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SOLUTIONS OF THE QUASI-EXACTLY SOLVABLE MATHIEU POTENTIAL BY THE ASYMPTOTIC ITERATION METHOD

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Abstract. In this paper, we study the Schrödinger equation with quasi-exactly solvable Mathieu potential which is belong to class X in Turbiner's classification. We calculate the energy eigenvalues as well as the corresponding wave functions within the framework of the asymptotic iteration method. We show that the method provides the exact results.

Key words: Schrödinger equation, quasi-exact solvability, Mathieu potential, asymptotic iteration method, *sl*(2) Lie algebra.

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1. INTRODUCTION

It is well known that exactly solvable (ES) models play an important role in many branches of physics, but in general, the Schrödinger equation cannot be solved exactly to determine the whole spectrum, except for a few potentials such as the harmonic oscillator and the hydrogen atom. Different methods and techniques for solving various wave equations with ES potentials as well as the most recent research in the area of ES linear and nonlinear models can be found in refs. [1–11]. During the last decades, an intermediate class between the ES potentials and the non-solvable potentials was introduced for which a finite number of eigenvalue spectrum can be calculated by purely algebraic means. For this reason, they were named quasi-exactly solvable (QES) [12–20]. Nowadays the QES problems have received a great amount of interest [21–26]. These potentials are distinguished by the fact that for them, the Hamiltonian can be expressed as a quadratic combination of the generators of a finite-dimensional Lie algebra that possesses a finite-dimensional representation space. As a result, the Hamiltonian can be written as a block diagonal matrix with at least one finite block. The solutions corresponding to this block can be always determined exactly. A complete classification of the QES potentials in one dimension have been done by Turbiner in ten classes [13, 14]. In the other hand, recently, the asymptotic iteration method (AIM) has been

introduced to obtain the solutions for an extended class of eigenvalue problems, including ES, QES and non-solvable ones [27–30]. Although many ES potentials have been solved by this method, there are few cases of QES potentials in the literature. In this paper, we apply AIM to calculate the eigenenergies of the class X potential in the Turbiner's classification, named QES Mathieu potential. Also, we solve it by using the Lie algebraic method. It is found that the two results coincide with each other.

This paper is organized as follows. In sect. 2, we briefly present an overview of AIM. In sect. 3, we apply AIM to find the eigenenergies as well as the corresponding wavefunctions of the Schrödinger equation for QES Mathieu potential. In Sect. 4, we solve the same problem by using the Lie algebraic method and therein we will make a comparison between the solutions obtained by the two methods. The paper ends with conclusions in sect. 5.

2. ASYMPTOTIC ITERATION METHOD

In this section we briefly review the AIM method of refs. [27–30]. AIM is proposed to solve the homogenous linear second-order differential equation of the form

$$\phi''(x) = \lambda_0(x)\phi'(x) + s_0(x)\phi(x), \tag{1}$$

where $\lambda_0(x)$ and $s_0(x)$ have sufficiently many continuous derivatives. In order to find a general solution to eq. (1), it can be iterated up to $(k+1)$-th and $(k+2)$-th derivatives, where $k = 1, 2, 3, \ldots$. Therefore, we have

$$\phi^{(k+1)}(x) = \lambda_{k-1}(x)\phi'(x) + s_{k-1}(x)\phi(x), \tag{2}$$

$$\phi^{(k+2)}(x) = \lambda_k(x)\phi'(x) + s_k(x)\phi(x), \tag{3}$$

where

$$\begin{aligned} \lambda_k(x) &= \lambda'_{k-1}(x) + s_{k-1}(x) + \lambda_0(x)\lambda_{k-1}(x), \\ s_k(x) &= s'_{k-1}(x) + s_0(x)\lambda_{k-1}(x). \end{aligned} \tag{4}$$

