

APPLICATIONS OF MAPLE SOFTWARE TO DERIVE EXACT SOLUTIONS  
OF GENERALIZED FIFTH-ORDER KORTEWEG-DE VRIES EQUATION  
WITH TIME-DEPENDENT COEFFICIENTS

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*Abstract.* In this paper, we present the applications of Maple Software to study the generalized fifth-order Korteweg-de Vries equation with time-dependent coefficients, which arises in modeling of physical phenomena. The nonlinear evolution equation  $u_t + \alpha(t)u^2u_x + \beta(t)u_xu_{xx} + \gamma(t)uu_{xxx} + \delta(t)u_{xxxx} = 0$ , where  $u$  is function of  $x$  and  $t$ , describes the interaction between a water wave and a floating ice cover and gravity-capillary wave. We apply the Lie group formalism to construct some new and physically important solutions.

*Key words:* Partial Differential Equations, Lie Classical Method, Exact solutions.

## 1. INTRODUCTION

Nonlinear partial differential equations (NPDEs) describe a wide variety of phenomena in hydrodynamics (*e.g.*, water surface gravity waves), plasma physics, nonlinear optics and photonics, Bose-Einstein condensation and several other fields; see, for example, a few recent reviews papers in these areas [1]-[13]. Due to the complexity of nonlinearity, obtaining the explicit and exact solutions for a real nonlinear physical equation is often difficult. Exact solutions of NPDEs provide much physical information and more insight into the physical aspects of the problem that lead to further applications of these NPDEs. The generalized fifth order Korteweg-de Vries (KdV) equation with time dependent variable coefficient  $t$  is

$$u_t + \alpha(t)u^2u_x + \beta(t)u_xu_{xx} + \gamma(t)uu_{xxx} + \delta(t)u_{xxxx} = 0, \quad (1)$$

where  $u$  is function of  $x$  and  $t$ . The generalized fifth-order KdV (fKdV) equation is a model equation for plasma waves, capillary-gravity water waves, and other dispersive phenomena when the cubic KdV-type dispersion is weak. The fKdV equation (1) describes the motion of long waves in shallow water under gravity and the one-dimensional nonlinear lattices. The nonlinear fifth-order KdV equation (1) is one of

the important mathematical models with wide range of applications in quantum mechanics. Zhibin *et al.* [14], studied the generalized fifth-order KdV model equation with constant coefficients and obtained solitary wave and soliton solutions. Wazwaz [15] found soliton solutions for several forms of fifth-order KdV equation with constant coefficients. Wazwaz [16] derived soliton solutions of variable coefficients fifth order KdV.

Nonlinear integrable equations with constant coefficients describe highly idealized physical systems. Therefore, equations with variable coefficients may provide various models for real physical phenomena, such as, in the propagation of small-amplitude surface waves, which runs on straits or large channels of slowly varying depth. As there are choices for parameters, the variable-coefficient nonlinear equations can be considered as generalization of the constant coefficients equations. The nonlinear wave equations with variable-coefficients are more realistic in various physical situations than their constant coefficients, since in realistic physical systems, no medium is homogenous due to the existence of inhomogeneities and non uniformities of boundaries. Here, if  $\alpha(t), \beta(t), \gamma(t)$  are constants and  $\delta(t) = 1$ , then the above fifth order evolution equations include well-known Kaup-Kupershmidt (KK), Lax, Sawada-Kotera (SK), Caudrey-Dodd-Gibbon (CDG) and Ito equations. For example when:

- (i)  $(\alpha, \beta, \gamma)$  are  $(30, 60, 270)$  and  $(20, 40, 120)$ , equation (1) is the fifth-order KdV equation, examined in Ref. [17].
- (ii)  $(\alpha, \beta, \gamma)$  are  $(5, 5, 5)$  and  $(-15, -15, 45)$ , equation (1) is the Sawada-Kotera equation, examined in Ref. [18, 19].
- (iii)  $(\alpha, \beta, \gamma)$  are  $(30, 75, 180)$  and  $(10, 25, 20)$ , equation (1) is the Kaup-Kupershmidt equation, examined in Ref. [20, 21].
- (iv)  $(\alpha, \beta, \gamma)$  are  $(30, 20, 10)$ , equation (1) is the Lax equation, examined in Ref. [22].
- (v)  $(\alpha, \beta, \gamma)$  are  $(2, 6, 3)$ , equation (1) is the Ito equation, examined in Ref. [23].
- (vi)  $(\alpha, \beta, \gamma)$  are  $(180, 30, 30)$ , equation (1) is the Caudrey-Dodd-Gibbon equation, examined in Ref. [24].

