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FIRST-ORDER TRANSITIONS INDUCED BY PERIODIC MAGNETIC INDUCTION IN MAGNETAR'S CRUST

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Abstract. In the present paper, we consider a non-rotating, strongly magnetized object, whose magnetic field is made of a strong magnetostatic component along Oz plus a magnetic induction of the form $B_x = B_0(t)\sin\kappa z$. We start with the Dirac equation for an electron in an uniform magnetic field alone, in order to obtain the orthonormal modes, for positively and negatively charged relativistic fermions. By assuming that the periodic component $B_x = B_0(t)\sin(\kappa z)$, which is supposed to exist in the magnetar's crust, can be treated as a perturbation, we compute the amplitudes and the rates per unit surface characterizing the up-up and down-up transitions.

Key words: Dirac equation, neutron stars, magnetars.

1. INTRODUCTION

Among the 1800 spindown powered radio pulsars known, the so-called *magnetars* [1] are by far the most excited objects, being characterized by a very strong X-ray emission and magnetic fields larger than the critical induction at which the cyclotron energy of an electron equals its rest mass energy.

In the last 20 years since Duncan and Thompson published their seminal work [2], the intense activity devoted to these slowly rotating neutron stars has led to many open questions regarding the configuration of the magnetic field inside and their internal structure.

Since the magnetars are bright thermal X-ray sources, the formation of free and bound pairs, by X and $gamma$ rays collisions,

$$\gamma + X \to e^- + e^+$$

has been a main target of investigations [3].

When one of the photons is already close to the threshold for single-photon pair creation, it is possible to produce an electron-positron pair even though the other one doesn't have a large energy.

In the case of the so-called *resonant scattering* mechanism, the target photon has an energy that can promote the electron, from $\omega_0 = m_0 c^2$ to the first Landau level

$$\omega_1 = m_0 c^2 \sqrt{1 + 2\frac{B}{B_Q}}\,, \text{ with } \hbar \frac{q B_Q}{m_0} = m_0 c^2 \;\Rightarrow\; B_Q = \frac{m_0^2 c^2}{q\hbar}\,.$$

For an electron of mass m_0, which does not move along Oz, the critical induction at which the cyclotron energy of an electron reaches the electron rest mass energy is $B_Q = 4.4 \cdot 10^{13}\ (G)$.

The excitation is followed immediately by de-excitation and the scattered photon will directly convert to an electron-positron pair if

$$h\nu_\gamma = \omega_1 - m_0 c^2 > 2 m_0 c^2\,,$$

meaning $B > 4B_Q$, which is a typical magnetic field induction inside magnetars.

Besides the first or second order processes of electron-positron pair production by photons, in a magnetic field, to which considerable attention has been given, the Compton scattering of the photon by an electron in strong magnetic fields has been also well studied [4].

In very strong magnetic inductions, when the incident photon energy is close to the Landau level of the intermediate particle, the probability of this process is significantly increased [5].

In what it concerns their internal structure, it is fully agreed by now that the magnetars are far from being composed of only neutrons since the extreme compression in the inner core may produce different types of subatomic particles. Once these fields are added to the normal matter [6], they are expected to lead to significant contributions in the measured stars' parameters [7].

In our previous works [8], we have considered both bosons and fermions evolving in a special electromagnetic configuration which is supposed to characterize a magnetar, pointing out transitions from oscillatory to exponentially growing modes along Oz. In the present paper, we start with the Dirac-type equation for positively and negatively charged particles evolving in a strong magnetic static field. Once one has the positive and negative energy solutions, one can study all kind of processes induced by magnetic perturbations.

2. THE PERTURBED ELECTROMAGNETIC FIELD

The theoretical description of the magnetic field comes from the basic equation derived by Goldreich and Reisenegger [9], describing the evolution of both isolated and accreting neutron stars, due to the Hall effect and Ohmic diffusion

$$\frac{\partial \vec{B}}{\partial t} = -\frac{c^2}{4\pi\sigma} \nabla \times \left[\nabla \times \vec{B}\right] + \nabla \times \left[-\frac{\vec{j}}{ne} \times \vec{B}\right]. \tag{1}$$

For some characteristic depth of the crust, L, the Ohmic decay time is given by $\tau_{Ohm} = 4\pi\sigma L^2/c^2$, while the Hall time is $\tau_{Hall} = 4\pi neL^2/(cB)$, where n is the electron number density. For a strong magnetic field, the Hall timescale is several orders of magnitude faster than the Ohmic one. As Goldreich and Reisenegger suggested, in 1992, by changing the field structure on Hall timescale, it may give rise to a turbulent cascade, enhancing the efficiency of Ohmic total energy decay, the energy being transferred from large to small scales.

Following Goldreich and Reisenegger's seminal work, numerous authors have studied this phenomenon via theoretical and numerical methods, in both spherical-shells [10] and Cartesian box geometries [11].

