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ON SOLITON DYNAMICS OF THE GENERALIZED FISHER EQUATION WITH TIME-DEPENDENT COEFFICIENTS

HOURIA TRIKI[1], ABDUL-MAJID WAZWAZ[2]

[1]Radiation Physics Laboratory, Department of Physics, Faculty of Sciences,
Badji Mokhtar University, P. O. Box 12, 23000 Annaba, Algeria
E-mail: houriatriki@gmail.com
[2]Department of Mathematics, Saint Xavier University, Chicago, IL 60655, USA
E-mail: wazwaz@sxu.edu

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Abstract. A generalized Fisher equation involving a nonlinear term of any order and time-dependent coefficients is investigated. We present exact analytical solutions describing periodic and solitary wave solutions using the modified sine-cosine method. A certain class of exact soliton-like solutions has been found by means of the auxiliary equation method. The conditions of existence and uniqueness of these solutions are given. We exploit the temporal variation of the coefficients to study the dynamics of solitary waves in presence of the linear dispersion effect and arbitrary power nonlinearity. We show numerically that the time-variation of the dependent coefficients provides a physical way to control the solitary wave profile. These solitary waves are expected to find practical application in inhomogeneous nonlinear systems that are described by Fisher-type equation.

Key words: Generalized Fisher equation, time-dependent coefficients, solitons.

1. INTRODUCTION

Envelope solitary waves propagating in inhomogeneous media have attracted considerable attention in recent years. Interesting results were particularly presented in the field of fiber-optic communication systems where the family of nonlinear Schrödinger equation with distributed coefficients is the basic model equation to describe the nonlinear wave dynamics [1]-[5]. Compared with the study of solitary wave dynamics in inhomogeneous fibers, the investigation of propagating waves in nonlinear media described, for instance, by the sine-Gordon equation, the Fisher equation, the Fitzhugh-Nagumo equation and many others, has received little attention. In Ref. [6], parametrically controlling solitary wave dynamics in the modified Korteweg-de Vries equation through the temporal variations of the distributed coefficients has been described.

A nonlinear solitary wave is an example of a stable wave packet that preserves its shape during propagation in a nonlinear dispersive medium [7]. The distinction between solitary wave and soliton solutions is that when any number of solitons interact they do not change form, and the only outcome of the interaction is a phase shift [8]. Solitons appear in many diverse fields such as nonlinear optics, plasma physics, biophysics, acoustics, and fluid mechanics. These particlelike excitations are stabilized by a balance of dispersion and nonlinearity [9]. Two different types of envelope solitons, bright and dark, can propagate in nonlinear dispersive media [10]. These physical objects have been studied extensively, both theoretically and experimentally [11]-[30].

The theoretical description of wave propagation in a given nonlinear medium is based on an analysis of nonlinear evolution equations with constant or variables coefficients. Due to the inhomogeneities of media and nonuniformities of boundaries, the variable-coefficient nonlinear evolution equations can be used to describe the real physical settings [31]. It is remarked that wave models with varying coefficients possess richer phenomena than those with constant coefficients. The constant coefficient nonlinear equation only approximately describe the dynamics of the physical system, however, the variable coefficient nonlinear equation can precisely describe the system's properties [32]. The finding of their solutions provides fundamental understanding of complex nonlinear phenomena arising in various dynamical systems. Furthermore, exact solutions allow one to calculate certain important physical quantities analytically as well as serving as diagnostics for numerical simulations [33]. It should be noted that the study of solitary waves in inhomogeneous systems has a number of interesting features that extend the results of the simplest case of homogeneous systems.

A variety of powerful methods, such as Backlund transformation, the inverse scattering method, bilinear transformation, the subsidiary ordinary differential equation method [34, 35], solitary wave ansatz method [36, 37], sine-cosine method [38, 39], and F-expansion method [40] were used to investigate various nonlinear dispersive and dissipative problems.