For instance, the Lax equation and the SK equation are completely integrable. These two equations generate  $N$ -soliton solutions. Another example is the KK equation which is known to be integrable and has bilinear representations. A fourth equation in this class is the Ito equation that is not completely integrable, but has a limited number of special conserved densities. Obviously, for arbitrary values of the constants  $\alpha, \beta$ , and  $\gamma$ , the Eq. (1) is not completely integrable, and therefore does not admit soliton solutions, which does not exclude the existence of solitary wave solutions. Note that with scales on  $u, x$  and  $t$ , the equations cannot be transformed into one another; they are fundamentally different.

Various methods have been developed to obtain solutions of nonlinear equations with constant coefficients and with time-dependent coefficients. Some of these

methods are Hirota bilinear method [25], the Bäcklund transformation method [26], the Darboux transformation [27], the inverse scattering method [28], Painlevé analysis [29], sine-cosine method [30], the sech-tanh method [31], the generalized symmetry method [32] and so on.

In this paper, we derive exact analytic solutions of variable-coefficient generalized fKdV equation. All the physical parameters in the solutions are evaluated as functions of varying coefficients. The exact analytic solutions of generalized fKdV equation with time-dependent coefficients have not been obtained yet. Moreover, by direct integration, it is difficult to solve the fKdV equation with higher degree nonlinear terms with variable coefficients. Exact solutions of nonlinear evolution equations are helpful for understanding the physical behavior of nonlinear phenomena and dynamical process modeled by these nonlinear models. Although, nonlinear models may possess constant or variable coefficients, which provide a rich variety of shape preserving waves and a series of interesting properties.

The paper is organized as follows. Section 2 is devoted to the Lie classical method, the symmetries of generalized variable-coefficient fKdV equation and the optimal system of basic vector fields. In Sec. 3, we derive similarity variables and similarity solutions. Corresponding to these similarity variables, some reduced ordinary differential equations (ODEs) and their exact solutions have been obtained. Some more exact solutions of nonlinear equation (1) are obtained by using  $(\frac{G'}{G})$  method in Sec. 4. In Sec. 5, some conclusions are drawn.

## 2. LIE CLASSICAL ANALYSIS OF VARIABLE COEFFICIENTS FKDV EQUATION

The Lie group method is also called *symmetry analysis*. A symmetry of a differential equation is an invertible transformation of the dependent and independent variables that does not change the original differential equation. A symmetry group of a system of differential equations is a group which transforms solutions of the system to other solutions. Once one has determined the symmetry group of a system of differential equations, a number of applications become available. Symmetries depend continuously on a parameter and form a group: the one-parameter group of transformations. The Lie classical method has been used to examine exact solutions of various NPDEs [33–44]. To start with, one can directly use the defining property of such a group and construct new solutions to the system from known ones. First, let us consider a one-parameter Lie group of infinitesimal transformations:

$$\begin{aligned} u^* &= u + \epsilon\phi(x, t, u) + O(\epsilon^2) \\ x^* &= x + \epsilon\xi_1(x, t, u) + O(\epsilon^2) \\ t^* &= t + \epsilon\xi_2(x, t, u) + O(\epsilon^2), \end{aligned} \tag{2}$$

which leaves the system (1) invariant. In other words, the transformations are such that if  $u$  is a solution of equation (1), then  $u^*$  is also a solution. The method for determining the symmetry group of (1) consists of finding the infinitesimals  $\phi, \xi_1$ , and  $\xi_2$ , which are functions of  $x, t, u$ . Assuming that the system (1) is invariant under the transformations (2), we get the following relations from the coefficients of the first order of  $\epsilon$ :

$$\begin{aligned} &\phi^t + \alpha(t)u^2\phi^x + 2\alpha(t)\phi uu_x + \xi_2\alpha'(t)u^2u_x + \beta(t)u_x\phi^{xx} + \beta(t)\phi^x u_{xx} \\ &+ \xi_2\beta'(t)u_x u_{xx} + \gamma(t)\phi u_{xxx} + \xi_2\gamma'(t)u\phi^{xxx} + \xi_2\gamma'(t)uu_{xx} \\ &+ \delta(t)\phi^{xxxx} + \delta'(t)u_x xxx = 0, \end{aligned} \quad (3)$$

where  $\phi^x, \phi^{xx}, \phi^{xxx}$ , and  $\phi^{xxxx}$  are extended (prolonged) infinitesimals acting on an enlarged space that includes all derivatives of the dependent variables  $u_x, u_{xx}, u_{xxx}, u_{xxx}$ , and  $u_{xxxx}$  (for more details the readers can refer to [35]). The infinitesimals are determined from invariance condition (3), by setting the coefficients of different differentials equal to zero. We obtain a large number of PDEs in  $\phi, \xi_1$  and  $\xi_2$  that need to be satisfied. The general solution of this large system provides the following forms for the infinitesimal elements  $\phi, \xi_1$ , and  $\xi_2$ :

