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APPROXIMATE SOLUTION OF NONLINEAR POISSON EQUATION BY FINITE DIFFERENCES METHOD

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Abstract. In this paper, we investigate a method to deduce the numerical solution for a large class of nonlinear Poisson equations. These types of problems come from steady reaction-diffusion and heat transfer equations, from elasticity, fluid mechanics, electrostatics, and geometry. Our method is based on finite differences and homotopy analysis. The numerical and exact solutions are compared and emphasize a very good approximation. Also, they show very good accordance between the convergence regions for absolute and relative errors. This fact is significant for the case when the exact solution is unknown.

Keywords: nonlinear Poisson equation, finite differences method, homotopy analysis, Gauss-Seidel algorithm.

1. INTRODUCTION

Most phenomena in nature's sciences are modeled by nonlinear differential equations. It is difficult to solve such problems, therefore their approximate resolution was a continuous challenge. The perturbation methods are depending on small/large physical parameters and thus they are valid only for weakly nonlinear problems [1–3]. On the other hand, many nonlinear problems do not contain such parameters. The non-perturbation techniques such as δ-expansion [4], Adomian decomposition method [5–9], homotopy perturbation method [10], variational iteration method [11] were widely applied to solve many nonlinear problems.

One of the most efficient methods used in nonlinear sciences is the homotopy analysis method (HAM). It can be applied to strong nonlinear problems and it was introduced in 1992 by Liao [12]. This technique was implemented with great success to magnetohydrodynamic problems for non-Newtonian fluids [13–15], viscous flow problems [16–18], nonlinear heat transfer [19]. Also, the method was used to obtain exact solutions for the coupled Ramani equations [20], to determine the approximate soliton solution of Kadomtsev-Petviashvili-II equation which models water waves with small surface tension [21], to perform successfully the numerical calculation of nonlinear coupled system of equations in the theory of thermo-elasticity [22], for the analytical treatment of Abel integral equations [23].

In this work, we study an efficient method to approximate the solution of a class of nonlinear Poisson equations in the hypothesis they have a solution.

For this purpose, we develop a combination between finite differences and homo-topy analysis method.

There are not many papers referring to applications of HAM to nonlinear partial differential equations. In [19, 24–26], the theory of general boundary element method is implemented and it is applied to high nonlinear problems.

Our paper is constructed in the following way: section 2 is referring to the technique of homotopy analysis applied to nonlinear Poisson equation. The finite differences method is involved for numerical resolution of the m-th order deformation equation. In section 3, we accomplish the numerical tests which prove very good accordance between the convergence regions for absolute and relative errors. This fact is especially important for the case the exact solution is not known. In application 3.2, we had to deduce some finite differences formulas in order to obtain better approximations.

The programs were accomplished in C/C++ programming language. The symbolic software products do not yield an analytic solution for Poisson equation.

2. DESCRIPTION OF THE METHOD

Consider the nonlinear Poisson problem in two dimensions:

$$\Delta u = F(x,u,\frac{\partial u}{\partial x_1},\frac{\partial u}{\partial x_2},\frac{\partial^2 u}{\partial x_1^2},\frac{\partial^2 u}{\partial x_1 \partial x_2},\frac{\partial^2 u}{\partial x_2^2}),$$

$$x = (x_1, x_2) \in D = (a,b)\times(c,d) \subset R^2 \tag{1}$$

$$u\big|_{\Gamma} = g\,. \tag{2}$$

The 3rd case can be analyzed likewise. The boundary value problem (1), (2) is related to problems which come from fluid mechanics, steady reaction-diffusion and heat transfer equations, electrostatics, elasticity, geometry.

