Romanian Reports in Physics, Vol. 68, No. 1, P. 341–352, 2016

OPTICAL SOLITONS IN BIREFRINGENT FIBERS BY LIE SYMMETRY ANALYSIS

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Received April 6, 2015

Abstract. The dynamics of optical solitons in birefringent fibers are addressed in this paper. The models are studied with both Kerr and parabolic laws of nonlinearity without four-wave mixing terms. The integration tool is the Lie symmetry approach. Similarity reductions to the derived ordinary differential equations are investigated by Lie classical method. These equations lead to exact solutions that are reported here.

Key words: Optical solitons; Birefringent fibers; Lie symmetry analysis.

1. INTRODUCTION

Optical solitons are one of the most fascinating areas of research in nonlinear fiber optics. There has been an overwhelming research activity that is being carried out in this area for the past few decades and this research is continuously burning bright. While most of the results were reported in the area of polarization-preserving fibers, it is equally necessary to gear attention to birefringent fibers, especially because there are fewer results known from this perspective. This paper therefore focuses on the dynamics of solitons in birefringent fibers. Both Kerr and parabolic laws of nonlinearity are taken into consideration.

When solitons propagate through optical fibers, there are several factors, such as fiber imperfections, random variations of fiber diameter and several additional issues, that lead to the split up of these pulses. This phenomenon is referred to as birefringence. One can also artificially introduce birefringence in an optical fiber. These split pulses lead to differential group delay. Thus there are *vector solitons* that are modeled by coupled nonlinear Schrödinger's equations (NLSEs). It is this vector

NLSE that will be introduced in the subsequent section.

The focus of this paper is the integrability issue of the nonlinear dynamical models. There are many integration tools that are available at present times in this regard. By implementing these powerful tools, several types of soliton-like solutions were retrieved and reported in the literature in the past decade [1]-[52]. This paper however applies one of the most powerful tools from applied mathematics. It is *Lie symmetry analysis* that is considered to be a classic and rich tool. This integration tool studies the models for birefringent fibers that appear with Kerr and parabolic laws of nonlinearity.

The Lie symmetry analysis is applied to study optical solitons in polarization preserving fibers on several occasions. These studies were reported in various papers [24, 25]. However, this powerful analysis is being applied to birefringent fibers, especially with parabolic law of nonlinearity, for the very first time. Needless to mention, the integrability aspects for birefringent fibers have been addressed in the past years using several different techniques [8, 21, 31, 34, 35].

2. GOVERNING EQUATIONS

In order to study the governing equations, there are two types of nonlinear media that will be considered. These are (1) Kerr law nonlinearity and (2) parabolic law nonlinearity. They are alternatively known as cubic nonlinearity and cubic-quintic nonlinear optical media, respectively. These are specifically discussed in the following subsections.

2.1. KERR LAW

For Kerr law nonlinear optical media, the intensity of light is dependent on the refractive index. In this case, the model for birefringence is given, *e.g.*, in Refs. [1, 4, 5, 8, 21]

$$iq_t + a_1 q_{xx} + \left(b_1 |q|^2 + c_1 |r|^2\right) q = 0 \tag{1}$$

$$ir_t + a_2 r_{xx} + \left(b_2 |r|^2 + c_2 |q|^2\right) r = 0 \tag{2}$$

In equations (1) and (2), a_j for $j = 1, 2$, represent the coefficients of group velocity dispersion while b_j represent the self-phase modulation (SPM) coefficients and finally c_j are the cross-phase modulation (XPM) terms. The first term, for both components, represents the linear temporal evolution term.