From the ratio of the $(k+2)$-th and $(k+1)$-th derivatives, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(\phi^{(k+1)}(x)\right) = \frac{\phi^{(k+2)}(x)}{\phi^{(k+1)}(x)} = \frac{\lambda_k(x)\left(\phi'(x) + \dfrac{s_k(x)}{\lambda_k(x)}\phi(x)\right)}{\lambda_{k-1}(x)\left(\phi'(x) + \dfrac{s_{k-1}(x)}{\lambda_{k-1}(x)}\phi(x)\right)}. \tag{5}$$

If we have, for sufficiently large k

$$\frac{s_k(x)}{\lambda_k(x)} = \frac{s_{k-1}(x)}{\lambda_{k-1}(x)} = \alpha_k(x), \tag{6}$$

which is the asymptotic aspect of the method, then eq. (5) is reduced to

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(\phi^{(k+1)}(x)\right) = \frac{\lambda_k(x)}{\lambda_{k-1}(x)}, \tag{7}$$

which yields

$$\phi^{(k+1)}(x) = C_1 \exp\left(-\int^x \frac{\lambda_k(t)}{\lambda_{k-1}(t)}\mathrm{d}t\right) = C_1\lambda_{k-1}(x)\exp\left(\int^x\left(\alpha_k(t)+\lambda_0(t)\right)\mathrm{d}t\right), \tag{8}$$

where C_1 is the integration constant. Substituting eq. (8) into (2), we obtain the general solution of eq. (1) as

$$\phi(x) = \exp\left(-\int^x \alpha_k(t)\mathrm{d}t\right)\left[C_2 + C_1\int^x \exp\left(\int^t\left(2\alpha_k(\tau)+\lambda_0(\tau)\right)\mathrm{d}\tau\right)\mathrm{d}t\right], \tag{9}$$

where the first part gives the solutions that are convergent and physical, whereas the second part yields the non-physical solutions that are divergent. Therefore taking $C_1 = 0$, the wave function generator is then given by

$$\phi_k(x) = C_2 \exp\left(-\int^x \frac{s_k(t)}{\lambda_k(t)}\mathrm{d}t\right). \tag{10}$$

The energy eigenvalues are determined from the quantization condition given by eq. (6), which, together with eq. (4), can also be written as follows

$$\delta_k(x) \equiv s_k(x)\lambda_{k+1}(x) - s_{k+1}(x)\lambda_k(x) = 0. \tag{11}$$

If the eigenvalue problem is ES, an explicit expression for energy can directly be obtained from the roots of this equation that depends only on the eigenvalues *E*. In contrast, if the eigenvalue problem is not analytically solvable, the termination condition (11) produces an expression that depends on both *x* and unknown *E*. In this case, for any given quantum number *n*, the energy eigenvalues can be calculated from the roots of eq. (11) at some suitable x_0 point. The chosen value of x_0 is arbitrary in principle and can be critical only to the speed of the convergence of the eigenvalues. This starting value may be determined generally as

the minimum value of the potential or the maximum value of the asymptotic wave function [30].

3. THE ASYMPTOTIC ITERATION METHOD FOR QES MATHIEU POTENTIAL

According to Turbiner's classification [14], the QES Mathieu potential belonging to class X has the form

$$V(x) = a^2 \sin^2(\alpha x) - \alpha a(2n+1)\cos(\alpha x) + \mu\alpha^2, \tag{12}$$

where $\alpha \geq 0$, $a \geq 0$ and $\mu = 0,1$. In this paper, we consider the case of $\mu = 1$ only. The case of $\mu = 0$ can be treated in a similar way. Thus, the one-dimensional time-independent Schrödinger equation for the potential (12) is written as

$$-\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} + \left(a^2 \sin^2(\alpha x) - \alpha a(2n+1)\cos(\alpha x) + \alpha^2\right)\psi(x) = E\psi(x). \tag{13}$$

