega_2)^2, b_2 = (A_1 + A_2)^2 + (\omega_1 - \omega_2)^2, b_3 = -4A_1 A_2, \text{ with } A_1, A_2, \omega_1, \omega_2, x_0, \phi_0, \text{ and } \phi_0$ being arbitrary constants.

Thus, vector one-, two-soliton, and even multisoliton symmetric and antisymmetric solutions of Eqs. (1) and (2) are given by the relation (3). From such soliton solutions one can see that the only difference, compared with standard NLS soliton solutions, is a phase factor $\sqrt{1 - \gamma^2}$, which is caused by both the PT-balanced gain/loss and the linear coupling effect. Indeed, Eq. (4) can be reduced to the standard NLS equation, i.e. $i\Phi_2 + \Phi_{xx} + 2|\Phi|^2 \Phi = 0$, by the transformation $\Phi = \sqrt{|\lambda_1 + \lambda_1^*|/2u^\pm} \exp (+i/\sqrt{1 - \gamma^2}) z).$ Thus, one can also obtain the same results from the standard NLS equation and one- and two-soliton solutions, and even multisoliton solutions are directly obtained by using the Darboux transformation.

In the following, we will discuss stability of these vector soliton solutions.
Using the linear stability analysis [61–63], one can show that if $|A_s| < A_{cr}$, the vector one-soliton solution (8) is stable, where the critical value $A_{cr}$ is given by

$$A_{cr}^2 = \frac{8(\chi + \chi_1)\sqrt{1 - \gamma^2}}{22\chi_1 - 10\chi + 2(25\chi_1 - 7\chi)(\chi + \chi_1)}.$$  \hspace{1cm} (10)

From it, one can see that the expression (10) is valid only in region of $\chi_1 < 0$ and $\chi_1 < \chi < -\chi_1$.

In order to examine the above theoretical result, we performed evolution of the vector one-soliton solution by numerically simulating Eqs. (1) and (2) with the initial conditions $\psi_1(x, 0)$ and $\psi_2(x, 0)$ given by the relations (3) and (8), and the results are summarized in Fig. 1. It should be pointed out that in our simulations the perturbations include two details. One of them is a uniform random perturbation for the initial conditions, where 5% random noise is added in the amplitude given by one-soliton solution (8) [see Figs. 1(a) and 1(b)]. The other one is a longitudinal...
uniform random perturbation for gain and loss parameters $\gamma_1$ and $\gamma_2$ with the average value $\gamma$, which can be considered as the uniform random fluctuation of the gain and loss parameters along the propagation direction $z$ around the parameter $\gamma$ [see Figs. 1(c) and 1(d)]. This is because the exact PT-symmetric waveguide does not exist in practical applications. For our choice of the parameters, the critical value $A_{cr} = 1.3572$, thus when $A_s = 1 < A_{cr}$, the stable evolution is exhibited in Figs. 1(e) and 1(f). However, in the case of $A_s > A_{cr}$, the vector one-soliton solution may be unstable, as shown in Fig. 2, where $A_s = 1.4$, which is larger than the critical value $A_{cr} = 1.3572$. From Fig. 2 it can be seen that the vector one-soliton solution break completely up when it propagate to a certain distance.
Note that the critical value given by the relation (10) is valid only for the case of \( \chi_1 < \chi < -\chi_1 \) with \( \chi_1 < 0 \). When this condition is not satisfied, stable vector one-soliton solution may also exist, as shown in Fig. 3, which presents the evolution plots of the vector one-soliton solution against both the initial random perturbation for the amplitude and the longitudinal random fluctuation for the gain/loss, with the system parameters \( \chi_1 = 2, \chi = -3, \) and \( \gamma = 0.5 \).

![Fig. 4](image)

Fig. 4 – The various dynamical evolutions of vector two-soliton solution. The bound states in (a) \( \omega_1 = \omega_2 = 0 \) and (b) \( \omega_1 = \omega_2 = 0.1 \); the separation evolution in (c) \( \omega_1 = -0.1 \) and \( \omega_2 = 0.1 \), and the elastic collision in (d) \( \omega_1 = 0.1, \omega_2 = -0.1 \). Here, the soliton parameters are \( A_1 = 1.1, A_2 = 1.0, x_{01} = -5, x_{02} = 5, \) and \( \phi_{01} = \phi_{02} = 0 \), and the system parameters are the same as in Fig. 1. In all panels, only the \( \psi_2 \) component of the mutually symmetric field \( (\psi_1, \psi_2) \) is displayed.