$$\begin{aligned} \xi_1 &= k_1x + k_2, \\ \xi_2 &= \frac{\int k_1\alpha(t)dt + k_3}{\alpha(t)}, \\ \phi &= 0, \end{aligned} \quad (4)$$

and  $\alpha(t), \beta(t), \gamma(t), \delta(t)$  are obtained from the following equations:

$$\begin{aligned} \xi_2\alpha'(t) - \alpha(t)\xi_{1x} + \alpha(t)\xi_{2t} &= 0, \\ \xi_2\beta'(t) - 3\beta(t)\xi_{1x} + \beta(t)\xi_{2t} &= 0, \\ \xi_2\gamma'(t) - 3\gamma(t)\xi_{1x} + \gamma(t)\xi_{2t} &= 0, \\ \xi_2\delta'(t) - 5\delta(t)\xi_{1x} + \delta(t)\xi_{2t} &= 0. \end{aligned} \quad (5)$$

where  $k_1, k_2$ , and  $k_3$  are arbitrary constants.

The Lie algebra associated with the equation (1) consists of the following three vector fields:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x} \\ X_2 &= \frac{1}{\alpha(t)} \frac{\partial}{\partial t} \\ X_3 &= x \frac{\partial}{\partial x} + \frac{\int \alpha(t)dt}{\alpha(t)} \frac{\partial}{\partial t}. \end{aligned} \quad (6)$$

and  $\beta(t), \gamma(t)$  and  $\delta(t)$  are given as:

$$\begin{aligned} \beta(t) &= c_1(k_1 \int \alpha(t)dt + k_3)^2 \alpha(t), \\ \gamma(t) &= c_2(k_1 \int \alpha(t)dt + k_3)^2 \alpha(t), \\ \delta(t) &= c_3(k_1 \int \alpha(t)dt + k_3)^4 \alpha(t), \end{aligned} \quad (7)$$

where  $c_1, c_2$ , and  $c_3$  are arbitrary constants.

## 2.1. OPTIMAL SYSTEM OF GENERATORS

A relation between two invariant solutions can be defined to hold true if the first one can be mapped to the other by applying a transformation group generated by a correct linear combination of the symmetry generators (operators) (6). The relation is an equivalence relation, since it is reflexive, symmetric, and transitive, which induces a natural partition on the set of all group invariant solutions into equivalence classes. We need only present one solution from each equivalence class, as the rest may be found by applying appropriate group symmetries; a complete set of such solutions is referred to as an "optimal system" of group invariant solutions. The problem of deriving an optimal system of group invariant solutions is equivalent to find an optimal system of generators or subalgebras spanned by these generators (or operators). The method used here is given by Ovsianikov and Olver in [39, 45], which basically consist of taking linear combinations of the generators (6), and reducing them to their simplest equivalent form by applying carefully chosen adjoint transformations. After following the method for optimal system of generators, we have the optimal system for equation (1):

$$\begin{aligned} & (i) X_3, \\ & (ii) X_2 + X_1, \\ & (iii) X_2 - X_1, \\ & (iv) X_2, \\ & (v) X_1. \end{aligned} \tag{8}$$

Because of the symmetry  $(x; t; u) \rightarrow (-x; t; u)$ , (ii) will map to (iii), thus in the optimal system, we examine reductions and exact solutions of the remaining four essential vector fields of the optimal system. Using these basic vector fields one can obtain reductions of equation (1) to ODEs after getting the similarity variable and similarity solution by solving the characteristic equations

$$\frac{dx}{\xi_1} = \frac{dt}{\xi_2} = \frac{du}{\phi}. \tag{9}$$

## 2.2. REDUCTION AND EXACT SOLUTIONS OF VARIABLE-COEFFICIENT FKDV EQUATION

One of the main purpose for calculating symmetry is to use them for obtaining symmetry reductions and similarity solutions. The goal of this subsection is to apply the symmetries calculated in the previous subsection to obtain symmetry reductions and exact solutions whenever it is possible.