Few years later, Rheinhardt and Geppert have proved within a linear stability analysis that, for specific background magnetic fields, the transfer of magnetic energy from a background (large-scale) field to small-scale modes may produce Hall instabilities, at some large wavenumber values [12]. They have considered a large-scale background field B_0, solution of the equation

$$\nabla \times \left[\left(\nabla \times \vec{B}_0 \right) \times \vec{B}_0 \right] = 0 \,, \tag{2}$$

which is almost constant in time, once we ignore the very gradual Ohmic decay. They have added a small perturbation, b, satisfying the linearized form of the equation whose solutions are either decaying or exponentially growing, when there is a transfer of energy from the background field to the perturbation. For a plane layer geometry, periodic in x and y and bounded in z, with either vacuum or perfectly conducting boundaries at the top and bottom, a magnetic induction of the form $\vec{B}_0 = h(z)\,\vec{e}_x$ is obviously satisfying the equation (2), for any form of the function $h(z)$.

As previously [8], we consider the magnetic field made up of a strong magnetostatic component $B_* \parallel Oz$, and a periodic *large scale* magnetic perturbation along Ox, depending on the $z-$coordinate as $B_x = b_0 \sin(\kappa z)$, where $\kappa \sim \pi/L$, inspired by the investigations performed by Wareing and Hollerbach [13].

One may easily check that the following orthogonal fields components,

$$\begin{aligned} E_y &= \frac{\kappa}{\sigma}\, b_0 \exp\left[-\frac{\kappa^2}{\sigma}\, t \right] \cos\left(\kappa z\right), \\ B_x &= b_0 \exp\left[-\frac{\kappa^2}{\sigma}\, t \right] \sin\left(\kappa z\right), \\ B_z &= B_* \,, \end{aligned} \tag{3}$$

are satisfying the basic equations

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \,,\;\; \nabla \cdot \vec{E} = 0 \,,\;\; \nabla \cdot \vec{B} = 0 \,,\;\; \nabla \times \vec{B} = \vec{j} + \frac{\partial \vec{E}}{\partial t}$$

and are sustained by two components of the four-potential obeying the Lorentz condition $\nabla \cdot \vec{A} = 0$, namely

$$A_x = -B_* y \, , \;\; A_y = \frac{b_0}{\kappa} e^{-\alpha t} \cos(\kappa z) \, , \tag{4}$$

where $\alpha = \frac{\kappa^2}{\sigma} \sim 1/\tau_{Ohm}$.

For a magnetic field frozen in the crust of extension $L \sim 1 \; (Km)$ and averaged electrical conductivity $\sigma \sim 10^{26} \; (s^{-1})$, the time variable being much less than the Ohmic decay time

$$t \ll \tau_{Ohm} = \frac{4\pi \sigma L^2}{c^2} = 4.4 \cdot 10^8 \; (yrs) \, ,$$

one may take $e^{-\alpha t} \approx 1$, so that A_y is given by the expression

$$A_y = \frac{b_0}{\kappa} \cos(\kappa z) \, . \tag{5}$$

3. CHARGED DIRAC FERMIONS IN STRONG STATIC MAGNETIC FIELDS

The positively charged relativistic fermions in a strong static magnetic field oriented along Oz are described by the usual Dirac equation

$$\left[\gamma^i D_i + M \right] \Psi_P = 0 \, , \tag{6}$$

where $D_i = \partial_i - iqA_i$ is the covariant derivative, with $q > 0$ and $A_x = -B_* y$, $B_* = const$. Unless we specify, we are going to employ natural units, *i.e.* $\hbar = c = 1$, and the following representation for the Dirac matrices,

$$\gamma^\mu = -i\beta \alpha^\mu \, , \;\; \gamma^4 = -i\beta \, , \tag{7}$$

with

$$\beta = \begin{pmatrix} \mathcal{I} & 0 \\ 0 & -\mathcal{I} \end{pmatrix} \, , \;\; \alpha^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix} . \tag{8}$$

We are looking for solutions of the Dirac equation (6), describing the positive energy states, of the form

$$\Psi^\sigma_{P,n}(\vec{x}, t) = e^{i(p_x x + p_z z - \omega t)} \phi^\sigma_P(y) \, , \tag{9}$$

where p_z and p_x are respectively the longitudinal and transverse momentum and n is for the energy level. The wave function is localized about the transverse coordinate y and is labeled by the *up* and *down* spin index, $\sigma = \pm 1$.