2.2. PARABOLIC LAW

This law arises, for example, in the nonlinear interaction between Langmuir waves and electrons and describes the nonlinear interaction between the high frequency Langmuir waves and the ion-acoustic waves by pondermotive forces. For such law, the governing model in birefringent fibers is [35]

$$iq_t + a_1 q_{xx} + \left(b_1|q|^2 + c_1|r|^2\right) q + \left(\alpha_1 |q|^4 + \beta_1 |q|^2 |r|^2 + \gamma_1 |r|^4\right) q = 0 \quad (3)$$

$$ir_t + a_2 r_{xx} + \left(b_2|r|^2 + c_2|q|^2\right) r + \left(\alpha_2 |r|^4 + \beta_2 |q|^2 |r|^2 + \gamma_2 |q|^4\right) r = 0 \quad (4)$$

Here the coefficients α_j, β_j, and γ_j arise from quintic nonlinear terms from the model with polarization-preserving fibers. The Lie symmetry analysis will be carried out in the following section.

3. LIE SYMMETRY ANALYSIS

The Lie method [11, 32, 33] of infinitesimal transformation groups has been widely used in equations of mathematical physics; some recent and important contributions were given in Refs. [26, 37]. The Lie classical method for finding symmetry reductions of partial differential equations (PDEs) is the *Lie group method* of infinitesimal transformations and the associated determining equations of an overdetermined linear system of PDEs.

3.1. KERR LAW

For complex-valued functions $q(x,t)$ and $r(x,t)$, the splitting into real and imaginary functions is carried out as

$$q(x,t) = u_1(x,t) + iu_2(x,t), \quad r(x,t) = v_1(x,t) + iv_2(x,t). \quad (5)$$

The equations (1) and (2) decompose into the following system of equations

$$-u_{2t} + a_1 u_{1xx} + (b_1(u_1^2 + u_2^2) + c_1(v_1^2 + v_2^2))u_1 = 0 \quad (6)$$

$$u_{1t} + a_1 u_{2xx} + (b_1(u_1^2 + u_2^2) + c_1(v_1^2 + v_2^2))u_2 = 0 \quad (7)$$

$$-v_{2t} + a_2 v_{1xx} + (c_2(u_1^2 + u_2^2) + b_2(v_1^2 + v_2^2))v_1 = 0 \quad (8)$$

$$v_{1t} + a_2 v_{2xx} + (c_2(u_1^2 + u_2^2) + b_2(v_1^2 + v_2^2))v_2 = 0 \quad (9)$$

Let us consider the Lie group of point transformations

$$\begin{aligned} t^* &= t + \epsilon\tau(x,t,u_1,u_2,v_1,v_2) + O(\epsilon^2) \\ x^* &= x + \epsilon\xi(x,t,u_1,u_2,v_1,v_2) + O(\epsilon^2) \\ u_1^* &= u_1 + \epsilon\eta_1(x,t,u_1,u_2,v_1,v_2) + O(\epsilon^2) \\ u_2^* &= u_2 + \epsilon\eta_2(x,t,u_1,u_2,v_1,v_2) + O(\epsilon^2) \\ v_1^* &= v_1 + \epsilon\phi_1(x,t,u_1,u_2,v_1,v_2) + O(\epsilon^2) \\ v_2^* &= v_2 + \epsilon\phi_2(x,t,u_1,u_2,v_1,v_2) + O(\epsilon^2), \end{aligned} \tag{10}$$

with small parameter $\epsilon \ll 1$.

The vector field associated with the above group of transformations can be written as

$$\begin{aligned} V =&\xi(x,t,u_1,u_2,v_1,v_2)\frac{\partial}{\partial x} + \tau(x,t,u_1,u_2,v_1,v_2)\frac{\partial}{\partial t} + \eta_1(x,t,u_1,u_2,v_1,v_2)\frac{\partial}{\partial u_1} \\ &+ \eta_2(x,t,u_1,u_2,v_1,v_2)\frac{\partial}{\partial u_2} + \phi_1(x,t,u_1,u_2,v_1,v_2)\frac{\partial}{\partial v_1} \\ &+ \phi_2(x,t,u_1,u_2,v_1,v_2)\frac{\partial}{\partial v_2}. \end{aligned} \tag{11}$$