Also, we performed several numerical simulations of the evolution of vector two-soliton solutions against the perturbations in the initial soliton amplitude and against the longitudinal random fluctuation for the gain/loss, as shown in Fig. 4. From it, one can see that when \( \omega_1 = \omega_2 \), they exhibit typical bound states of vector two-soliton solutions [see Figs. 4(a) and 4(b)]. When \( \omega_1 < 0 \) and \( \omega_2 > 0 \), they present the characteristic feature of a separation evolution [see Fig. 4(c)] (the separation distance between the two peaks increases with the propagation length \( z \)), and while when \( \omega_1 > 0 \) and \( \omega_2 < 0 \) exhibit a typical elastic collision [see Fig. 4(d)].
Fig. 5 – The supersoliton in the soliton chain composed of 60 symmetric solitons with initial separation $\Delta x = 9$. (a) An elastic overtaking collision between the supersolitons excited by $k_2 = 1$ and $k_{10} = 0.5$. (b) A head-on collision of the supersolitons excited by $k_2 = -k_{59} = 1$. Here, the system parameters are the same as in Fig. 1. In all panels, only the $\psi_2$ component of the mutually symmetric field ($\psi_1, \psi_2$) is displayed.

Thus, using stable individual symmetric solitons, one can construct a soliton chain with alternating signs of adjacent solitons to demonstrate the vector multisoliton Newton’s cradle dynamics in it [66, 70]. Figure 5 presents the propagation of supersolitons in the PT-symmetric coupler, where the soliton chain is of the form $(u, v), (-u, -v), (u, v), \ldots, (-1)^{n-1}(u, v)$, which guarantees repulsion between neighboring solitons and forms a bound state, and the supersoliton is initially excited by kicking one soliton, i.e., multiplying its two components by $\exp(ikx)$ with kick strength $k$. From Fig. 5 one can see that the supersolitons in the soliton chain exhibit an elastic overtaking collision [see Fig. 5(a)] and a head-on collision [see Fig. 5(b)].

4. VECTOR SOLITON SOLUTION ON FINITE BACKGROUND AND VECTOR TALBOT RECURRENCE EFFECTS

In this section, we will discuss vector soliton solutions on finite background for Eqs. (1) and (2). For this purpose, we take $u_{\pm B} = \left(\sqrt{2}/\sqrt{|\chi_1 + \chi|}\right)A e^{i\varphi_{\pm}}$ as an initial seed solution in Eq. (7), where $\varphi_{\pm} = \omega x + (2A^2 - \omega^2 \pm \sqrt{1 - \gamma^2}) z$. After solving the linear equation (5) and in terms of Darboux transformation (7) with the spectral parameter $\lambda = A_s/2 + i\omega_s/2$, we can obtain the soliton solution on finite background in the form

$$u_{\pm B}(x, z) = \frac{\sqrt{2}}{\sqrt{|\chi_1 + \chi|}} \left( A + A_s \frac{a \cosh \theta + \cos \varphi}{\cosh \theta + a \cos \varphi} + i A_s \frac{b \sinh \theta + c \sin \varphi}{\cosh \theta + a \cos \varphi} \right) e^{i\varphi_{\pm}}.$$

(11)
Here $\theta$ and $\varphi$ are given by

$$\begin{align*}
\theta &= -2MIx - [2MR A_s - 2MI(\omega_s + \omega)]z - x_0, \\
\varphi &= 2MRx - [2MI A_s + 2MR(\omega_s + \omega)]z - \varphi_0,
\end{align*}$$