**Vector field** (i)  $X_3$

Corresponding to this vector field, the forms of the similarity variable and similarity solution are as follows:  $\zeta = \frac{x}{\int \alpha(t) dt}$ ,  $u(x, t) = F(\zeta)$ .

and coefficient functions  $\beta(t)$ ,  $\gamma(t)$  and  $\delta(t)$ , for this vector field are given as:

$$\begin{aligned}\beta(t) &= c_1 \left( \int \alpha(t) dt \right)^2 \alpha(t), \\ \gamma(t) &= c_2 \left( \int \alpha(t) dt \right)^2 \alpha(t), \\ \delta(t) &= c_3 \left( \int \alpha(t) dt \right)^4 \alpha(t),\end{aligned}\tag{10}$$

where  $c_1, c_2$ , and  $c_3$  are arbitrary constants. Using these substitutions in equation (1), we get the following reduced ODE:

$$-\zeta F'(\zeta) + F(\zeta)^2 F'(\zeta) + c_1 F'(\zeta) F''(\zeta) + c_2 F(\zeta) F'''(\zeta) + c_3 F''''(\zeta) = 0, \tag{11}$$

Now, we evaluate the solution of equation (11) in the power series of the form

$$F(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^n. \tag{12}$$

Substituting (12) into (11) and comparing coefficients, we have

$$\begin{aligned}a_{n+5} &= \frac{1}{c_3(n+1)(n+2)(n+3)(n+4)(n+5)} \left( -na_n - \sum_{k=0}^n \sum_{j=0}^k (n+1-k) a_j a_{k-j} a_{n+1-k} \right. \\ &\quad - c_1 \sum_{k=0}^n (n+2-k)(n+1-k)(k+1) a_{k+1} a_{n+2-k} \\ &\quad \left. - c_2 \sum_{k=0}^n (n+3-k)(n+2-k)(n+1-k) a_{n+3-k} a_k \right),\end{aligned}\tag{13}$$

$n = 0, 1, 2, \dots$

From (13), we can get all coefficients  $a_n, n \geq 1$  of power series (11)

$$\begin{aligned}a_5 &= \frac{1}{120c_3} (-a_0^2 a_1 - 2c_1 a_1 a_2 - 6c_2 a_3 a_0), \\ a_6 &= \frac{1}{720c_3} (-a_1 - 2(a_0^2 a_2 + a_0 a_1^2) - c_1(6a_1 a_3 + 4a_2^2) - c_2(24a_4 a_0 + 6a_3 a_1)), \\ a_7 &= \frac{1}{2520c_3} (-2a_2 - (3a_0^2 a_3 + 6a_0 a_1 a_2 + a_1^3) - c_1(12a_1 a_4 + 18a_2 a_3) \\ &\quad - c_2(60a_5 a_0 + 24a_4 a_1 + 6a_3 a_2)),\end{aligned}\tag{14}$$

and so on. Hence, for arbitrary chosen constant numbers  $a_0, a_1, a_2, a_3$ , and  $a_4$  the other terms of the sequence  $\{a_n\}_{n=0}^{\infty}$  can be determined successively from (13) in a unique manner. This implies that for equation (11), there exists a power series solution (12) with the coefficients given by (13).

Now, the power series solution of (11) can be written as

$$\begin{aligned}
u(x, t) = & a_0 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3 + a_4\zeta^4 + \frac{1}{120c_3}(-a_0^2a_1 - 2c_1a_1a_2 - 6c_2a_3a_0)\zeta^5 \\
& + \sum_{n=1}^{\infty} \frac{1}{c_3(n+1)(n+2)(n+3)(n+4)(n+5)} \left( -na_n - \sum_{k=0}^n \sum_{j=0}^k (n+1-k)a_j a_{k-j} a_{n+1-k} \right. \\
& - c_1 \sum_{k=0}^n (n+2-k)(n+1-k)(k+1)a_{k+1}a_{n+2-k} - \\
& \left. c_2 \sum_{k=0}^n (n+3-k)(n+2-k)(n+1-k)a_{n+3-k}a_k \right) \zeta^{n+5}
\end{aligned} \tag{15}$$

The exact solution of fKdV equation (1) with variable coefficients is

$$\begin{aligned}
u(x, t) = & a_0 + a_1 \left( \int \frac{x}{\alpha(t)dt} \right) + a_2 \left( \int \frac{x}{\alpha(t)dt} \right)^2 + a_3 \left( \int \frac{x}{\alpha(t)dt} \right)^3 + a_4 \left( \int \frac{x}{\alpha(t)dt} \right)^4 \\
& + \frac{1}{120c_3}(-a_0^2a_1 - 2c_1a_1a_2 - 6c_2a_3a_0) \left( \int \frac{x}{\alpha(t)dt} \right)^5 + \sum_{n=1}^{\infty} \frac{1}{c_3(n+1)(n+2)(n+3)(n+4)(n+5)} \\
& \left( -na_n - \sum_{k=0}^n \sum_{j=0}^k (n+1-k)a_j a_{k-j} a_{n+1-k} - c_1 \sum_{k=0}^n (n+2-k)(n+1-k)(k+1) \right. \\
& \left. a_{k+1}a_{n+2-k} - c_2 \sum_{k=0}^n (n+3-k)(n+2-k)(n+1-k)a_{n+3-k}a_k \right) \left( \int \frac{x}{\alpha(t)dt} \right)^{n+5},
\end{aligned} \tag{16}$$

where  $a_0, a_1, a_2, a_3, a_4, c_1, c_2,$  and  $c_3$  are arbitrary constants.