The symmetries of equations (6)-(9) will be generated by the vector field of the form (11). Applying the second prolongation $\mathrm{pr}^{(2)}V$ of V to equations (6)-(9), we find that the coefficient functions $\xi, \tau, \eta_1, \eta_2, \phi_1$, and ϕ_2 must satisfy the invariance conditions

$$\begin{aligned} &-\eta_2^t + a_1\eta_1^{xx} + b_1(3u_1^2\eta_1 + u_2^2\eta_1 + 2u_1u_2\eta_2) \\ &+ c_1(v_1^2\eta_1 + 2u_1v_1\phi_1 + v_2^2\eta_1 + 2u_1v_2\phi_2) = 0, \\ &\eta_1^t + a_1\eta_2^{xx} + b_1(3u_2^2\eta_2 + u_1^2\eta_2 + 2u_1u_2\eta_1) \\ &+ c_1(v_1^2\eta_2 + 2u_2v_1\phi_1 + v_2^2\eta_2 + 2u_2v_2\phi_2) = 0, \\ &-\phi_2^t + a_2\phi_1^{xx} + b_2(3v_1^2\phi_1 + v_2^2\phi_1 + 2v_1v_2\phi_2) \\ &+ c_2(u_1^2\phi_1 + 2v_1u_1\eta_1 + u_2^2\phi_1 + 2v_1u_2\eta_2) = 0, \\ &\phi_1^t + a_2\phi_2^{xx} + b_2(3v_2^2\phi_2 + v_1^2\phi_2 + 2v_1v_2\phi_1) \\ &+ c_2(u_1^2\phi_2 + 2v_2u_1\eta_1 + u_2^2\phi_2 + 2v_2u_2\eta_2) = 0, \end{aligned} \tag{12}$$

where $\eta_1^t, \eta_2^t, \eta_1^{xx}, \eta_2^{xx}, \phi_1^t, \phi_2^t, \phi_1^{xx}$, and ϕ_2^{xx} are extended infinitesimals acting on a jet space that includes derivatives of the dependent variables.

We substitute the values of $\eta_1^t, \eta_2^t, \eta_1^{xx}, \eta_2^{xx}, \phi_1^t, \phi_2^t, \phi_1^{xx}$, and ϕ_2^{xx} in the system (12) and we replace the values of $u_{1t}, u_{2t}, v_{1t}, v_{2t}$ from equations (6)-(9). Equating to zero the coefficients of derivative terms and solving the obtained system of partial differential equations, we have

$$\begin{aligned}
\xi &= c_2 + \frac{x}{2}c_5 + tc_6 \\
\tau &= c_1 + tc_5 \\
\eta_1 &= -u_2c_3 - \frac{u_1}{2}c_5 - \frac{xu_2}{2a_1}c_6 \\
\eta_2 &= u_1c_3 - \frac{u_2}{2}c_5 + \frac{u_1x}{2a_1}c_6 \\
\phi_1 &= v_2c_4 - \frac{v_1}{2}c_5 - \frac{v_2x}{2a_2}c_6 \\
\phi_2 &= -v_1c_4 - \frac{v_2}{2}c_5 + \frac{v_1x}{2a_2}c_6,
\end{aligned} \tag{13}$$

where c_1, c_2, c_3, c_4, c_5, and c_6 are arbitrary constants.

The infinitesimal generators of the corresponding Lie algebra are given by

$$\begin{aligned}
V_1 &= \frac{\partial}{\partial t}, \\
V_2 &= \frac{\partial}{\partial x}, \\
V_3 &= -u_2\frac{\partial}{\partial u_1} + u_1\frac{\partial}{\partial u_2}, \\
V_4 &= v_2\frac{\partial}{\partial v_1} - v_1\frac{\partial}{\partial v_2}, \\
V_5 &= \frac{x}{2}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{u_1}{2}\frac{\partial}{\partial u_1} - \frac{u_2}{2}\frac{\partial}{\partial u_2} - \frac{v_1}{2}\frac{\partial}{\partial v_1} - \frac{v_2}{2}\frac{\partial}{\partial v_2}, \\
V_6 &= t\frac{\partial}{\partial x} - \frac{xu_2}{2a_1}\frac{\partial}{\partial u_1} + \frac{xu_1}{2a_1}\frac{\partial}{\partial u_2} - \frac{xv_2}{2a_2}\frac{\partial}{\partial v_1} + \frac{xv_1}{2a_2}\frac{\partial}{\partial v_2}.
\end{aligned} \tag{14}$$