The Fisher's equation [41]-[44]

$$u_t = u_{xx} + u\,(1-u)\,, \tag{1}$$

describes the process of interaction between diffusion and reaction [45]. Fisher proposed this equation as a model for the propagation of a mutant gene with $u(x,t)$ denoting the density of advantageous. This equation is encountered in chemical kinetics and population dynamics, which includes problems such as nonlinear evolution of a population in one-dimensional habitual and neutron population in a nuclear reaction [46, 47]. Quite recently, the applicability of the Fisher equation to bacterial population dynamics is studied with the help of explicit analytic solutions for the spatial distribution of a stationary bacterial population under a static mask [48].

In this work, we further generalize this equation by considering the case of arbitrary power nonlinearity and time-varying coefficients so that

$$u_t + \alpha(t)u_x = \beta(t)u_{xx} + \gamma(t)u\,(1-u^n)\,, \tag{2}$$

where $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ are arbitrary functions of the time variable t, while n is any positive integer. Here the effect of linear dispersion is taken into account by the term $\alpha(t)\,u_x$. The latter can be removed by a suitable change of the coordinate frame, but here it has been kept explicitly to contrast its effect with the nonlinear and dispersion terms.

The model (2) describes nonlinear wave propagation in inhomogenous media governed by the Fisher-type equation. The Fisher equation with variable coefficients is one of the most important generic models that have captured attention of many researchers in different fields of physics [49]-[51]. In Ref. [50], the phenomenon of wavefront jump or ignition ahead of the reaction front for a piecewise constant reaction rate has been analyzed by using the generalized Fisher equation involving diffusion transport with a finite velocity and a spatially nonuniform reaction rate. Besides, the variable coefficient Fisher equation is used to describe nonlinear phenomena such as thermonuclear fusion and hydronium physics, etc. [32]. Notice that a generalization of Fisher's equation involving memory effects where the decay time of the memory function is an exponential function in time is used to study memory effects in transport [51].

Generally, Eq. (2) is not integrable by the inverse scattering method. It is of interest to construct exact solutions for this wave equation for a better understanding of nonlinear phenomena arising in dynamical systems described by this model.

The present paper is concerned with traveling wave solutions of (2). Solitons and periodic solutions will be determined using the modified sine-cosine method under various circumstances. We will show that the change of the parameters will drastically change the solitary wave profile which can exhibit interesting structures. For completeness, we also find certain soliton-typed explicit solutions by means of the auxiliary equation method. To our knowledge, the study of the generalized Fisher equation with time-dependent coefficients has not been widespread. Particularly, the investigation of solitary wave dynamics in the Fisher-type equation with arbitrary power nonlinearities and variable coefficients has not been previously presented.

2. TRAVELING WAVE SOLUTIONS

In this section, we construct analytic solutions of the generalized wave equation (2) using the modified sine-cosine method and the auxiliary equation method, in the presence of the nonlinear term of any order. The results can therefore be used in a variety of many types of nonlinearity. The effects of explicit time-dependence of model parameters will be also discussed.

2.1. THE SINE-COSINE METHOD

To reach the goal of finding analytically exact solutions of (2), we first use the modified sine-cosine method which admits the use of the assumptions [39, 52]

$$u(x,t) = \lambda(t)\cos^p(\mu\xi), \quad \xi = x - c(t)t \tag{3}$$

and

$$u(x,t) = \lambda(t)\sin^p(\mu\xi), \quad \xi = x - c(t)t \tag{4}$$

for some parameters $\lambda(t)$, μ, and p that are to be determined. Here μ is the wave number and $c(t)$ is the wave speed.
The assumption (3) gives

$$u_t = \frac{d\lambda(t)}{dt}\cos^p(\mu\xi) - \lambda(t)p\mu\left(-c(t) - \frac{dc(t)}{dt}t\right)\cos^{p-1}(\mu\xi)\sin(\mu\xi), \tag{5}$$

$$u_x = -\lambda(t)p\mu\cos^{p-1}(\mu\xi)\sin(\mu\xi), \tag{6}$$

$$u_{xx} = \lambda(t)\mu^2 p(p-1)\cos^{p-2}(\mu\xi) - \lambda(t)\mu^2 p^2\cos^p(\mu\xi), \tag{7}$$

and the assumption (4) gives

$$u_t = \frac{d\lambda(t)}{dt}\sin^p(\mu\xi) + \lambda(t)p\mu\left(-c(t) - \frac{dc(t)}{dt}t\right)\sin^{p-1}(\mu\xi)\cos(\mu\xi), \tag{8}$$

$$u_x = \lambda(t)p\mu\sin^{p-1}(\mu\xi)\cos(\mu\xi), \tag{9}$$

$$u_{xx} = \lambda(t)\mu^2 p(p-1)\sin^{p-2}(\mu\xi) - \lambda(t)\mu^2 p^2\sin^p(\mu\xi), \tag{10}$$