**Vector field** (ii)  $X_2 + X_1$

Corresponding to this vector field, the forms of the similarity variable and similarity solution are as follows:  $\zeta = x - \int \alpha(t)dt$ ,  $u(x, t) = F(\zeta)$ , and coefficient functions for this vector field are

$$\begin{aligned}
\beta(t) &= c_1\alpha(t), \\
\gamma(t) &= c_2\alpha(t), \\
\delta(t) &= c_3\alpha(t)
\end{aligned} \tag{17}$$

Using these substitutions in equation (1), we get the following reduced ODE:

$$-F'(\zeta) + F(\zeta)^2 F'(\zeta) + c_1 F'(\zeta) F''(\zeta) + c_2 F(\zeta) F'''(\zeta) + c_3 F''''(\zeta) = 0 \tag{18}$$

On solving this ODE, we have

$$F(\zeta) = \frac{24c_3c_4^4 + 1}{c_4^2c_1} - \frac{3(24c_3c_4^4 + 1) \tanh(c_5 + c_4\zeta)^2}{c_4^2c_1}, \tag{19}$$

where  $c_2 = \frac{64c_4^8c_3c_1^2 + 576c_4^8c_3^2 - 4c_1^2c_4^4 + 48c_3c_4^4 + 1}{8c_1c_4^2(24c_3c_4^4 + 1)}$  and hence, the solution of the fKdV equation (1) with variable coefficients is

$$u(x, t) = \frac{24c_3c_4^4 + 1}{c_4^2c_1} - \frac{3(24c_3c_4^4 + 1) \tanh(c_5 + c_4(x - \int \alpha(t)dt))^2}{c_4^2c_1}, \tag{20}$$

where  $c_2$  is given as above and  $c_1, c_3, c_4$  are arbitrary constants.

**Vector field (iii)  $X_2$** 

Corresponding to this vector field, we get only constant solution.

**Vector field (iv)  $X_1$** 

Corresponding to this vector field, the forms of the similarity variable and similarity

solution are as follows:  $\zeta = x$ ,  $u(x, t) = F(\zeta)$ ,

and coefficient functions for vector field  $X_2$  are

$$\begin{aligned}\beta(t) &= c_1\alpha(t), \\ \gamma(t) &= c_2\alpha(t), \\ \delta(t) &= c_3\alpha(t)\end{aligned}\tag{21}$$

Using these substitutions in equation (1), we obtain the following reduced ODE:

$$F(\zeta)^2 F'(\zeta) + c_1 F'(\zeta) F''(\zeta) + c_2 F(\zeta) F'''(\zeta) + c_3 F''''(\zeta) = 0\tag{22}$$

After solving this ODE, we have

$$F(\zeta) = c_4^2 \left( 8c_2 - \frac{8}{3}c_1 + (-12c_2 + 4c_1) \tanh(c_3 + c_4(\zeta))^2 \right),\tag{23}$$

where  $c_2 = -\frac{1}{9}c_1^2 + \frac{1}{3}c_1c_3$ .

The exact solution for this vector field of the fKdV equation with variable coefficients is

$$u(x, t) = c_4^2 \left( 8c_2 - \frac{8}{3}c_1 + (-12c_2 + 4c_1) \tanh(c_3 + c_4(x))^2 \right),\tag{24}$$

where  $c_2$  is given as above and  $c_1, c_3, c_4$  are arbitrary constants.

### 3. SOME MORE EXACT SOLUTIONS OF VARIABLE-COEFFICIENT FKDV EQUATION BY THE $\left(\frac{G'}{G}\right)$ METHOD

The  $\left(\frac{G'}{G}\right)$ -expansion method to find exact solutions of NPDEs has been recently developed. Some of the recent work for finding exact solutions of NPDEs by using this method is given in [46–49]. Here, we will obtain some more exact solutions of variable-coefficient fKdV equation *via* generalized  $\left(\frac{G'}{G}\right)$ -expansion method. To study the traveling wave solutions of equation (1), we take a plane wave transformation in the form

$$u(x, t) = u(\xi), \xi = kx + \int \tau(t) dt.\tag{25}$$

Equation (1) is carried to an ODE, which takes the following form

$$\tau(t)u(\xi) + k\alpha(t)u(\xi)^2u'(\xi) + k^3\beta(t)u'(\xi)u''(\xi) + k^3\gamma(t)u\xi u'''(\xi) + \delta(t)u''''(\xi) = 0,\tag{26}$$