With

$$\phi_P(y) = \begin{bmatrix} \zeta_P(y) \\ \eta_P(y) \end{bmatrix} \, , \tag{10}$$

where ζ_P and η_P are two-component spinors, the Dirac equation leads to the following system of coupled differential equations

$$\begin{aligned} &(\mathrm{a}) \quad \sigma^2\zeta_P' + i\left[(p_x+qB_*y)\,\sigma^1 + p_z\sigma^3\right]\zeta_P = i\,[\omega+M]\,\eta_P\,; \\ &(\mathrm{b}) \quad \sigma^1\eta_P' + i\left[(p_x+qB_*y)\,\sigma^1 + p_z\sigma^3\right]\eta_P = i\,[\omega-M]\,\zeta_P\,, \end{aligned} \tag{11}$$

where $'$ stands for the derivative with respect to y. From the first equation in (11), we express $\eta_P(y)$ as

$$\eta_P = \frac{1}{\omega+M}\left\{-i\sigma^2\zeta_P' + \left[(p_x+qB_*y)\,\sigma^1 + p_z\sigma^3\right]\zeta_P\right\}, \tag{12}$$

and, by replacing it and its derivative in (11.b), we come to the following second-order differential equation for the other spinor,

$$\zeta_P'' + \left[\omega^2 - p_z^2 - M^2 + iqB_*\sigma^2\sigma^1 - (p_x+qB_*y)^2\right]\zeta_P \,=\, 0\,. \tag{13}$$

With the new variable

$$s = \frac{1}{\sqrt{qB_*}}\,(p_x+qB_*y) = \frac{y}{\ell_B} + \ell_B p_x\,, \tag{14}$$

where $\ell_B = 1/\sqrt{qB_*}$ is the magnetic length, the equation (13) splits into the following two equations for the ζ_P-components

$$\frac{d^2\zeta_P^A}{ds^2} + \left[\frac{\omega^2-p_z^2-M^2}{qB_*} \pm 1 - s^2\right]\zeta_P^A = 0\;, \text{ with } A = 1,2\,, \tag{15}$$

which are usual oscillator-type equations. Thus, we have come to the dispersion relation

$$\omega_n^2 = p_z^2 + M^2 + 2nqB_*\,, \tag{16}$$

and to the following form of the spinor ζ

$$\zeta_{P,n}(s) = \left[\begin{array}{c} \mathcal{I}_n(s) \\ \mathcal{I}_{n-1}(s) \end{array}\right], \tag{17}$$

where $\mathcal{I}_n(s)$ are the Hermite associated functions

$$\mathcal{I}_n(s) = \tilde{C}_n e^{-s^2/2} H_n(s)\,,$$

with the normalization constant

$$\tilde{C}_n = (qB_*)^{1/4}\left(2^n\, n!\,\sqrt{\pi}\right)^{-1/2}\;.$$

In what it concerns the other spinor, η, one has to go back with (17) and its derivative into the expression (12) which turns into

$$\eta_P(s) = \frac{\sqrt{qB_*}}{\omega + M}\left[-i\sigma^2 \frac{d\zeta_P}{ds} + s\sigma^1 \zeta_P + \frac{p_z}{\sqrt{qB_*}}\sigma^3 \zeta_P\right]$$
$$= \frac{1}{\omega + M}\left[\begin{array}{c} \left(p_z + \sqrt{2nqB_*}\right)\mathcal{I}_n \\ \left(-p_z + \sqrt{2nqB_*}\right)\mathcal{I}_{n-1} \end{array}\right]$$

once we have employed the relations

$$\mathcal{I}'_{n-1} - x\mathcal{I}_{n-1} = -\sqrt{2n}\mathcal{I}_n \ ; \ \mathcal{I}'_n + x\mathcal{I}_n = \sqrt{2n}\mathcal{I}_{n-1} \, .$$

Putting everything together, one is able to write down the full wave function describing the positively charged fermions evolving in a constant magnetic field as

$$\Psi_{P,n}(\vec{x},t) = e^{i(p_x x + p_z z - \omega t)}\frac{1}{L}\sqrt{\frac{\omega + M}{2\omega}}\left[\begin{array}{c} \mathcal{I}_n(s) \\ \mathcal{I}_{n-1}(s) \\ \frac{1}{\omega+M}\left(p_z + \sqrt{2nqB_*}\right)\mathcal{I}_n(s) \\ \frac{1}{\omega+M}\left(-p_z + \sqrt{2nqB_*}\right)\mathcal{I}_{n-1}(s) \end{array}\right], \tag{18}$$

where s is given by (14), $\sqrt{(\omega + M)/(2\omega)}$ is the normalization factor and L^3 is the volume. This leads to the following up and down positive energy ortho-normal modes

$$\Psi^+_{P,n,p_z}(\vec{x},t) = e^{i(p_x x + p_z z - \omega t)}\frac{1}{L}\sqrt{\frac{\omega + M}{2\omega}}\left[\begin{array}{c} \mathcal{I}_n \\ 0 \\ \frac{p_z}{\omega+M}\mathcal{I}_n \\ \frac{\sqrt{2nqB_*}}{\omega+M}\mathcal{I}_{n-1} \end{array}\right]_{n\geq 1} \tag{19}$$

and respectively

$$\Psi^-_{P,n,p_z}(\vec{x},t) = e^{i(p_x x + p_z z - \omega t)}\frac{1}{L}\sqrt{\frac{\omega + M}{2\omega}}\left[\begin{array}{c} 0 \\ \mathcal{I}_{n-1} \\ \frac{\sqrt{2nqB_*}}{\omega+M}\mathcal{I}_n \\ -\frac{p_z}{\omega+M}\mathcal{I}_{n-1} \end{array}\right]_{n\geq 1} \tag{20}$$