The vector fields given by (14), form a Lie algebra as

$$\begin{aligned}
[V_1,V_2] = [V_1,V_3] &= [V_1,V_4] = 0, \\
[V_2,V_3] &= [V_2,V_4] = 0, \\
[V_3,V_4] = [V_3,V_5] &= [V_3,V_6] = 0, \\
[V_4,V_5] &= [V_4,V_6] = 0, \\
[V_1,V_5] = V_1,\ [V_1,V_6] &= V_2, \\
[V_2,V_5] = \frac{V_2}{2},\ [V_2,V_6] &= \left(\frac{V_3}{2a_1} - \frac{V_4}{2a_2}\right), \\
[V_5,V_6] &= \frac{V_6}{2}.
\end{aligned} \tag{15}$$

To obtain the symmetry reductions of equations (6)-(9), we have to solve the characteristic equation

$$\frac{dx}{\xi}=\frac{dt}{\tau}=\frac{du_1}{\eta_1}=\frac{du_2}{\eta_2}=\frac{dv_1}{\phi_1}=\frac{dv_2}{\phi_2}, \tag{16}$$

where $\xi,\tau,\eta_1,\eta_2,\phi_1$, and ϕ_2 are given by (13).

To solve (16), we consider the following cases:

(i) V_6 (ii) V_5 (iii) $V_2+k_1V_3$

Case (i) V_6

Solving the characteristic equation (16), we have the following similarity variables

$$\begin{aligned} &\xi=t,\\ &q(x,t)=u_1(x,t)+iu_2(x,t)=F_1(\xi)\exp\left(i\left(\frac{x^2}{4a_1t}+F_2(\xi)\right)\right) \end{aligned} \tag{17}$$

$$r(x,t)=v_1(x,t)+iv_2(x,t)=G_1(\xi)\exp\left(i\left(\frac{x^2}{4a_2t}+G_2(\xi)\right)\right), \tag{18}$$

where ξ is a new independent variable and F_1,F_2,G_1,G_2 are new dependent variables.

Substituting equations (18) along with equations (17) into equations (1) and (2), we immediately obtain the reduced equations, which after separating real and imaginary parts read

$$\begin{aligned} &2\xi F_1'+F_1=0\\ &-2\xi F_1F_2'+2b_1\xi F_1^3+2c_1\xi F_1G_1^2=0\\ &2\xi G_1'+G_1=0\\ &-2\xi G_1G_2'+2b_2\xi G_1^3+2c_2\xi G_1F_1^2=0. \end{aligned} \tag{19}$$

We obtain the following solution of ordinary differential equations (ODEs) (19)

$$\begin{aligned} &F_1=\frac{C_1}{\sqrt{\xi}}\\ &F_2=\left(b_1C_1^2+c_1C_2^2\right)\log(\xi)+C_3\\ &G_1=\frac{C_2}{\sqrt{\xi}}\\ &F_2=\left(b_2C_2^2+c_2C_1^2\right)\log(\xi)+C_4, \end{aligned} \tag{20}$$

where C_1,C_2,C_3, and C_4 are arbitrary constants.

The corresponding solution of the main system of equations (1) and (2) is given by

$$
\begin{aligned}
q(x,t) &= \frac{C_1}{t}\exp\left(i\left(\frac{x^2}{4a_1t}+\left(b_1C_1^2+c_1C_2^2\right)\log(t)+C_3\right)\right)\\
r(x,t) &= \frac{C_2}{t}\exp\left(i\left(\frac{x^2}{4a_2t}+\left(b_2C_2^2+c_2C_1^2\right)\log(t)+C_4\right)\right).
\end{aligned}
\tag{21}
$$