where the prime denotes the differential with respect to  $\xi$ . In view of the  $\left(\frac{G'}{G}\right)$  method, we introduce the anstaz

$$u(\xi) = \sum_{i=0}^n a_i \left(\frac{G'}{G}\right)^i, \quad (27)$$

where  $a_i$  are constants to be determined later and  $G = G(\xi)$  satisfies

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0. \quad (28)$$

Considering the homogeneous balance between  $u(\xi)^2 u'(\xi)$  and  $u''''(\xi)$  in (26), yields  $n = 2$ . The solution of equation (1) can be expressed by

$$u(\xi) = a_0 + a_1 \left(\frac{G'}{G}\right) + a_2 \left(\frac{G'}{G}\right)^2, \quad (29)$$

where  $a_0, a_1$ , and  $a_2$  are constants to be determined. Substituting (29) with (28) into (26), collecting the coefficients of  $\left(\frac{G'}{G}\right)$  we obtain a set of algebraic equations for  $a_0, a_1, a_2$ , and  $\tau(t)$  and solving this system with the aid of Maple Package we obtain the three sets of solutions as

#### Case – I

$$\begin{aligned} a_0 &= \frac{2}{3}a_2\mu + \frac{1}{12}a_2\lambda^2, a_1 = a_2\lambda, \\ \tau(t) &= \frac{k^3((576k^2\mu^2 + 36k^2\lambda^4 - 288k^2\lambda^2\mu)\delta(t) + (16a_2\mu^2 + a_2\lambda^4 - 8a_2\lambda^2\mu)\beta(t))}{24} \\ \alpha(t) &= \frac{-6k^2(60\delta(t)k^2 + \beta(t)a_2 + 2a_2\gamma(t))}{a_2^2}. \end{aligned} \quad (30)$$

#### Case – II

$$\begin{aligned} a_1 &= a_2\lambda, \alpha(t) = -\frac{6k^2\gamma(t)}{a_2}, \beta(t) = -\frac{60k^2\delta(t) + a_2\gamma(t)}{a_2} \\ \tau(t) &= -k^3((k^2\lambda^4 a_2 + 16k^2 a_2 \mu^2 - 8k^2 a_2 \lambda^2 \mu)\delta(t) + (-2a_2^2 \mu^2 - 6a_0^2 + a_0 a_2 \lambda^2 \\ &\quad - a_2^2 \lambda^2 \mu + 8a_0 a_2 \mu)\gamma(t)). \end{aligned} \quad (31)$$

#### Case – III

$$\begin{aligned} a_0 &= \frac{1}{12}a_2(8\mu + \lambda^2), a_1 = a_2\lambda, \alpha(t) = \frac{3k^2(48k^2\delta(t) - a_2\gamma(t))}{a_2^2}, \\ \tau(t) &= -\frac{k^3((512k^2\mu^2 + 32k^2\lambda^4 - 256k^2\lambda^2\mu)\delta(t) + (16a_2\mu^2 + a_2\lambda^4 - 8a_2\lambda^2\mu)\gamma(t))}{16}, \\ \beta(t) &= -\frac{3}{2}\frac{56k^2\delta(t) + a_2\gamma(t)}{a_2}. \end{aligned} \quad (32)$$

From Case-I and equations (28) and (29), we find the following solutions of equation

(1), as follows. When  $\lambda^2 - 4\mu > 0$ , we can obtain the following solutions,

$$\begin{aligned} u_{11}(\xi) &= \frac{2}{3}a_2\mu + \frac{1}{12}a_2\lambda^2 \\ &+ a_2\lambda \left( -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2} \frac{C_1 \sinh \frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi) + C_2 \cosh \frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi)}{C_1 \cosh \frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi) + C_2 \sinh \frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi)} \right) \\ &+ a_2 \left( -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2} \frac{C_1 \sinh \frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi) + C_2 \cosh \frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi)}{C_1 \cosh \frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi) + C_2 \sinh \frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi)} \right)^2, \end{aligned} \quad (33)$$