The technique of HAM enables the numerical solving of nonlinear partial differential equations with boundary conditions:

$$\boldsymbol{N}[u(x)] = 0 \tag{3}$$

$$\boldsymbol{N}_1[u(x)]\big|_{\Gamma} = 0\,. \tag{4}$$

The method provides a homotopy series

$$\phi(x,p)=\sum_{s=0}^{\infty} u_s(x)p^s\,, \tag{5}$$

so that when p varies from 0 to 1, the series solution ϕ varies from the initial guess $\phi(x,0)=u_0(x)$ to the exact solution $\phi(x,1)=u(x)$. The method states the zero-order deformation equation [12]:

$$(1-p)\boldsymbol{L}[\phi(x,p)-u_0(x)]=p\hbar H(x)\boldsymbol{N}[\phi(x,p)]. \tag{6}$$

For the boundary conditions (4), the method yields

$$(1-p)\boldsymbol{L}_1[\phi(x,p)-u_0(x)]=p\hbar H_1(x)\boldsymbol{N}_1[\phi(x,p)]. \tag{7}$$

The method offers a great freedom to select the auxiliary linear operators $\boldsymbol{L}, \boldsymbol{L}_1$ the initial approximation u_0 and the auxiliary functions

$$H, H_1\neq 0, H, H_1\in C(\overline{D}).$$

The convergence-control parameter $\hbar\neq 0$ can be selected to provide the convergence of homotopy series for $p=1$:

$$u(x)=\phi(x,1)=\sum_{s=0}^{\infty} u_s(x). \tag{8}$$

The s^{th} order homotopy derivative $D_s(\phi)=\frac{1}{s!}\frac{\partial^s\phi}{\partial p^s}\Big|_{p=0}$ in eq. (6) provides

$$\boldsymbol{L}[u_s(x)-\chi_s u_{s-1}(x)]=\hbar H(x)D_{s-1}\boldsymbol{N}([\phi]), \tag{9}$$

where $\chi_s=\begin{cases}0, & s\leq 1\\ 1, & s\geq 2\end{cases}$. The eq. (9) is called the s^{th} order deformation equation.

Concerning about the problem (1), (2), it is natural to choose the auxiliary linear operator

$$\boldsymbol{L}[\phi(x,p)]=\Delta[\phi(x,p)]. \tag{10}$$

We define the nonlinear operator

$$\boldsymbol{N}[\phi]=\Delta[\phi]-F\left(x,\phi,\frac{\partial\phi}{\partial x_1},\frac{\partial\phi}{\partial x_2},\frac{\partial^2\phi}{\partial x_1^2},\frac{\partial^2\phi}{\partial x_1\partial x_2},\frac{\partial^2\phi}{\partial x_2^2}\right). \tag{11}$$

The boundary condition (2) yields

$$\boldsymbol{N}_1[\phi] = \phi\Big|_{\Gamma} - g, \quad \boldsymbol{L}_1[\phi] = \phi\Big|_{\Gamma} .$$

We select $u_0(x)$ in such a way $u_0\Big|_{\Gamma} = g$. The s^{th} order equation for boundary conditions provides

$$u_1\Big|_{\Gamma} = \hbar H_1[u_0\Big|_{\Gamma} - g] = 0 \tag{12}$$

$$u_s\Big|_{\Gamma} = (1+\hbar H_1)u_{s-1}\Big|_{\Gamma} = 0, \quad s \ge 2 . \tag{13}$$

From eq. (9) we deduce

$$\Delta u_1 = \hbar\, H(x)\left[\Delta u_0 - F(x, u_0, \frac{\partial u_0}{\partial x_1}, \frac{\partial u_0}{\partial x_2}, \frac{\partial^2 u_0}{\partial x_1^2}, \frac{\partial^2 u_0}{\partial x_1 \partial x_2}, \frac{\partial^2 u_0}{\partial x_2^2})\right] \tag{14}$$

and ($s \ge 2$)

$$\Delta u_s = \left(1+\hbar H(x)\right)\Delta u_{s-1} - \hbar H(x) D_{s-1}[F(x, \phi, \frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, \frac{\partial^2 \phi}{\partial x_1^2}, \frac{\partial^2 \phi}{\partial x_1 \partial x_2}, \frac{\partial^2 \phi}{\partial x_2^2})] . \tag{15}$$