Case (ii) V_5

The corresponding similarity variables are

$$
\begin{aligned}
\xi &= \frac{x^2}{t}\\
u_1 &= \frac{1}{x}F_1(\xi)\\
u_2 &= \frac{1}{x}F_2(\xi)\\
v_1 &= \frac{1}{x}G_1(\xi)\\
v_2 &= \frac{1}{x}G_2(\xi),
\end{aligned}
\tag{22}
$$

where ξ is a new independent variable and $F_1(\xi), F_2(\xi), G_1(\xi), G_2(\xi)$ are new dependent variables. Substituting the similarity variables (22) in the system of equations (6)-(9), we have

$$
\begin{aligned}
&\xi^2F_2'+2a_1(F_1-\xi F_1'+2\xi^2F_1'')+\left(b_1(F_1^2+F_2^2)+c_1(G_1^2+G_2^2)\right)F_1=0\\
&-\xi^2F_1'+2a_1(F_2-\xi F_2'+2\xi^2F_2'')+\left(b_1(F_1^2+F_2^2)+c_1(G_1^2+G_2^2)\right)F_2=0\\
&\xi^2G_2'+2a_2(G_1-\xi G_1'+2\xi^2G_1'')+\left(c_2(F_1^2+F_2^2)+b_2(G_1^2+G_2^2)\right)G_1=0\\
&-\xi^2G_1'+2a_2(G_2-\xi G_2'+2\xi^2G_2'')+\left(c_2(F_1^2+F_2^2)+b_2(G_1^2+G_2^2)\right)G_2=0.
\end{aligned}
\tag{23}
$$

Using (22), the solution of the main system of equations (1)-(2) is given by

$$
\begin{aligned}
q(x,t) &= 1/x\left(F_1(\frac{x^2}{t})+iF_2(\frac{x^2}{t})\right)\\
r(x,t) &= 1/x\left(G_1(\frac{x^2}{t})+iG_2(\frac{x^2}{t})\right),
\end{aligned}
\tag{24}
$$

where F_1, F_2, G_1, and G_2 are given by (23).

Case (iii) $V_2+k_1V_3$

The similarity variables are given by

$$
\begin{aligned}
\xi &= t\\
q(x,t) &= u_1(x,t)+iu_2(x,t)=F_1(\xi)\exp(i(xk_1+F_2(\xi)))\\
r(x,t) &= v_1(x,t)+iv_2(x,t)=G_1(\xi)\exp(i(xk_2+G_2(\xi))),
\end{aligned}
\tag{25}
$$

where ξ is a new independent variable and F_1, F_2, G_1, G_2 are the new dependent variables.

Substituting (25) in the main system of equations (1)-(2), we obtain the reduced system of ODEs as follows

$$\begin{aligned} &iF_1{}' - F_1 F_2' - (a_1 k_1^2 - b_1 F_1^2 - c_1 G_1^2) F_1 = 0 \\ &iG_1{}' - G_1 G_2' - (a_2 k_2^2 - b_2 G_1^2 - c_2 F_1^2) G_1 = 0, \end{aligned} \tag{26}$$

where $'$ denotes the derivative with respect to ξ. Solving the system (26), we obtain

$$\begin{aligned} F_1(\xi) &= C_1 \\ G_1(\xi) &= C_2 \\ F_2(\xi) &= (-a_1 k_1^2 + b_1 C_1^2 + c_1 C_2^2) t + C_3 \\ G_2(\xi) &= (-a_2 k_1^2 + b_2 C_2^2 + c_2 C_1^2) t + C_4, \end{aligned} \tag{27}$$

where C_1, C_2, C_3, and C_4 are arbitrary constants.