when  $\lambda^2 - 4\mu < 0$ , we can obtain the following solutions,

$$\begin{aligned} u_{12}(\xi) &= \frac{2}{3}a_2\mu + \frac{1}{12}a_2\lambda^2 \\ &+ a_2\lambda \left( -\frac{\lambda}{2} + \frac{\sqrt{-\lambda^2+4\mu}}{2} \frac{-C_1 \sin \frac{1}{2}(\sqrt{-\lambda^2+4\mu}\xi) + C_2 \cos \frac{1}{2}(\sqrt{-\lambda^2+4\mu}\xi)}{C_1 \cos \frac{1}{2}(\sqrt{-\lambda^2+4\mu}\xi) + C_2 \sin \frac{1}{2}(\sqrt{-\lambda^2+4\mu}\xi)} \right) \\ &+ a_2 \left( -\frac{\lambda}{2} + \frac{\sqrt{-\lambda^2+4\mu}}{2} \frac{-C_1 \sin \frac{1}{2}(\sqrt{-\lambda^2+4\mu}\xi) + C_2 \cos \frac{1}{2}(\sqrt{-\lambda^2+4\mu}\xi)}{C_1 \cos \frac{1}{2}(\sqrt{-\lambda^2+4\mu}\xi) + C_2 \sin \frac{1}{2}(\sqrt{-\lambda^2+4\mu}\xi)} \right)^2, \end{aligned} \quad (34)$$

when  $\lambda^2 - 4\mu = 0$ , we can obtain the following solutions,

$$u_{13}(\xi) = \frac{2}{3}a_2\mu + \frac{1}{12}a_2\lambda^2 + a_2\lambda \left( -\frac{\lambda}{2} + \frac{C_2}{C_1+C_2\xi} \right) + a_2 \left( -\frac{\lambda}{2} + \frac{C_2}{C_1+C_2\xi} \right)^2, \quad (35)$$

where

$$\xi = kx + \frac{1}{24}k^3 \times [(576k^2\mu^2 + 36k^2\lambda^4 - 288k^2\lambda^2\mu) \int \delta(t)dt + (16a_2\mu^2 + a_2\lambda^4 - 8a_2\lambda^2\mu) \int \beta(t)dt].$$

From Case-II and equations (28) and (29), we find the following solutions of equation (1), as follows. When  $\lambda^2 - 4\mu > 0$ , we can obtain the following solutions,

$$\begin{aligned} u_{21}(\xi) &= a_0 + a_2\lambda \left( -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2} \frac{C_1 \sinh \frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi) + C_2 \cosh \frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi)}{C_1 \cosh \frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi) + C_2 \sinh \frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi)} \right) \\ &+ a_2 \left( -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2} \frac{C_1 \sinh \frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi) + C_2 \cosh \frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi)}{C_1 \cosh \frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi) + C_2 \sinh \frac{1}{2}(\sqrt{\lambda^2-4\mu}\xi)} \right)^2, \end{aligned} \quad (36)$$

when  $\lambda^2 - 4\mu < 0$ , we can obtain the following solutions,

$$\begin{aligned} u_{22}(\xi) &= a_0 + a_2\lambda \left( -\frac{\lambda}{2} + \frac{\sqrt{-\lambda^2+4\mu}}{2} \frac{-C_1 \sin \frac{1}{2}(\sqrt{-\lambda^2+4\mu}\xi) + C_2 \cos \frac{1}{2}(\sqrt{-\lambda^2+4\mu}\xi)}{C_1 \cos \frac{1}{2}(\sqrt{-\lambda^2+4\mu}\xi) + C_2 \sin \frac{1}{2}(\sqrt{-\lambda^2+4\mu}\xi)} \right) \\ &+ a_2 \left( -\frac{\lambda}{2} + \frac{\sqrt{-\lambda^2+4\mu}}{2} \frac{-C_1 \sin \frac{1}{2}(\sqrt{-\lambda^2+4\mu}\xi) + C_2 \cos \frac{1}{2}(\sqrt{-\lambda^2+4\mu}\xi)}{C_1 \cos \frac{1}{2}(\sqrt{-\lambda^2+4\mu}\xi) + C_2 \sin \frac{1}{2}(\sqrt{-\lambda^2+4\mu}\xi)} \right)^2, \end{aligned} \quad (37)$$

when  $\lambda^2 - 4\mu = 0$ , we can obtain the following solutions,

$$u_{23}(\xi) = a_0 + a_2\lambda \left( -\frac{\lambda}{2} + \frac{C_2}{C_1+C_2\xi} \right) + a_2 \left( -\frac{\lambda}{2} + \frac{C_2}{C_1+C_2\xi} \right)^2, \quad (38)$$

where  $\xi = kx - k^3((m(x,t) + n(x,t)))$  and

$$m(x,t) = (k^2\lambda^4 a_2 + 16k^2\mu^2 a_2 - 8k^2\lambda^2\mu a_2) \int \delta(t)dt$$

$$n(x,t) = (-2a_2^2\mu^2 - 6a_0^2 + a_0a_2\lambda^2 - a_2^2\lambda^2\mu + 8a_0a_2\mu) \int \gamma(t)dt.$$