Substitution of (5)-(7) into the wave equation (2) leads

$$\begin{aligned}
&\frac{d\lambda(t)}{dt}\cos^p(\mu\xi) - \lambda(t)p\mu\left(-c(t) - \frac{dc(t)}{dt}t\right)\cos^{p-1}(\mu\xi)\sin(\mu\xi)\\
&-\alpha(t)\lambda(t)p\mu\cos^{p-1}(\mu\xi)\sin(\mu\xi) - \beta(t)\lambda(t)\mu^2 p(p-1)\cos^{p-2}(\mu\xi)\\
&+\beta(t)\lambda(t)\mu^2 p^2\cos^p(\mu\xi) - \gamma(t)\lambda(t)\cos^p(\mu\xi) + \gamma(t)\lambda^{n+1}(t)\cos^{p(n+1)}(\mu\xi) = 0.
\end{aligned} \tag{11}$$

Equating the exponents and the coefficients of like powers of cosine function in (11) leads to

$$p(p-1) \neq 0, \tag{12}$$

$$p - 2 = p(n+1), \tag{13}$$

$$\frac{d\lambda(t)}{dt} + \beta(t)\lambda(t)\mu^2 p^2 - \gamma(t)\lambda(t) = 0, \tag{14}$$

$$-\lambda(t)p\mu\left(-c(t)-\frac{dc(t)}{dt}t\right)-\alpha(t)\lambda(t)p\mu=0, \tag{15}$$

$$-\beta(t)\lambda(t)\mu^2 p(p-1)+\gamma(t)\lambda^{n+1}(t)=0, \tag{16}$$

Solving this system yields

$$p\neq 0,1, \tag{17}$$

$$p=-\frac{2}{n}, \tag{18}$$

$$\lambda(t)=\lambda_0\exp\left[\int\left\{\gamma\left(t\right)-\frac{4\mu^2}{n^2}\beta\left(t\right)\right\}dt\right], \tag{19}$$

$$c(t)=\frac{1}{t}\int\alpha(t)dt, \tag{20}$$

$$\lambda(t)=\left[\frac{2\left(n+2\right)\mu^2\beta(t)}{n^2\gamma(t)}\right]^{\frac{1}{n}}, \tag{21}$$

where λ_0 is an integration constant related to the initial pulse amplitude.

Equating the two values of the wave amplitude from (19) and (21) yields the constraining relation:

$$\lambda_0\exp\left[\int\left\{\gamma\left(t\right)-\frac{4\mu^2}{n^2}\beta\left(t\right)\right\}dt\right]=\left[\frac{2\left(n+2\right)\mu^2\beta(t)}{n^2\gamma(t)}\right]^{\frac{1}{n}}, \tag{22}$$

which means that the parameters $\beta(t)$, $\gamma\left(t\right)$, μ, and λ_0 are not independent and the existing solutions are obtained in the framework of this relationship.