Considering a solution of the form

$$\psi(x) = \sin(\alpha x) e^{\frac{a}{\alpha}\cos(\alpha x)} \phi(x), \tag{14}$$

and inserting it into eq. (13), we get

$$\frac{\mathrm{d}^2\phi(z)}{\mathrm{d}z^2} + \left(\frac{2\alpha a z^2 + 3\alpha^2 z - 2\alpha a}{\alpha^2(z^2-1)}\right)\frac{\mathrm{d}\phi(z)}{\mathrm{d}z} + \left(\frac{2\alpha a(1-n)z + 2\alpha^2 - E}{\alpha^2(z^2-1)}\right)\phi(z) = 0, \tag{15}$$

where $z = \cos(\alpha x)$. We intend to to solve this differential equation by two different methods, the AIM method and the Lie algebraic method which will be discussed in the next section. The AIM can be applied to this problem with

$$\lambda_0(z) = \frac{-2\alpha a z^2 - 3\alpha^2 z + 2\alpha a}{\alpha^2(z^2-1)}, \qquad s_0(z) = \frac{2\alpha a(n-1)z - 2\alpha^2 + E}{\alpha^2(z^2-1)}, \tag{16}$$

from which, together with the quantization condition (11), we can calculate the energy eigenvalues of the system. Since the problem is not ES, we have to select a suitable z_0 point and solve the eq. (11) for z_0 to find the eigenvalues of energy. In this work, we observed that the best initial value for z_0 is $z_0 = 0$. Therefore, at the end of the iterations, we put $z_0 = 0$. To obtain the numerical results, we take

$\alpha = a = 1$. Our numerical results for n up to 5 are reported in Table 1. Also, using eq. (10) within the framework of the AIM, we can compute the wave functions as follows. For $n = 1$, the energy eigenvalue is $E_1^{(1)} = 2$. Substituting $E_1^{(1)}$ into (16) and then using eq. (10), we obtain $\phi(z) = 1$. Thus, the ground state wave function from eq. (14) is given by

$$\psi_{E_1}^{(1)}(x) = \sin(x) e^{\cos(x)}. \tag{17}$$

Similarly, for $n = 2$, the energy eigenvalues are $E_1^{(2)} = 1$ and $E_2^{(2)} = 6$. Substituting these values into (16) and then using eq. (10), we obtain $\phi(x) = \cos(x) + 2$ and $\phi(x) = 2\cos(x) - 1$, respectively. The wave functions of the first excited states from eq. (14) are then given by

$$\psi_{E_1}^{(2)}(x) = 2\sin(x)\left(1 + \frac{\cos(x)}{2}\right) e^{\cos(x)}, \tag{18}$$

$$\psi_{E_2}^{(2)}(x) = -\sin(x)\left(1 - 2\cos(x)\right) e^{\cos(x)}. \tag{19}$$

In the next section, we attempt to reproduce these results using the Lie algebraic approach.

4. THE LIE ALGEBRAIC METHOD FOR QES MATHIEU POTENTIAL

In this section, using the Lie algebraic approach of quasi-exact solvability, we obtain the energy spectrum of eq. (15) through the *sl*(2) algebraization. It is easily shown that by considering the generators of $sl(2)$ as

$$\begin{aligned} J_{n-1}^{+} &= z^2 d_z - (n-1)z, \\ J_{n-1}^{0} &= z d_z - \frac{n-1}{2}, \\ J_{n-1}^{-} &= d_z, \quad n = 1, 2, 3, ..., \end{aligned} \tag{20}$$

and after some algebra, eq. (15) can be expressed as

$$\begin{aligned} &H\phi = E\phi, \\ &H = C_1 J_{n-1}^{+} J_{n-1}^{-} + C_2 J_{n-1}^{-} J_{n-1}^{-} + C_3 J_{n-1}^{+} + C_4 J_{n-1}^{-} + C_5 J_{n-1}^{0} + C_6. \end{aligned} \tag{21}$$

Substituting (20) into (21), results in the following differential form

$$\left\{ P(z)\frac{\mathrm{d}^2}{\mathrm{dz}^2} + Q(z)\frac{\mathrm{d}}{\mathrm{dz}} + R(z) \right\} \phi(z) = 0, \tag{22}$$

where

$$\begin{aligned} P(z) &= C_1 z^2 + C_2, \\ Q(z) &= C_3 z^2 + \left(C_5 - (\mathrm{n}-1)C_1\right) z + C_4, \\ R(z) &= \left((1-n)C_3\right) z + \left(C_6 - \frac{n-1}{2} C_5 - E \right). \end{aligned} \tag{23}$$