We construct a grid of the domain D by dividing $[a,b]$ in n subintervals and $[c,d]$ in m subintervals. Denote

$$h = \frac{b-a}{n}, \quad x_{1i} = a + ih, \quad i = \overline{0,n}$$
$$k = \frac{d-c}{m}, \quad x_{2j} = c + jk, \quad j = \overline{0,m}$$
$$u_{sij} \cong u_s(x_{1i}, x_{2j}), \quad s \ge 0, \quad i = \overline{0,n}, \quad j = \overline{0,m} \cdot$$

The problems (14, 12) and (15, 13) are Poisson equations with Dirichlet conditions because the right term is known:

$$\begin{aligned} &\Delta u_s = f, \\ &u_s\Big|_{\Gamma} = 0 \quad \text{for } s \ge 1 . \end{aligned} \tag{16}$$

We apply the finite difference scheme of order $O(h^2)$:

$$\frac{\partial^2 u_s}{\partial x_1^2}(x_{1i}, x_{2j}) \cong \frac{u_{s,i-1,j} - 2u_{sij} + u_{s,i+1,j}}{h^2}, \quad i = \overline{1,n-1}, j = \overline{1,m-1}$$

and similar for $\frac{\partial^2 u_s}{\partial x_2^2}(x_{1i}, x_{2j})$. They produce a linear algebraic system which can be solved by the iterative Gauss-Seidel algorithm [27]. In our applications, we apply the algorithm in the frame ($\lambda = (\frac{h}{k})^2$, N_1 = the number of iterations):

$$u_{s,i,j}^{(0)} = u_{s-1,i,j}$$
$$u_{s,i,j}^{(r+1)} = \frac{1}{2(\lambda+1)}\left[-h^2 f(x_{1i}, x_{2j}) + u_{s,i,j}^{(r)} + u_{s,i-1,j}^{(r+1)} + \lambda(u_{s,i,j+1}^{(r)} + u_{s,i,j-1}^{(r+1)})\right]$$

for $s \geq 1,\ r = \overline{0, N_1},\ j = \overline{1, m-1},\ i = \overline{1, n-1}$.

The sequence of problems (14), (12) and (15), (13) produce the approximate homotopy series solution, which is computed in the nodes of the grid by the relation

$$\phi(x_{1i}, x_{2j}, p) \cong \phi_{hk}(x_{1i}, x_{2j}, p) = \sum_{s=0}^{\infty} u_{s,i,j} p^s,\ i = \overline{0,n},\ j = \overline{0,m}. \qquad (17)$$

We define the initial approximation by the values in the nodes of the grid:

$$u_{0ij} = \begin{cases} g(x_{1i}, x_{2j}), \text{ for } (x_{1i}, x_{2j}) \in \Gamma \\ \tilde{g}(x_{1i}, x_{2j}), \text{ for } (x_{1i}, x_{2j}) \in \mathrm{Int}(D) \end{cases} \qquad (18)$$

for $i = \overline{0,n},\ j = \overline{0,m}$.

The function $\tilde{g}$ will be selected according to the rule of solution expression [12] which accounts the governing equation and the boundary conditions.

3. APPLICATIONS

APPLICATION 3.1. We analyze the nonlinear partial differential problem:

$$\Delta \quad u = 3u^2 \text{ in } D = (0,1)\times(0,1)$$
$$u\Big|_{x_1=0} = \frac{4}{(3+x_2)^2},\ u\Big|_{x_1=1} = \frac{4}{(4+x_2)^2} \qquad (19)$$
$$u\Big|_{x_2=0} = \frac{4}{(3+x_1)^2},\ u\Big|_{x_2=1} = \frac{4}{(4+x_1)^2}.$$

It has the solution $u(x_1, x_2) = \dfrac{4}{(3 + x_1 + x_2)^2}$.

Tsai examined this boundary value problem using homotopy method of fundamental solutions. The technique proposed in [28] is valid in the case we know the fundamental solution and the analytical particular solutions of the augmented polyharmonic spline (APS) associated with the considered operator are known.