Similar results are also obtained by using the sine method (4). This leads to the following periodic solutions

$$u_1(x,t)=\left\{\frac{2\left(n+2\right)\mu^2\beta(t)}{n^2\gamma(t)}\sec^2\left[\mu\left(x-\int\alpha(t)dt\right)\right]\right\}^{\frac{1}{n}}, \tag{23}$$

and

$$u_2(x,t)=\left\{\frac{2\left(n+2\right)\mu^2\beta(t)}{n^2\gamma(t)}\csc^2\left[\mu\left(x-\int\alpha(t)dt\right)\right]\right\}^{\frac{1}{n}}, \tag{24}$$

However, if setting $\mu=i\mu_1$ we obtain the solitons solutions

$$u_3(x,t)=\left\{-\frac{2\left(n+2\right)\mu_1^2\beta(t)}{n^2\gamma(t)}\mathrm{sech}^2\left[\mu_1\left(x-\int\alpha(t)dt\right)\right]\right\}^{\frac{1}{n}}, \tag{25}$$

and

$$u_4(x,t)=\left\{-\frac{2\left(n+2\right)\mu_1^2\beta(t)}{n^2\gamma(t)}\mathrm{csch}^2\left[\mu_1\left(x-\int\alpha(t)dt\right)\right]\right\}^{\frac{1}{n}}. \tag{26}$$

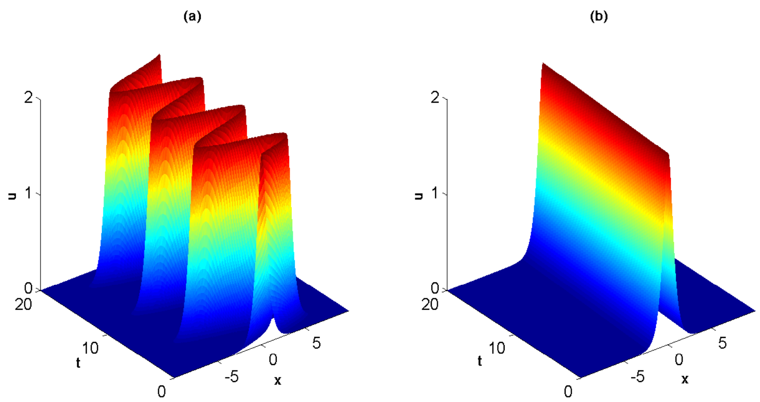

Fig. 1 – Evolution of the solitary wave solution (25) with parameters as follows: $n = 2$, $\mu = 2$, $\beta(t) = -1$, $\gamma(t) = 2$; (a) $\alpha(t) = 4\cos(t)$, (b) $\alpha(t) = 0.05$.

The above solutions clearly indicate that the pulse amplitude is not constant and it is affected by the time-varying coefficients $\beta(t)$ and $\gamma(t)$, while the pulse velocity depends only on the linear dispersion coefficient $\alpha(t)$.

Let us now proceed by investigating the dynamics of solitary waves through the temporal variations of the dependent coefficients $\alpha(t)$, $\beta(t)$, and $\gamma(t)$. In order to understand the influence of such parameters on the propagating waves, here we take the bright-type solitary wave solution (25) as an example. We carefully examine the role of the linear dispersion coefficient $\alpha(t)$ and the quantity $\beta(t)/\gamma(t)$ in the change of the solitary wave profile. We start with the case where $\alpha(t)$ is variant with respect to the time variable t while $\beta(t)/\gamma(t)$ is constant. For a given set of $\alpha(t)$, one can see in Figs. 1 and 2 that the solitary wave takes different shapes while the amplitude and width are not modulated by the variable coefficient $\alpha(t)$.

Let us now fix $\alpha(t)$ and consider the case where $\beta(t)/\gamma(t)$ is a time-varying function. Figure 3 presents the evolution plots of the solution (25) for different functions of $\beta(t)/\gamma(t)$. We see that the soliton profile present a localized structure in $x - t$ plane.

By comparing Figs. 1 and 2 with 3, we find that the quantity $\beta(t)/\gamma(t)$ has a stronger influence on the solitary wave propagation than the dispersion coefficient $\alpha(t)$.