The corresponding solution of the main system of equations (1)-(2) is given by

$$\begin{aligned} q(x,t) &= C_1 \exp\left(i(k_1 x + (-a_1 k_1^2 + b_1 C_1^2 + c_1 C_2^2) t + C_3)\right) \\ r(x,t) &= C_2 \exp\left(i(k_2 x + (-a_2 k_1^2 + b_2 C_2^2 + c_2 C_1^2) t + C_4)\right). \end{aligned}$$

Case (iv) $V_1 + \mu V_3$

Solving the characteristic equation, the similarity variables are

$$\begin{aligned} \xi &= x \\ q(x,t) &= u_1(x,t) + i u_2(x,t) = F_1(\xi) \exp\left(i(\mu t + F_2(\xi))\right) \\ r(x,t) &= v_1(x,t) + i v_2(x,t) = G_1(\xi) \exp\left(i(\mu t + G_2(\xi))\right), \end{aligned} \tag{28}$$

where ξ is a new independent variable and F_1, F_2, G_1, G_2 are the new dependent variables. Using (28) in system (1)-(2), we obtain the following system of ODEs

$$2F_1' F_2' + F_1 F_2'' = 0 \tag{29}$$

$$a_1 F_1'' - \mu F_1 - a_1 F_1 {F_2'}^2 + \left(b_1 F_1^2 + c_1 G_1^2\right) F_1 = 0 \tag{30}$$

$$2G_1' G_2' + G_1 G_2'' = 0 \tag{31}$$

$$a_2 G_1'' - \mu G_1 - a_2 G_1 {G_2'}^2 + \left(b_2 G_1^2 + c_2 F_1^2\right) G_1 = 0, \tag{32}$$

where $'$ denotes the derivative with respect to ξ. Solving the equations (29) and (31), we have

$$\begin{aligned} F_2' &= \frac{C_1}{F_1^2} \\ G_2' &= \frac{C_2}{G_1^2}, \end{aligned} \tag{33}$$

where C_1, C_2 are arbitrary constants. Using (33) in equations (30)-(32), we have

$$\begin{aligned} a_1 F_1'' - \mu F_1 - \frac{a_1 C_1^2}{F_1^3} + \left(b_1 F_1^2 + c_1 G_1^2\right) F_1 = 0 \\ a_2 G_1'' - \mu G_1 - \frac{a_2 C_2^2}{G_1^3} + \left(b_2 G_1^2 + c_2 F_1^2\right) G_1 = 0. \end{aligned} \tag{34}$$

The corresponding solution of the main system (1)-(2) is given by

$$\begin{aligned} q(x,t) = F_1(\xi) \exp\left(i\left(\mu t + \int (\frac{C_1}{F_1^2(\xi)} d\xi + C_3)\right)\right) \\ r(x,t) = G_1(\xi) \exp\left(i\left(\mu t + \int (\frac{C_2}{G_1^2(\xi)} d\xi + C_4)\right)\right), \end{aligned} \tag{35}$$

where ξ is given by (28) and F_1, G_1 are given by (34).

3.2. PARABOLIC LAW

In this case the following symmetries of equations (3)-(4) are obtained

$$\begin{aligned} &\xi = C_1,\ \tau = C_2 \\ &\eta_1 = -u_2 C_3,\ \eta_2 = u_1 C_3 \\ &\phi_1 = \phi_2 = 0, \end{aligned} \tag{36}$$

where C_1, C_2, and C_3 are arbitrary constants.

The corresponding vector fields are

$$\begin{aligned} V_1 &= \frac{\partial}{\partial x} \\ V_2 &= \frac{\partial}{\partial t} \\ V_3 &= -u_2 \frac{\partial}{\partial u_1} + u_1 \frac{\partial}{\partial u_2}. \end{aligned} \tag{37}$$

The similarity variables for equations (3)-(4), corresponding to the vector field $\mu_2 V_1 + \mu_1 V_2 + \mu_3 V_3$ are given as

$$\begin{aligned} \xi &= \mu_1 x - \mu_2 t \\ q &= F_1(\xi) \exp\left(i(\mu_3 x + F_2(\xi))\right) \\ r &= G_1(\xi) \exp\left(i G_2(\xi)\right), \end{aligned} \tag{38}$$

where ξ is the new independent variable and F_1, F_2, G_1, G_2 are the new dependent variables. Using the similarity variables (38) in system of equations (3)-(4) and separating the real and imaginary parts, we obtain