where

$$\omega_n^2 = p_z^2 + M^2 + 2nqB_* \, .$$

Let us turn now to the negatively charged fermions, described by the same Dirac equation (6), with the gauge derivative $D_j = \partial_j + i|q|A_j$ ($q < 0$) and $A_x = -B_* y$. As previously, once the solution of the Dirac equation for the positive energy states of negatively charged particles is taken as

$$\Psi^\sigma_{N,n}(\vec{x},t) = e^{i(p_x x + p_z z - \omega t)}\phi^\sigma_N(y) \, , \tag{21}$$

where

$$\phi_N(y) = \begin{bmatrix} \zeta_N(y) \\ \eta_N(y) \end{bmatrix}, \tag{22}$$

the Dirac equation leads to the following system of coupled differential equations

$$\begin{array}{ll} \text{(a)} & \sigma^2 \zeta_N' + i\left[(p_x - |q|B_* y)\,\sigma^1 + p_z \sigma^3\right] \zeta_N = i\,[\omega + M]\,\eta_N\,; \\ \text{(b)} & \sigma^1 \eta_N' + i\left[(p_x - |q|B_* y)\,\sigma^1 + p_z \sigma^3\right] \eta_N = i\,[\omega - M]\,\zeta_N\,. \end{array} \tag{23}$$

In terms of the new variable

$$\tau = \frac{1}{\sqrt{|q|B_*}}\left(|q|B_* y - p_x\right), \tag{24}$$

the spinors ζ_N and η_N being now

$$\zeta_{N,n}(\tau) = \begin{bmatrix} \mathcal{I}_{n-1}(\tau) \\ \mathcal{I}_n(\tau) \end{bmatrix}, \tag{25}$$

and

$$\eta_{N,n}(\tau) = \frac{1}{\omega + M} \begin{bmatrix} \left(p_z - \sqrt{2n|q|B_*}\right) \mathcal{I}_{n-1}(\tau) \\ \left(-p_z - \sqrt{2n|q|B_*}\right) \mathcal{I}_n(\tau) \end{bmatrix},$$

where $\mathcal{I}_n(\tau)$ are the normalized Hermite associated functions, the full positive energy wave functions, solutions to the Dirac equation describing the negatively charged particles evolving in a constant magnetic field are

$$\Psi_{N,n,p_z}(\vec{x},t) = e^{i(p_x x + p_z z - \omega t)}\,\frac{1}{L}\sqrt{\frac{\omega + M}{2\omega}} \begin{bmatrix} \mathcal{I}_{n-1}(\tau) \\ \mathcal{I}_n(\tau) \\ \frac{1}{\omega+M}\left(p_z - \sqrt{2n|q|B_*}\right) \mathcal{I}_{n-1}(\tau) \\ \frac{1}{\omega+M}\left(-p_z - \sqrt{2n|q|B_*}\right) \mathcal{I}_n(\tau) \end{bmatrix} \tag{26}$$

corresponding to the dispersion relation

$$\omega_n^2 = p_z^2 + M^2 + 2n|q|B_*\,. \tag{27}$$

Finally, the up and down positive energy ortho-normal modes are given by

$$\Psi^{+}_{N,n,p_z}(\vec{x},t) = e^{i(p_x x + p_z z - \omega t)}\,\frac{1}{L}\sqrt{\frac{\omega + M}{2\omega}} \begin{bmatrix} \mathcal{I}_{n-1} \\ 0 \\ \frac{p_z}{\omega+M}\mathcal{I}_{n-1} \\ -\frac{\sqrt{2n|q|B_*}}{\omega+M}\mathcal{I}_n \end{bmatrix}_{n \geq 1} \tag{28}$$

and respectively

$$\Psi^{-}_{N,n,p_z}(\vec{x},t) = e^{i(p_x x + p_z z - \omega t)} \frac{1}{L}\sqrt{\frac{\omega+M}{2\omega}} \begin{bmatrix} 0 \\ \mathcal{I}_n \\ -\frac{\sqrt{2n|q|B_*}}{\omega+M}\mathcal{I}_{n-1} \\ -\frac{p_z}{\omega+M}\mathcal{I}_n \end{bmatrix}_{n\geq 1} \qquad (29)$$

and they get a simplified form in the frame where $p_z = 0$ [14].

4. FIRST-ORDER PERTURBATIVE APPROACH

If the magnetic perturbation $B_x = b_0 \sin(\kappa z)$ comes into play, the Dirac equation (6) becomes

$$\left[\gamma^i \partial_i + iqB_* y \gamma^1 + M\right]\Psi = iq\frac{b_0}{\kappa}\cos(\kappa z)\gamma^2 \Psi\,, \qquad (30)$$

where the term in the right hand side comes from the covariant derivative $D_2 = \partial_y - iqA_y$, with A_y given by (5).