From these results one concludes that, the solitary wave profile can be effectively controlled through the temporal variations of the parameters α and β/γ. We still do not possess a formal proof of its stability; however, detailed stability analyses of this solitary wave under some initial perturbations by employing the numerical

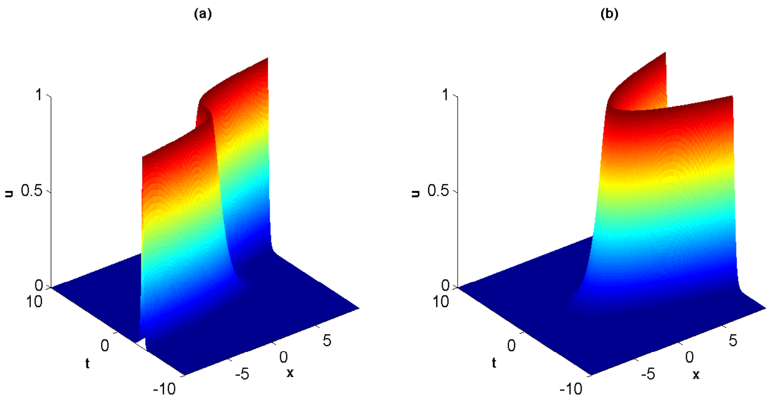

Fig. 2 – Evolution of the solitary wave solution (25) with parameters as follows: $n = 2$, $\mu = 1$, $\beta(t) = -1$, $\gamma(t) = 2$; (a) $\alpha(t) = 0.6t^2$, (b) $\alpha(t) = 0.8t$.

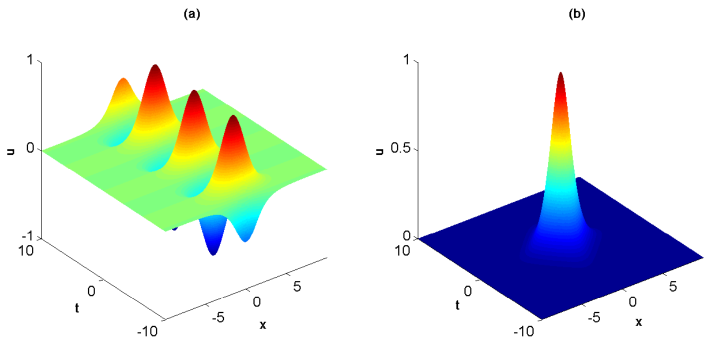

Fig. 3 – Evolution of the solitary wave solution (25) with parameters as follows: $n = 2$, $\mu = 1$, $\alpha(t) = 0.1$; (a) $\beta(t) = 10^{-6}\sin^2(t)$, $\gamma(t) = -2$, (b) $\beta(t) = \mathrm{sech}^2(t)$, $\gamma(t) = -2$.

simulation methods are now under investigation.

2.2. THE AUXILIARY EQUATION METHOD

In what follows, making use of the auxiliary differential technique, we obtain some new soliton-like solutions for the generalized Fisher equation with variable coefficients (2). Let us first introduce the new variable:

$$u(x,t) = v^{\frac{1}{n}}(x,t), \tag{27}$$

Substituting (27) into (2) yields an equation for v as

$$nvv_t + n\alpha(t)vv_x - \beta(t)(1-n)v_x^2 - n\beta(t)vv_{xx} - n^2\gamma(t)v^2 + n^2\gamma(t)v^3 = 0, \tag{28}$$

Next, we adopt the ansatz of Zhao *et al.* [53] of the form

$$v(t) = f(t) + g_1(t)\varphi(\xi) + g_2(t)\varphi^2(\xi), \quad \xi = p(t)x + q(t), \tag{29}$$

where $f(t)$, $g_1(t)$, $g_2(t)$, $p(t)$, and $q(t)$ are all unknown functions of t. We let $\varphi(\xi)$ satisfies the auxiliary ordinary differential equation [54]:

$$\left(\frac{d\varphi}{d\xi}\right)^2 = q_4\varphi^4 + q_3\varphi^3 + q_2\varphi^2, \tag{30}$$

where q_2, q_3, and q_4 are constants. Importantly, the choice of the ansatz solution (29) is dictated by considering the balance between the terms vv_{xx} and v^3 in (28).