with the coefficients being $a = -A(2MI + A_s)/\{A^2 + [(\omega - \omega_s)/2 + MR]^2 + (MI + A_s/2)^2\}$, $b = -A(\omega - \omega_s + 2MR)/\{A^2 + [(\omega - \omega_s)/2 + MR]^2 + (MI + A_s/2)^2\}$, and $c = \{A^2 - [(\omega - \omega_s)/2 + MR]^2 - (MI + A_s/2)^2\}/\{A^2 + [(\omega - \omega_s)/2 + MR]^2 + (MI + A_s/2)^2\}$. And other coefficient is $M = M_R + iM_I$ with $M_R = \{(\omega_s - \omega)^2 + 4A^2 - A_s^2\}^2 + 4A_s^2(\omega_s - \omega)^2 \pm [((\omega_s - \omega)^2 + 4A^2 - A_s^2)]^{1/2}/\sqrt{8}$ and $M_I = \{(\omega_s - \omega)^2 + 4A^2 - A_s^2\}^2 + 4A_s^2(\omega_s - \omega)^2 \pm [((\omega_s - \omega)^2 + 4A^2 - A_s^2)]^{1/2}/\sqrt{8}$, where $A, \omega, A_s$, and $\omega_s$ are the arbitrary real constants. Without loss of generality we assume here that $A$ and $A_s$ are non-negative constants. The solution (11) includes two special cases. One of them can be reduced to the one-soliton solution (8) as the amplitude $A$ vanishes. The other one can be reduced to the continuous-wave solution $(\sqrt{2}/\sqrt{\chi_1 + \chi})Ae^{i\omega z}$ when the soliton amplitude $A_s$ is zero. Therefore, in general, the exact solution $u_B^\pm(x, z)$ presents a soliton embedded in a continuous-wave background and can be used to describe the interaction between the soliton and the continuous wave in the nonlinear regime. Therefore the symmetric and antisymmetric vector soliton solutions on finite background for Eqs. (1) and (2) can be given by the relation (3).

Now, we consider a special case, i.e., $\omega = \omega_s = 0$, $A = 1$, and $\alpha = A_s^2/4$, where $\alpha$ is the quarter of the peak power for one-soliton solution. In this case, the solution (11) can be reduced to

$$u_B^\pm(x, z) = \frac{\sqrt{2}}{\sqrt{\chi_1 + \chi}} \left[ \Omega^2 \cosh(\beta Z) + i\beta \sinh(\beta Z) \right] \frac{2}{\cos(\beta Z) - 2\sqrt{\alpha} \cos(\Omega X)} - 1, \quad (12)$$

where $\Omega = 2\sqrt{1-\alpha}$ and $\beta = 4\sqrt{\alpha(1-\alpha)}$, and $Z = z - z_0$, $X = x - x_0$ with $z_0$ and $x_0$ being arbitrary real constants. It should be pointed out that the solution (12) includes a single governing parameter $\alpha$ and can describe the different physical behaviors for the different $\alpha$ [71].

For $0 < \alpha < 1$, the solution describes the Akhmediev breather, where $\Omega$ and $\beta$ are the real constants, which represent the modulation frequency and exponential growth, respectively. In this case, the solution (12) is periodic with period $2\pi/\Omega$ along the $x$ axis and is localized in longitudinal direction, which can be considered as a modulation instability process. Also, the maximally compressed pulse train is attained at the point of maximum compression $z = z_0$, which is of the form

$$u_{AB}^\pm(x, z_0) = \frac{\sqrt{2}}{\sqrt{\chi_1 + \chi}} \left[ \frac{2(1-\alpha)}{1 - \sqrt{\alpha} \cos(\Omega X)} - 1 \right]. \quad (13)$$
with the maximal amplitude $4\sqrt{2\alpha}/\sqrt{|\chi_1 + \chi|}$. Taking the initial state as the maximally compressed pulse train (13), nonlinear Talbot recurrence effects have been demonstrated by numerically simulating the standard NLS equation in Refs. [72, 73]. Indeed, in the linear regimes, the Talbot effects in PT-symmetric photonic lattices have been proposed in Ref. [74].

Fig. 6 – The vector Talbot recurrence effect excited by the linearly modulated continuous waves (14) and (15). (a) and (b) the intensity evolution plots for the $\psi_1$ and $\psi_2$ components, respectively; (c) the corresponding intensity profiles for the $\psi_2$ component at certain distances, and (d) the intensity evolution plot for the $\psi_2$ component at $x = 0$ (the solid curve) and at $x = -\pi/(F\Omega)$ (the dotted curve). Here, $x_0 = 0$, $\alpha = 0.5$, and $F = 1.7$, and the system parameters are the same as in Fig. 1.