According to (11), we define the nonlinear operator

$$\boldsymbol{N}[\phi(x,p)] = \Delta[\phi(x,p)] - 3\phi^2(x,p)$$

with ϕ defined in (5). We deduce

$$D_s[\phi^2] = D_s\left[\sum_{t=0}^{\infty}\left(\sum_{i=0}^{t} u_i u_{t-i}\right)p^t\right] = \sum_{i=0}^{s} u_i u_{s-i} \,. \tag{20}$$

The deformation equations (14), (15) become

$$\Delta u_1 = \hbar\, H(x_1, x_2)\left[\Delta u_0 - 3u_0^2\right], \text{ in } D \tag{21}$$

$$u_1\Big|_{\Gamma} = 0$$

and ($s \geq 2$)

$$\Delta u_s = \left(1 + \hbar H(x_1, x_2)\right)\Delta u_{s-1} - 3\hbar H(x_1, x_2)\sum_{i=0}^{s-1} u_i u_{s-1-i}\,, \text{ in } D \tag{22}$$

$$u_s\Big|_{\Gamma} = 0\,.$$

The solution of our problem can be written as

$$u(x_1, x_2) = \sum_{i,j=0}^{\infty} a_{ij}\,(x_1 - \frac{1}{2})^i (x_2 - \frac{1}{2})^j \,.$$

Taking in consideration the governing equation, the boundary conditions and the rule of solution expression, we select the initial approximation:

$$u_{0ij} = \begin{cases} g(x_{1i}, x_{2j}), & \text{for } (x_{1i}, x_{2j}) \in \Gamma \\ 0, & \text{for } (x_{1i}, x_{2j}) \in \text{Int}(D) \end{cases}.$$

The simulation was implemented for $H(x) = 1$. The approximate solution of N^{th} order is computed by the relation

$$u(x)=\phi(x,1)\cong\phi_{hkN}(x,1;\hbar)=\sum_{i=0}^{N}u_i(x;\hbar)\overset{def}{=}U_{hkN}(x,\hbar)\,. \tag{23}$$

We state the absolute and relative errors at N^{th} order approximation:

$$\begin{aligned}&\text{err}_{\text{abs}}(x,\hbar,N)=\left|U_{hkN}(x,\hbar)-u(x)\right|,\\&\text{err}_{\text{rel}}(x,\hbar,N)=\left|U_{hkN}(x,\hbar)-U_{hk,N-1}(x,\hbar)\right|,\ x\in D.\end{aligned} \tag{24}$$

Table 1

The absolute errors for different orders of approximation

(m, n)	N order of approximation	Control convergence parameter	$\sup_{x\in D} err_{abs}$
(10,10)	10	–0.885	$4.394927\cdot10^{-5}$
	15	–1.29	$1.216543\cdot10^{-5}$
	20	–1.14	$7.557942\cdot10^{-6}$
	40	–1.584	$7.286129\cdot10^{-6}$
(20,20)	10	–0.975	$3.191858\cdot10^{-5}$
	15	–1.72	$2.005488\cdot10^{-6}$
	20	–1.41	$1.743130\cdot10^{-6}$
	40	–1.185	$1.282305\cdot10^{-6}$

Figures 1 and 2 are referring to the graphs of the functions

$$\hbar\mapsto\sup\{\text{err}_{\text{abs}}(x,\hbar,N)\,/\,x\in\bar{D}\}$$
$$\hbar\mapsto\sup\{\text{err}_{\text{rel}}(x,\hbar,N)\,/\,x\in\bar{D}\}$$

for different selections of m, n, N. The representations emphasize very good accordance between the two convergence regions. It is really important for the cases when the exact solution is not known.

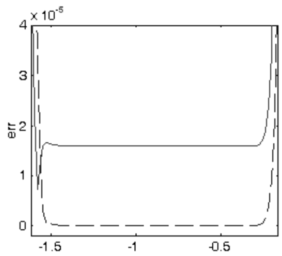

Fig. 1 – $\hbar$-curve of err_{abs} (solid line) and err_{rel} (dashed line) for $N=40, m=n=10$.