As in the perturbation theory, the field equation (30) can be written as

$$\hat{H}_0\,\Psi = \hat{V}\,\Psi\,, \qquad (31)$$

pointing out the operator

$$\hat{V} = iq\frac{b_0}{\kappa}\cos(\kappa z)\gamma^2\,, \qquad (32)$$

describing the (perturbed) effective potential.

One may deal with the first-order transition amplitudes between the up-up, down-down or up-down states whose unperturbed wave functions are (19, 20, 28, 29) and their charged conjugates, induced by the magnetic periodic induction $B_x = b_0 \sin(\kappa z)$.

For the positive energy mode (19), with $\gamma^2 = -i\beta\alpha^2$ and $\bar{\Psi} = \Psi^\dagger \beta$, the up-up scattering amplitude defined by

$$\begin{aligned} \mathcal{A}^{uu} &= \frac{qb_0}{\kappa}\sum\int d^4x\, a_+^\dagger(\vec{p}\,', n')\, a_+(\vec{p}, n)\, e^{-i(p'_x x + p'_z z)}\, e^{i\omega' t} e^{i(p_x x + p_z z)} e^{-i\omega t} \\ &\quad \cos(\kappa z)\left[\phi^+_{n'}(y)\right]^\dagger \alpha^2 \left[\phi^+_n(y)\right]\,, \end{aligned} \qquad (33)$$

is computed in the Appendix section. This can be written in terms of the following main contributions

$$
\begin{aligned}
&\mathcal{A}^{uu}_{p_z,p'_z,0} = \frac{qb_0}{2\kappa}\,\frac{1}{2L^2}\,2\pi\delta\left(p'_x - p_x\right)2\pi\delta\left(p'_z - p_z - \kappa\right) \\
&\left\{\sqrt{\frac{2qB_*}{p_z^2 - 2qB_*}}\left[2\pi\delta\left(p'_z - \sqrt{p_z^2 - 2qB_*}\right) + 2\pi\delta\left(p'_z + \sqrt{p_z^2 - 2qB_*}\right)\right]\right. \\
&\left. - \sqrt{\frac{2qB_*}{p_z^2 + 2qB_*}}\left[2\pi\delta\left(p'_z - \sqrt{p_z^2 + 2qB_*}\right) + 2\pi\delta\left(p'_z + \sqrt{p_z^2 + 2qB_*}\right)\right]\right\} \\
&+ \frac{qb_0}{2\kappa}\,\frac{1}{2L^2}\,2\pi\delta\left(p'_x - p_x\right)2\pi\delta\left(p'_z - p_z + \kappa\right) \\
&\left\{\sqrt{\frac{2qB_*}{p_z^2 - 2qB_*}}\left[2\pi\delta\left(p'_z - \sqrt{p_z^2 - 2qB_*}\right) + 2\pi\delta\left(p'_z + \sqrt{p_z^2 - 2qB_*}\right)\right]\right. \\
&\left. - \sqrt{\frac{2qB_*}{p_z^2 + 2qB_*}}\left[2\pi\delta\left(p'_z - \sqrt{p_z^2 + 2qB_*}\right) + 2\pi\delta\left(p'_z + \sqrt{p_z^2 + 2qB_*}\right)\right]\right\},
\end{aligned} \tag{34}
$$

pointing out two types of transitions:

I. The first ones, corresponding to

$$p_z = -\frac{qB_*}{\kappa} - \frac{\kappa}{2} \to p'_z = -\frac{qB_*}{\kappa} + \frac{\kappa}{2},$$

and

$$p_z = \frac{qB_*}{\kappa} + \frac{\kappa}{2} \to p'_z = \frac{qB_*}{\kappa} - \frac{\kappa}{2},$$

are characterized by the following transition rate per unit surface

$$\frac{d\mathcal{P}^{uu}_I}{d\Sigma dt} = \left(\frac{qb_0}{\kappa}\right)^2 \frac{qB_*}{|2qB_* - \kappa^2|}. \tag{35}$$

II. The second ones,

$$p_z = \frac{qB_*}{\kappa} - \frac{\kappa}{2} \to p'_z = \frac{qB_*}{\kappa} + \frac{\kappa}{2},$$

and

$$p_z = -\frac{qB_*}{\kappa} + \frac{\kappa}{2} \to p'_z = -\frac{qB_*}{\kappa} - \frac{\kappa}{2},$$

have a transition rate per unit surface given by

$$\frac{d\mathcal{P}^{uu}_{II}}{d\Sigma dt} = \left(\frac{qb_0}{\kappa}\right)^2 \frac{qB_*}{2qB_* + \kappa^2}. \tag{36}$$