It should be pointed out that this maximally compressed pulse train given by Eq. (13) is a nonlinearly modulated continuous wave, which can hardly be generated in experiments. Here, we consider a linearly modulated continuous wave as the initial state to demonstrate the vector Talbot recurrence in PT-symmetric coupled
waveguide. The initial states are of the form [71, 75, 76]

\[
\psi_1(x,0) = \frac{\sqrt{2}(-i\gamma \pm \sqrt{1-\gamma^2})}{\sqrt{|\chi_1 + \chi|}} [1 + A \cos (F\Omega X)],
\]

(14)

\[
\psi_2(x,0) = \frac{\sqrt{2}}{\sqrt{|\chi_1 + \chi|}} [1 + A \cos (F\Omega X)],
\]

(15)

where \(F\) is a frequency modulation factor and \(A = 2\sqrt{\alpha}\) is the modulation intensity, which implies that the component \(\psi_2(x,0)\) has the same maximal amplitude with that given by Eq. (13).

A lot of simulations show that for a given governing parameter \(0 < \alpha < 1\), the pattern of the vector Talbot recurrence effect can occur by suitably choosing the frequency modulation factor \(F\). As an example, Fig. 6 presents the recurrence patterns of the initial input states at the Talbot length \(z_T \approx 3.0680\) and at the half Talbot length \(z_T/2\), respectively, when \(\alpha = 0.5\) and \(F = 1.7\). To explain better this phenomenon, Fig. 6(c) depicts the corresponding intensity distributions for the \(\psi_2\) component at \(z = 0\), \(z = z_T\) (which is overlapping with the black solid curve at \(z = 0\)) and \(z = z_T/2\), where the exact state given by (13) is also displayed for comparison. Fig. 6(d) presents the intensity evolution plots for the \(\psi_2\) component at the center position \(x = 0\) and at the half period position \(x = -\pi/(\Omega F)\), respectively. From them one clearly see that the initial input states reemerge at the Talbot length [see the blue circles in Fig. 6(d)] and the half Talbot length with a half period shift [see the black triangles in Fig. 6(d)], respectively, as shown in Ref. [72].

For \(\alpha > 1\), the parameters \(\Omega\) and \(\beta\) in (12) are pure imaginary numbers such that the hyperbolic functions and trigonometric functions in Eq. (12) become the trigonometric functions and hyperbolic functions, respectively, which lead to the contrasting localization and periodicity characteristics and the physical explanation differed from the Akhmediev breather. In this case, the solution (12) represents a bright soliton propagating on top of a finite background and exhibits the characteristic of periodic oscillation along the propagation direction, and is generally called as the Kuznetsov-Ma soliton. Similarly, a maximally compressed pulse is attained at \(z = z_0\), which is of the form

\[
u_{KM}^\pm (x, z_0) = \frac{\sqrt{2}}{\sqrt{|\chi_1 + \chi|}} \left[\frac{2(\alpha - 1)}{\sqrt{\alpha \cosh(\Omega X)} - 1} - 1\right],
\]

(16)

where \(\Omega = 2\sqrt{\alpha - 1}\). Experimentally, the Kuznetsov-Ma soliton can be excited by the strongly modulated continuous wave by fitting its profile at the point of the minimal intensity [71]. Here, in order to perform Talbot recurrence effect in PT-symmetric coupled waveguide, we fit the maximally compressed pulse (16) by choos-
Fig. 7—The vector Talbot recurrence effect excited by the linearly modulated continuous waves (17) and (18). (a) and (b) the intensity evolution plots for the \( \psi_1 \) and \( \psi_2 \) components, respectively; (c) the corresponding intensity profiles for the \( \psi_2 \) component at certain distances, and (d) the intensity evolution plots for the \( \psi_2 \) component at 0 and at \( x = -\pi / (F \Omega) \). Here \( \alpha = 1.4 \) and \( F = F_c \approx 3.2102 \), and other parameters are the same as in Fig. 1.