Substituting (29) into (2) along with (30), collecting the coefficients of φ^i, where $i = 0,..,6$, and $\varphi^j\varphi'$ to zero, where $j = 0,1,2,3$, and setting them to zero we get the following set of equations:

$$nf\left\{f_t - \gamma nf + \gamma nf^2\right\} = 0, \tag{31}$$

$$n(f_t g_1 + fg_{1t}) - \beta nq_2 fg_1 p^2 + \gamma n^2\left(3f^2 g_1 - 2fg_1\right) = 0, \tag{32}$$

$$\begin{aligned} &n(f_t g_2 + fg_{2t} + g_1 g_{1t}) - \beta(1-n)q_2 g_1^2 p^2 \\ &- \beta np^2\left\{\frac{3}{2}q_3 fg_1 + 4q_2 fg_2 + q_2 g_1^2\right\} \\ &+ \gamma n^2\left(3f^2 g_2 + 3fg_1^2 - g_1^2 - 2fg_2\right) = 0, \end{aligned} \tag{33}$$

$$\begin{aligned} &n(g_1 g_{2t} + g_2 g_{1t}) - \beta p^2(1-n)\left(q_3 g_1^2 + 4q_2 g_1 g_2\right) \\ &- \beta np^2\left\{2q_4 fg_1 + \frac{3}{2}q_3 g_1^2 + 5q_3 fg_2 + 5q_2 g_1 g_2\right\} \\ &+ \gamma n^2\left(g_1^3 + 6fg_1 g_2 - 2g_1 g_2\right) = 0, \end{aligned} \tag{34}$$

$$\begin{aligned}&ng_2g_{2t}-\beta p^2(1-n)\left(4q_2g_2^2+4q_3g_1g_2+q_4g_1^2\right)\\&-\beta np^2\left\{2q_4g_1^2+\frac{13}{2}q_3g_1g_2+6q_4fg_2+4q_2g_2^2\right\}\\&+\gamma n^2\left(3g_1^2g_2+3fg_2^2-g_2^2\right)=0,\end{aligned}\tag{35}$$

$$-4\beta p^2(1-n)\left(q_3g_2^2+q_4g_1g_2\right)-\beta np^2\left(8q_4g_1g_2+5q_3g_2^2\right)+3\gamma n^2g_1g_2^2=0,\tag{36}$$

$$-4\beta(1-n)q_4g_2^2p^2-6\beta np^2q_4g_2^2+\gamma n^2g_2^3=0,\tag{37}$$

$$nfg_1\left[p_tx+q_t+\alpha p\right]=0,\tag{38}$$

$$n\left(2fg_2+g_1^2\right)\left[p_tx+q_t+\alpha p\right]=0,\tag{39}$$

$$3ng_1g_2\left[p_tx+q_t+\alpha p\right]=0,\tag{40}$$

$$2ng_2^2\left[p_tx+q_t+\alpha p\right]=0.\tag{41}$$

where the indice t represent the derivative with respect to the time variable.
By solving the above algebraic equations we obtain explicit expressions of the wave parameters as

$$f(t)=\frac{e^{n\int\gamma(t)dt}}{1+e^{n\int\gamma(t)dt}},\tag{42}$$

$$g_1(t)=\frac{q_3(n+2)e^{n\int\gamma(t)dt}\left(1+e^{n\int\gamma(t)dt}\right)^{-1}}{nq_2+q_3(n+2)\left(1+e^{n\int\gamma(t)dt}\right)},\tag{43}$$

$$g_2(t)=\frac{8q_4^2(n+2)e^{n\int\gamma(t)dt}\left(1+e^{n\int\gamma(t)dt}\right)^{-1}}{16nq_2q_4-(4-n)q_3^2+8q_4^2(n+2)\left(1+e^{n\int\gamma(t)dt}\right)},\tag{44}$$

$$p(t)=\sqrt{\frac{\gamma n^2}{2q_4\beta(n+2)}g_2},\tag{45}$$

and

$$q(t)=\sqrt{\frac{\gamma n^2}{2q_4\beta(n+2)}}\times\int dtQ(t)-\int dt\alpha(t)p(t),\tag{46}$$

where the function $Q(t)$ can be easily expressed in terms of the functions $f(t)$, $g_1(t)$, and $g_2(t)$.