The down-down transitions being characterized by similar amplitudes and rates, we shall end our discussion by treating the down-up case. Using the wave functions (19) and (20), the transition amplitude,

$$\begin{aligned}\mathcal{A}^{du} &= \frac{qb_0}{\kappa}\sum\int d^4x\, a_+^\dagger(\vec{p}\,',n')\,a_+(\vec{p},n)\,e^{-i(p_x'x+p_z'z)}\,e^{i\omega' t}e^{i(p_xx+p_zz)}e^{-i\omega t}\\ &\quad \cos(\kappa z)\left[\phi_{n'}^+(y)\right]^\dagger\alpha^2\left[\phi_n^+(y)\right], \qquad (37)\end{aligned}$$

is derived in the Appendix section as being

$$\begin{aligned}\mathcal{A}^{du} &= -i\,\frac{qb_0}{2\kappa}\,\frac{1}{2L^2}\,2\pi\delta\left(p_x'-p_x\right)\frac{\kappa}{\sqrt{p_z^2+2qB_*}}\\ &\times\ 2\pi\left[\delta\left(p_z'-p_z-\kappa\right)-\delta\left(p_z'-p_z+\kappa\right)\right]\\ &\times\ 2\pi\left[\delta\left(p_z'-\sqrt{p_z^2+2qB_*}\right)+\delta\left(p_z'+\sqrt{p_z^2+2qB_*}\right)\right],\end{aligned}$$

leading to the transitions

$$\begin{aligned} p_z &= \frac{qB_*}{\kappa}-\frac{\kappa}{2}\to p_z'=\frac{qB_*}{\kappa}+\frac{\kappa}{2}\,;\\ p_z &= -\frac{qB_*}{\kappa}+\frac{\kappa}{2}\to p_z'=-\frac{qB_*}{\kappa}-\frac{\kappa}{2}\,,\end{aligned}$$

characterized by the same rate per unit surface, namely

$$\frac{d\mathcal{P}^{du}}{d\Sigma dt}=\left(\frac{qb_0}{\kappa}\right)^2\frac{\kappa^4}{\left(2qB_*+\kappa^2\right)^2}\,. \qquad (38)$$

5. CONCLUSIONS

Almost 20 years after Duncan and Thompson introduced the notion of *magnetar*, for what we call today the anomalous X-ray pulsars and the Soft gamma repeaters, particles moving in magnetic fields with an induction larger than the critical value at which the cyclotron energy of an electron equals the electron rest mass energy, have been a main target of investigation.

Recently, the resonant magnetic Compton scattering, characterized by an increased probability, compared to the one in the Thompson scattering, has been proposed as a possible mechanism of converting the strong magnetic fields into radiation [15].

Since, in such analysis, the initial and final electrons as well as the intermediate ones are described by the Dirac equation for an electron in an uniform magnetic field, in the section 3 of the present paper, we derive the positive energy ortho-normal modes, (19, 20) and respectively (28, 29), for positively and negatively charged relativistic fermions evolving in a strong static magnetic field alone.

Next, we have added to the strong magnetostatic component, B_*, a periodic perturbation $B_x = B_0(t)\sin(\kappa z)$. In choosing this form we have been inspired by Rheinhardt and Geppert's papers [12], who considered an infinitely extended slab on the x and y coordinates, with finite thickness only in the $z-$direction. The background magnetic field described in their work by a function bounded in z, has been taken later as a periodic function of the form $B_x = b_0\sin(\kappa z)$ [13].

When such an additional term comes into play, the Dirac equation (6) turns into (30) and can be solved within the perturbation theory, the operator

$$\hat{V} = iq\frac{b_0}{\kappa}\cos(\kappa z)\gamma^2\,,$$

being the effective potential.

Using the positive energy modes (19, 20), we have computed the scattering amplitudes and the transition rates per unit surface. For $\kappa^2 \ll qB_*$, which is a normal condition for the magnetar's strong magnetic field ($\ell_B \ll L \sim \pi/\kappa$), the *up-up* rates, (35) and (36), are almost not affected B_*, being

$$\frac{d\mathcal{P}^{uu}}{d\Sigma dt} \approx \frac{1}{2}\left(\frac{qb_0}{\kappa}\right)^2\left[1\pm\frac{\kappa^2}{2qB_*}\right],$$

while the down-up transitions, characterized by the rate (38), *i.e.*

$$\frac{d\mathcal{P}^{du}}{d\Sigma dt} \approx \frac{\kappa^2}{4}\left(\frac{b_0}{B_*}\right)^2,$$

are significantly suppressed for $B_* \gg b_0$.

One may conclude by saying that the perturbative periodic magnetic field is initiating the transitions which are energetically supported by the huge magnetic static induction, B_*. The electron with the energy spectrum (27), promoted from ω_0 to the first Landau level with ω_1 can produce, by de-excitation, photons that will directly convert to an electron-positron pair [3], since, for $B_* \geq 10^{11}$ (T) which is a typical magnetic induction inside the magnetar's crust, the following condition is fulfilled

$$h\nu = \omega_1 - \omega_0 \sim \sqrt{qB_*\hbar c^2} > 2m_0c^2\,.$$

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6. APPENDIX

The up-up scattering amplitude, for the positive energy mode (19), defined in (33),