The linearly modulated continuous waves in the form

\[
\psi_1(x, 0) = \frac{\sqrt{2}(\gamma \pm 1 - \gamma^2)}{\sqrt{1 + \chi}} \left[ \sqrt{\alpha} + A \cos \left( F \Omega X \right) \right],
\]

\[
\psi_2(x, 0) = \frac{\sqrt{2}}{\sqrt{1 + \chi}} \left[ \sqrt{\alpha} + A \cos \left( F \Omega X \right) \right],
\]

where \( A = \sqrt{\alpha} + 1 \) and when \( F = F_c \equiv \arccos(\sqrt{2} - 1) / \arccosh(\sqrt{2\alpha} - \sqrt{2} + 1) / \sqrt{\alpha} \); note that it has the same FWHM with that given by (16).

Figure 7 presents the evolution patterns of the vector Talbot recurrence effect excited by the linearly modulation continuous waves (17) and (18) with \( \alpha = 1.4 \). Similarly, we also depicted the corresponding intensity distributions for the \( \psi_2 \) component at \( z = 0 \), \( z = z_T \), and \( z = 2z_T \) with \( z_T \approx 1.292 \), respectively, as shown in Fig. 7(c). Figure 7(d) presents the intensity evolution plots for the \( \psi_2 \) component at the center position \( x = 0 \) and at the half period position \( x = -\pi / (F \Omega) \). From them one...
can see that the initial states reemerge only at $z = z_T$ [see the blue solid circles in Fig. 7(d)], but does not reemerge at the half Talbot length.

It should be emphasized that the initial states (14) and (15) with $0 < \alpha < 1$ differ from the initial states (17) and (18) with $\alpha > 1$. We note that that the linearly modulated continuous waves given by (17) and (18) have a stronger modulation than the initial states (14) and (15). Also, we point out that the initial states (17) and (18) fit very well the expression (16), including the maximal amplitude and the FWHM, but the initial states (14) and (15) fit only the maximal amplitude.

Finally, we consider the asymptotic case $\alpha \to 1$. In this case, the solution (12) can be reduced into the Peregrine solution in the form

$$u_{PS}^\pm = \frac{\sqrt{2}}{\sqrt{|\chi_1 + \chi_2|}} \left( \frac{4 + 16iZ}{1 + 4X^2 + 16Z^2} - 1 \right) e^{i(2 \pm \sqrt{1 - \gamma^2})Z},$$

which is the superposition of a continuous wave solution and a rational fraction function and forms a maximally compressed pulse at the point of maximum compress-
sion. This implies that the maximally compressed pulse can be excited by a small localized (single peak) perturbation pulse of the continuous-wave background. Thus, one can obtain the vector Peregrine solution by employing the relation (3). Here, as an example, we take the initial condition as a Gaussian-type perturbation on the continuous-wave background [77]

\[
\psi_1(x, 0) = \frac{\sqrt{2}(-i\gamma \pm \sqrt{1 - \gamma^2})}{\sqrt{1 + \chi}} \left(1 + \delta e^{-\sigma x^2}\right), \tag{20}
\]

\[
\psi_2(x, 0) = \frac{\sqrt{2}}{\sqrt{1 + \chi}} \left(1 + \delta e^{-\sigma x^2}\right), \tag{21}
\]

where \(\delta\) is a small modulation intensity and \(\sigma\) is a parameter related to the width of the perturbation pulse. Fig. 8 presents the evolution plots of a small localized perturbation on the continuous wave. From it one can see that the maximally compressed pulse is attained at \(z = 6.839\), and then the pulse exhibits a generic splitting feature during propagation.

5. CONCLUSION

In summary, the vector soliton solutions, including vector multisoliton solutions and the vector soliton solutions on finite background, for the coupled NLS equations with linear-coupling and gain-loss effects have been presented by employing the Darboux transformation. For the vector multisoliton solutions, the stability has been discussed, and the results have shown that vector soliton solutions are stable in certain regions against both the initial random perturbation for the amplitude and the longitudinal random fluctuation along the propagation direction for the gain-loss. Furthermore, the elastic overtaking collision and the head-on collision of the supersolitons have been demonstrated. Also, based on the vector soliton solutions on a finite background, including the vector Akhmediev breather and vector Kuznetsov-Ma soliton, the nonlinear Talbot recurrence effects excited by the linearly modulated continuous waves have been discussed, and the results have shown that the recurrence patterns of the nonlinear Talbot effects can be drawn by suitably choosing the frequency modulation factor. Finally, the evolution scenarios of the vector Peregrine solution have been also investigated by initially exciting a small localized perturbation on a continuous-wave background.

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