$$\begin{aligned}
a_1\mu_1^2 F_1 F_2'' + 2a_1\mu_1^2 F_1' F_2' + 2a_1\mu_1\mu_3 F_1' - \mu_2 F_1' &= 0 \\
\mu_2 F_1 F_2' + a_1\mu_1^2 F_1'' - a_1\mu_3^2 F_1 - 2a_1\mu_1\mu_3 F_1 F_2' - a_1\mu_1^2 F_1 F_2''^2 + b_1 F_1^3 + c_1 F_1 G_1^2 & \\
+\alpha_1 F_1^5 + \beta_1 F_1^3 G_1^2 + \lambda F_1 G_1^4 &= 0 \\
-\mu_2 G_1' + 2a_2\mu_1^2 G_1' G_2' + a_2\mu_1^2 G_1 G_2'' &= 0 \\
\mu_2 G_1 G_2' + a_2\mu_1^2 G_1'' - a_2\mu_1^2 G_1 {G_2'}^2 + b_2 G_1^3 + c_2 G_1 F_1^2 + \alpha_2 G_1^5 & \\
+\beta_2 G_1^3 F_1^2 + \lambda G_1 F_1^4 &= 0,
\end{aligned} \tag{39}$$

where $'$ denotes the derivative with respect to ξ.

We obtain the following solution of the ODE system (39)

$$\begin{aligned}
F_1 &= l_1 \\
F_2 &= l_2 + l_3\xi \\
G_1 &= m_1 \\
G_2 &= m_2\xi + m_3
\end{aligned} \tag{40}$$

where l_1, l_2, l_3, m_1, m_2, and m_3 are arbitrary constants that satisfy the following algebraic equations

$$\begin{aligned}
\mu_2 l_1 l_2 - a_1 l_1 \mu_3^2 - 2a_1\mu_1\mu_3 l_1 l_2 + b_1 l_1^3 + c_1 l_1 m_1^2 + \alpha_1 l_1^5 + \beta_1 l_1^3 m_1^2 + \lambda_1 l_1 m_1^4 &= 0 \\
\mu_2 m_1 m_2 - a_2\mu_1^2 m_1 m_2^2 + b_2 m_1^3 + c_2 m_1 l_1^2 + \alpha_2 m_1^5 + \beta_2 m_1^3 l_1^2 + \lambda m_1 l_1^4 &= 0.
\end{aligned} \tag{41}$$

The corresponding solution of the main system of equations (3)-(4) is given by

$$\begin{aligned}
q(x,t) &= l_1 \exp(i(\mu_3 x + l_2 + l_3(\mu_1 x - \mu_2 t))) \\
r(x,t) &= m_1 \exp(i(m_2 + m_3(\mu_1 x - \mu_2 t))),
\end{aligned} \tag{42}$$

where l_1, l_2, l_3, m_1, m_2, and m_3 satisfy the algebraic system of equations (41).

4. CONCLUSIONS

In this paper, we have considered the dynamics of birefringent fibers by using the Lie symmetry analysis. Two types of nonlinear media that have been considered are those with Kerr law nonlinearity and parabolic law nonlinearity. The system of PDEs under consideration was analyzed by the Lie symmetry method to reduce it to ODEs. Corresponding to each reduction, certain exact solutions of the nonlinear PDEs are obtained. These solutions are very useful in the soliton community especially in the field of nonlinear optics and plasma physics where different modifications of the standard NLSEs frequently arise. Later, these results will be extended. The Lie symmetry analysis will be applied to birefringent fibers for Kerr law and parabolic law with four-wave-mixing (FWM) terms taken into consideration. This will lead to far more involved analysis than that in the present paper. The analysis and results of this paper stand on a strong footing to study dense wavelength division multiplexing (DWDM) systems, with and without FWM terms. The results of those studies are also on the horizon and will be reported soon. Finally, different perturbation terms will also be included to paint a complete picture to the area of optical solitons with birefringence as well as in DWDM systems. Later, this study will also be applied to optical couplers. All of these results are currently awaited.

Acknowledgements. This research is funded by Qatar National Research Fund (QNRF) under the grant number NPRP 6-021-1-005. The authors (AB & MB) thankfully acknowledge this support from QNRF.

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