$$
\begin{aligned}
\mathcal{A}^{uu} &= \frac{qb_0}{\kappa} \sum \int d^4x \, a_+^{\dagger}(\vec{p}\,', n') \, a_+(\vec{p}, n) \, e^{-i(p'_x x + p'_z z)} \, e^{i\omega' t} e^{i(p_x x + p_z z)} e^{-i\omega t} \\
& \quad \cos(\kappa z) \left[\phi_{n'}^{+}(y)\right]^{\dagger} \alpha^2 \left[\phi_n^{+}(y)\right] \\
&= \frac{qb_0}{2\kappa} \, 2\pi\delta\left(p'_x - p_x\right) 2\pi\delta\left(\omega' - \omega\right) 2\pi \left[\delta\left(p'_z - p_z - \kappa\right) + \delta\left(p'_z - p_z + \kappa\right)\right] \\
& \quad \frac{1}{2L^2} \sum_n \sqrt{\frac{(\omega+M)(\omega'+M)}{\omega\omega'}} \left[\frac{\sqrt{2n'qB_*}}{\omega'+M} \, \delta_{n,n'-1} - \frac{\sqrt{2nqB_*}}{\omega+M} \, \delta_{n-1,n'}\right],
\end{aligned}
$$

can be firstly written in terms of the following contributions:

$$\begin{aligned}\mathcal{A}^{uu}_{p_z,p'_z,n} &= \frac{qb_0}{2\kappa}\frac{1}{2L^2}2\pi\delta\left(p'_x-p_x\right)2\pi\delta\left(p'_z-p_z-\kappa\right)\\ &\sum_{n=0}^{\infty}\Bigg[2\pi\delta\left(\omega'_{n+1}-\omega_n\right)\sqrt{\frac{2(n+1)qB_*}{\omega'_{n+1}\omega_n}}\sqrt{\frac{\omega_n+M}{\omega'_{n+1}+M}}\\ &-2\pi\delta\left(\omega'_n-\omega_{n+1}\right)\sqrt{\frac{2(n+1)qB_*}{\omega'_n\omega_{n+1}}}\sqrt{\frac{\omega'_n+M}{\omega_{n+1}+M}}\Bigg]\\ &+\frac{qb_0}{2\kappa}\frac{1}{2L^2}2\pi\delta\left(p'_x-p_x\right)2\pi\delta\left(p'_z-p_z+\kappa\right)\\ &\sum_{n=0}^{\infty}\Bigg[2\pi\delta\left(\omega'_{n+1}-\omega_n\right)\sqrt{\frac{2(n+1)qB_*}{\omega'_{n+1}\omega_n}}\sqrt{\frac{\omega_n+M}{\omega'_{n+1}+M}}\\ &-2\pi\delta\left(\omega'_n-\omega_{n+1}\right)\sqrt{\frac{2(n+1)qB_*}{\omega'_n\omega_{n+1}}}\sqrt{\frac{\omega'_n+M}{\omega_{n+1}+M}}\Bigg].\end{aligned}$$

The distance between the Landau levels being quite large, we consider transitions of the up fermion between the $n=0$ and $n=1$ energy levels, as in the resonant scattering mechanism. Thus, the above amplitude gets a much simpler form,

$$\begin{aligned}\mathcal{A}^{uu}_{p_z,p'_z,0} &= \frac{qb_0}{2\kappa}\frac{1}{2L^2}2\pi\delta\left(p'_x-p_x\right)2\pi\delta\left(p'_z-p_z-\kappa\right)\\ \times&\left[2\pi\delta\left(\omega'_1-\omega_0\right)\sqrt{\frac{2qB_*}{\omega'_1\omega_0}}\sqrt{\frac{\omega_0+M}{\omega'_1+M}}-2\pi\delta\left(\omega'_0-\omega_1\right)\sqrt{\frac{2qB_*}{\omega'_0\omega_1}}\sqrt{\frac{\omega'_0+M}{\omega_1+M}}\right]\\ +&\frac{qb_0}{2\kappa}\frac{1}{2L^2}2\pi\delta\left(p'_x-p_x\right)2\pi\delta\left(p'_z-p_z+\kappa\right)\\ \times&\left[2\pi\delta\left(\omega'_1-\omega_0\right)\sqrt{\frac{2qB_*}{\omega'_1\omega_0}}\sqrt{\frac{\omega_0+M}{\omega'_1+M}}-2\pi\delta\left(\omega'_0-\omega_1\right)\sqrt{\frac{2qB_*}{\omega'_0\omega_1}}\sqrt{\frac{\omega'_0+M}{\omega_1+M}}\right],\end{aligned}$$

where the initial and final energies are

$$\begin{aligned}\omega_0=\sqrt{p_z^2+M^2}\,,\ \omega_1=\sqrt{p_z^2+M^2+2qB_*}\,,\\ \omega'_0=\sqrt{p_z'^2+M^2}\,,\ \omega'_1=\sqrt{p_z'^2+M^2+2qB_*}\,.\end{aligned}$$