Comparing eq. (22) with eq. (15) gives

$$\begin{aligned} &C_1 = -C_2 = \alpha^2, \quad C_3 = -C_4 = 2\alpha a, \\ &C_5 = (n+2)\alpha^2, \quad C_6 = \left(\frac{n^2 + n + 2}{2} \right) \alpha^2, \end{aligned} \tag{24}$$

from which, together with eq. (21), we obtain the Lie algebraic differential operator $H \in U_{sl(2)}$ as

$$\begin{aligned} &H\phi(z) = E\phi(z), \\ &H = \alpha^2 J^+_{n-1} J^-_{n-1} - \alpha^2 J^-_{n-1} J^-_{n-1} + 2\alpha a \left(J^+_{n-1} - J^-_{n-1} \right) + \left(n+2\right) \alpha^2 J^0_{n-1} \\ &\qquad + \left(\frac{n^2 + n + 2}{2} \right) \alpha^2. \end{aligned} \tag{25}$$

It follows from this that the operator H possesses a finite-dimensional invariant subspace spanned by the basis $\left\{1, z, z^2, ..., z^{n-1}\right\}$. Thus, the corresponding eigenfunctions are given by

$$\phi_{n-1}(z) = \sum_{m=1}^{n} c_{m-1} z^{m-1}, \quad n = 1, 2, 3, ..., \tag{26}$$

which allows us to compute a finite part of the spectrum algebraically. In continue, we take $\alpha = a = 1$ and so from eq. (25), the eigenvalue for $n = 1$ is calculated as $E_1^{(1)} = 2$. Thus, the ground state wave function from eqs. (14) and (26) is obtained as

$$\psi^{(1)}_{E_1}(x) = c_0 \sin(x) e^{\cos(x)}, \tag{27}$$

where the coefficient c_0 is the expansion constant. For $n = 2$, from eq. (25), we obtain

$$\begin{pmatrix} 2-E & -2 \\ -2 & 5-E \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = 0. \tag{28}$$

Thus, the QES states have the energy eigenvalues as $E_1^{(2)} = 1$ and $E_2^{(2)} = 6$ and the corresponding wave functions from eqs. (14) and (26) are therefore given by

$$\psi_{E_1}^{(2)}(x) = c_0 \sin(x)\left(1+\frac{\cos(x)}{2}\right)e^{\cos(x)}, \tag{29}$$

$$\psi_{E_2}^{(2)}(x) = c_0 \sin(x)\left(1-2\cos(x)\right)e^{\cos(x)}, \tag{30}$$

which are the same with those obtained by AIM except for a constant factor. Following this process, any finite part of the QES spectra can be obtained algebraically. We have calculated some of the excited energy levels and the results have written in Table 1. Comparison of the results of the two methods shows that they are the same.

Table 1

Comparison between the eigenenergies computed by the two methods (AIM and Lie algebraic) for the QES Mathieu potential

n	E_{AIM} eq. (11)	E_{QES} eq. (25)
1	2	2
2	1 6	1 6
3	–0.179 6 11.179	–0.179 6 11.179
4	–1.4646 5.7771 11.5417 18.1458	–1.4646 5.7771 11.5417 18.1458
5	–2.8228 5.3667 11.8727 18.4706 27.1129	–2.8228 5.3667 11.8727 18.4706 27.1129

5. CONCLUSIONS

By using two different methods, namely the AIM and the Lie algebraic methods, we have investigated the solutions of the Schrödinger equation with QES Mathieu potential. We have applied the numerical treatment within the framework of AIM to find energy eigenvalues as well as the corresponding wave functions. It is shown that the results are identical with the exact results obtained by the Lie algebraic method.

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