Now, from Case-III and equations (28) and (29), we find the following solutions of equation (1), as follows. When  $\lambda^2 - 4\mu > 0$ , we can obtain the following solutions,

$$\begin{aligned} u_{31}(\xi) &= \frac{1}{2}a_2(8\mu + \lambda^2) \\ &+ a_2\lambda \left( -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} \frac{C_1 \sinh \frac{1}{2}(\sqrt{\lambda^2 - 4\mu}\xi) + C_2 \cosh \frac{1}{2}(\sqrt{\lambda^2 - 4\mu}\xi)}{C_1 \cosh \frac{1}{2}(\sqrt{\lambda^2 - 4\mu}\xi) + C_2 \sinh \frac{1}{2}(\sqrt{\lambda^2 - 4\mu}\xi)} \right) \\ &+ a_2 \left( -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} \frac{C_1 \sinh \frac{1}{2}(\sqrt{\lambda^2 - 4\mu}\xi) + C_2 \cosh \frac{1}{2}(\sqrt{\lambda^2 - 4\mu}\xi)}{C_1 \cosh \frac{1}{2}(\sqrt{\lambda^2 - 4\mu}\xi) + C_2 \sinh \frac{1}{2}(\sqrt{\lambda^2 - 4\mu}\xi)} \right)^2, \end{aligned} \quad (39)$$

when  $\lambda^2 - 4\mu < 0$ , we can obtain the following solutions,

$$\begin{aligned} u_{22}(\xi) &= \frac{1}{2}a_2(8\mu + \lambda^2) \\ &+ a_2\lambda \left( -\frac{\lambda}{2} + \frac{\sqrt{-\lambda^2 + 4\mu}}{2} \frac{-C_1 \sin \frac{1}{2}(\sqrt{-\lambda^2 + 4\mu}\xi) + C_2 \cos \frac{1}{2}(\sqrt{-\lambda^2 + 4\mu}\xi)}{C_1 \cos \frac{1}{2}(\sqrt{-\lambda^2 + 4\mu}\xi) + C_2 \sin \frac{1}{2}(\sqrt{-\lambda^2 + 4\mu}\xi)} \right) \\ &+ a_2 \left( -\frac{\lambda}{2} + \frac{\sqrt{-\lambda^2 + 4\mu}}{2} \frac{-C_1 \sin \frac{1}{2}(\sqrt{-\lambda^2 + 4\mu}\xi) + C_2 \cos \frac{1}{2}(\sqrt{-\lambda^2 + 4\mu}\xi)}{C_1 \cos \frac{1}{2}(\sqrt{-\lambda^2 + 4\mu}\xi) + C_2 \sin \frac{1}{2}(\sqrt{-\lambda^2 + 4\mu}\xi)} \right)^2, \end{aligned} \quad (40)$$

when  $\lambda^2 - 4\mu = 0$ , we can obtain the following solutions,

$$u_{23}(\xi) = \frac{1}{2}a_2(8\mu + \lambda^2) + a_2\lambda \left( -\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2\xi} \right) + a_2 \left( -\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2\xi} \right)^2, \quad (41)$$

where

$$\xi = kx - \frac{1}{16}k^3 \times \left[ (512k^2\mu^2 + 32k^2\lambda^4 - 256k^2\lambda^2\mu) \int \delta(t)dt + (16a_2\mu^2 + a_2\lambda^4 - 8a_2\lambda^2\mu) \int \gamma(t)dt \right].$$

#### 4. DISCUSSION AND CONCLUDING REMARKS

In this paper, the variable-coefficient fKdV equation was investigated by using Lie symmetry analysis method. We have obtained similarity reductions and exact solutions based on the Lie group method by generating the group of infinitesimals with the help of Maple Software. Especially, all the similarity reductions and exact solutions are given for the first time in this paper, to the best of our knowledge. The group invariant solutions to the reduced ODEs of variable-coefficient fKdV equation are considered based on optimal system. These similarity solutions possess significant features in physical systems and cannot be derived from the method of dynamical systems.

**Remark 1:** As the nonlinear equation (1) describes physically important nonlinear equations, such as Kaup-Kupershmidt, Lax, Sawada-Kotera, Caudrey-Dodd-Gibbon, and Ito equations, the similarity solutions of these equations can be obtained by substituting particular values of  $\alpha(t)$ ,  $\beta(t)$ ,  $\gamma(t)$ , and  $\delta(t)$ .

**Remark 2:** We have obtained exact explicit solutions of highly nonlinear variable-coefficient fKdV equation by using generalized  $\left(\frac{G'}{G}\right)$ -expansion method, for the first

time to our knowledge. These solutions are expressed in terms of the hyperbolic functions, trigonometric functions, and rational functions. Here, we found new exact solutions that might be useful for applications in mathematical physics and applied mathematics.

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