Once we have used the relations

$$\begin{aligned}\delta\left(\omega'_1-\omega_0\right) &= \frac{\omega_0}{\sqrt{p_z^2-2qB_*}}\left[\delta\left(p'_z-\sqrt{p_z^2-2qB_*}\right)+\delta\left(p'_z+\sqrt{p_z^2-2qB_*}\right)\right],\\ \delta\left(\omega'_0-\omega_1\right) &= \frac{\omega_1}{\sqrt{p_z^2+2qB_*}}\left[\delta\left(p'_z-\sqrt{p_z^2+2qB_*}\right)+\delta\left(p'_z+\sqrt{p_z^2+2qB_*}\right)\right],\end{aligned}$$

one gets a more transparent form,

$$\begin{aligned}
&\mathcal{A}^{uu}_{p_z,p'_z,0} = \frac{qb_0}{2\kappa}\,\frac{1}{2L^2}\,2\pi\delta\left(p'_x - p_x\right)2\pi\delta\left(p'_z - p_z - \kappa\right)\\
&\left\{\sqrt{\frac{2qB_*}{p_z^2 - 2qB_*}}\left[2\pi\delta\left(p'_z - \sqrt{p_z^2 - 2qB_*}\right) + 2\pi\delta\left(p'_z + \sqrt{p_z^2 - 2qB_*}\right)\right]\right.\\
&\left. - \sqrt{\frac{2qB_*}{p_z^2 + 2qB_*}}\left[2\pi\delta\left(p'_z - \sqrt{p_z^2 + 2qB_*}\right) + 2\pi\delta\left(p'_z + \sqrt{p_z^2 + 2qB_*}\right)\right]\right\}\\
&+ \frac{qb_0}{2\kappa}\,\frac{1}{2L^2}\,2\pi\delta\left(p'_x - p_x\right)2\pi\delta\left(p'_z - p_z + \kappa\right)\\
&\left\{\sqrt{\frac{2qB_*}{p_z^2 - 2qB_*}}\left[2\pi\delta\left(p'_z - \sqrt{p_z^2 - 2qB_*}\right) + 2\pi\delta\left(p'_z + \sqrt{p_z^2 - 2qB_*}\right)\right]\right.\\
&\left. - \sqrt{\frac{2qB_*}{p_z^2 + 2qB_*}}\left[2\pi\delta\left(p'_z - \sqrt{p_z^2 + 2qB_*}\right) + 2\pi\delta\left(p'_z + \sqrt{p_z^2 + 2qB_*}\right)\right]\right\},
\end{aligned}$$

pointing out the transitions characterized by the probabilities (35) and (36).

Similarly, using the wave functions (19) and (20), the down-up transition amplitude (37) can be computed as

$$\begin{aligned}
\mathcal{A}^{du} &= \frac{qb_0}{\kappa}\sum\int d^4x\, a^\dagger_+(\vec{p}\,',n')\,a_+(\vec{p},n)\,e^{-i(p'_x x + p'_z z)}\,e^{i\omega' t}e^{i(p_x x + p_z z)}e^{-i\omega t}\\
&\quad \cos(\kappa z)\left[\phi^+_{n'}(y)\right]^\dagger \alpha^2\left[\phi^+_n(y)\right]\\
&= -i\,\frac{qb_0}{2\kappa}\,\frac{1}{2L^2}\,2\pi\delta\left(p'_x - p_x\right)2\pi\delta\left(\omega' - \omega\right)\\
&\quad 2\pi\left[\delta\left(p'_z - p_z - \kappa\right) + \delta\left(p'_z - p_z + \kappa\right)\right]\\
&\quad \sum_n\sqrt{\frac{(\omega + M)(\omega' + M)}{\omega\omega'}}\left[\frac{p'_z}{\omega' + M} - \frac{p_z}{\omega + M}\right]\delta_{n-1,n'}\,.
\end{aligned}$$

Again, by employing the relations

$$\begin{aligned}
&\omega'^{\,2}_{n-1} - \omega_n^2 = p'^2_z - p_z^2 - 2qB_*\,,\\
&\delta\left(\omega'_{n-1} - \omega_n\right)\\
&= \frac{\omega_n}{\sqrt{p_z^2 + 2qB_*}}\left[\delta\left(p'_z - \sqrt{p_z^2 + 2qB_*}\right) + \delta\left(p'_z + \sqrt{p_z^2 + 2qB_*}\right)\right],
\end{aligned}$$

the amplitude's expression becomes

$$\begin{aligned}
\mathcal{A}^{du} &= -i\,\frac{qb_0}{2\kappa}\,\frac{1}{2L^2}\,2\pi\delta\left(p_x'-p_x\right)\frac{\kappa}{\sqrt{p_z^2+2qB_*}} \\
&\times\; 2\pi\left[\delta\left(p_z'-p_z-\kappa\right)-\delta\left(p_z'-p_z+\kappa\right)\right] \\
&\times\; 2\pi\left[\delta\left(p_z'-\sqrt{p_z^2+2qB_*}\right)+\delta\left(p_z'+\sqrt{p_z^2+2qB_*}\right)\right],
\end{aligned}$$

pointing out the transitions characterized by (38).