Also, we are left with a constraining relation

$$g_1(t) = \frac{q_3}{2q_4} g_2(t). \tag{47}$$

Using the results for $g_1(t)$ and $g_2(t)$, the constraining relation yields

$$(4-n)q_3^2 = 12nq_2q_4 + 4q_4(n+2)(2q_4 - q_3)\left(1 + e^{n\int\gamma(t)dt}\right). \tag{48}$$

Regarding the auxiliary ordinary differential equation (30) we can see that it possesses the following exact solutions [54]:
(i) When $q_2 > 0$:

$$\varphi(\xi) \equiv \varphi_1(\xi) = \frac{-q_2q_3\text{sech}^2\left(\pm\frac{\sqrt{q_2}}{2}\xi\right)}{q_3^2 - q_2q_4\left(1 - \tanh\left(\pm\frac{\sqrt{q_2}}{2}\xi\right)\right)^2}, \tag{49}$$

(ii) When $q_3^2 - 4q_2q_4 > 0$, $q_2 > 0$:

$$\varphi(\xi) \equiv \varphi_2(\xi) = \frac{2q_2\text{sech}\left(\sqrt{q_2}\xi\right)}{\sqrt{q_3^2 - 4q_2q_4} - q_3\text{sech}\left(\sqrt{q_2}\xi\right)}, \tag{50}$$

Substituting the expressions (42)-(46) into (29), inserting the resulting equation into (27), and considering the explicit solutions (49) and (50), we obtain the following new soliton-like solutions for (2):
Type 1: Taking the solution (49), we get a first new soliton-like solution for (2):

$$\begin{aligned} u = \Bigg\{ & \frac{e^{n\int\gamma(t)dt}}{1 + e^{n\int\gamma(t)dt}} + \frac{q_3(n+2)e^{n\int\gamma(t)dt}\left(1 + e^{n\int\gamma(t)dt}\right)^{-1}}{nq_2 + q_3(n+2)\left(1 + e^{n\int\gamma(t)dt}\right)}\varphi_1(\xi) \\ & + \frac{8q_4^2(n+2)e^{n\int\gamma(t)dt}\left(1 + e^{n\int\gamma(t)dt}\right)^{-1}}{16nq_2q_4 - (4-n)q_3^2 + 8q_4^2(n+2)\left(1 + e^{n\int\gamma(t)dt}\right)}\left[\varphi_1(\xi)\right]^2 \Bigg\}^{\frac{1}{n}}, \end{aligned} \tag{51}$$

which exist provided that $q_2 > 0$ and n is any positive integer.

Type 2: Choosing the solution (50), we obtain another new soliton-like solution for

(2):

$$u=\left\{\frac{e^{n\int\gamma(t)dt}}{1+e^{n\int\gamma(t)dt}}+\frac{q_3(n+2)e^{n\int\gamma(t)dt}\left(1+e^{n\int\gamma(t)dt}\right)^{-1}}{nq_2+q_3(n+2)\left(1+e^{n\int\gamma(t)dt}\right)}\varphi_2(\xi)\right.$$
$$\left.+\frac{8q_4^2(n+2)e^{n\int\gamma(t)dt}\left(1+e^{n\int\gamma(t)dt}\right)^{-1}}{16nq_2q_4-(4-n)q_3^2+8q_4^2(n+2)\left(1+e^{n\int\gamma(t)dt}\right)}\left[\varphi_2(\xi)\right]^2\right\}^{\frac{1}{n}}, \tag{52}$$

which exist provided that $q_3^2-4q_2q_4>0$, $q_2>0$ and n is any positive integer.

3. CONCLUSION

The modified sine-cosine method is employed to investigate a generalized Fisher equation with a nonlinear term of any order and variable coefficients. Exact solitons and periodic wave solutions were determined. Conditions for the existence of these solutions have been reported. The auxiliary differential equation technique is also used to obtain some new soliton-like solutions for the model. Numerical experiments with various functional forms for the variable coefficients have shown that the soliton's shape can be effectively controlled through the temporal variations of these parameters. In particular, new and interesting solitary wave profiles are obtained and shown to depend on the varying coefficients that must be appropriately chosen. The solutions can effectively be used to investigate related nonlinear physical problems where this equation arises.

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