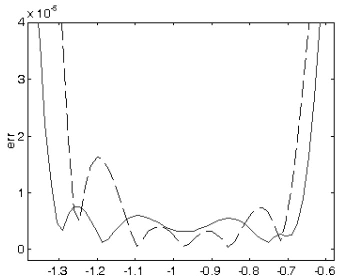

Fig. 2 – $\hbar$-curve of err_{abs} (solid) and err_{rel} (dashed) for $N=40, m=n=20$.

Figure 3 is referring to the graph of the function

$$D \ni x \mapsto err_{abs}(x, -1.185, 40).$$

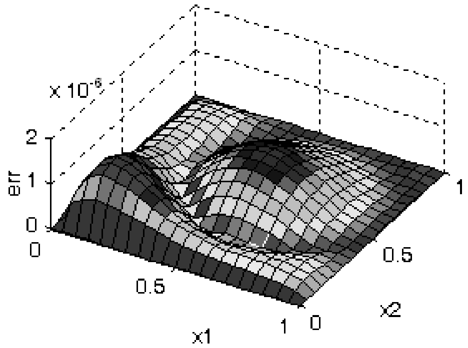

Fig. 3 – Absolute error at 40th order approximation for $m = n = 20$.

APPLICATION 3.2. Although this problem is manufactured, we selected it because it is a strong nonlinear partial differential equation with a known exact solution. The difficulty is not diminished by this choice.

We proceed with our problem: find $u \in C^2(D) \cap C(\overline{D})$ so that

$$\Delta u + (x_1^2 + x_2^2)u = \sin(u \frac{\partial u}{\partial x_1}) - \sin(\frac{1}{2} x_2 \cdot \sin 2x_1 x_2), \quad (x_1, x_2) \in D = (0,1) \times (0,1)$$

$$\begin{matrix} u\big|_{x_1=0} = 0, & u\big|_{x_1=1} = \sin x_2 \\ u\big|_{x_2=0} = 0, & u\big|_{x_2=1} = \sin x_1 \end{matrix} . \tag{25}$$

The exact solution of (25) is $u(x_1, x_2) = \sin(x_1 x_2)$.

Taking into consideration the eqs. (11), (25), we define

$$\boldsymbol{N}[\phi] = \Delta[\phi] - (x_1^2 + x_2^2)\phi - \sin(\phi \frac{\partial \phi}{\partial x_1}) + \sin(\frac{1}{2} x_2 \cdot \sin 2x_1 x_2). \tag{26}$$

The rule of solution expression determines the initial guess in the inner nodes of the grid to be $\widetilde{g}(x) = 0$. The relation (14) gives the first order deformation equation:

$$\Delta u_1 = \hbar\, H(x) \left[\Delta u_0 + (x_1^2 + x_2^2)u_0 + \sin(\frac{1}{2} x_2 \cdot \sin 2x_1 x_2) - \sin(u_0 \frac{\partial u_0}{\partial x_1}) \right]$$

$$u_1\Big|_{\Gamma} = 0\,. \tag{27}$$

Define now the auxiliary homotopy series

$$\psi(x,p) = \sum_{s=0}^{\infty} \psi_s(x)p^s = \phi\frac{\partial\phi}{\partial x_1} = \sum_{s=0}^{\infty}\left(\sum_{j=0}^{s} \frac{\partial u_j}{\partial x_1}u_{s-j}\right)p^s\,. \tag{28}$$

At this stage, we take into account (28) and the recurrence relations [29]:

$$\begin{aligned}
&D_0(\sin\psi) = \sin(\psi_0) = \sin(u_0\frac{\partial u_0}{\partial x_1})\\
&D_0(\cos\psi) = \cos(\psi_0) = \cos(u_0\frac{\partial u_0}{\partial x_1})\\
&D_s(\sin\psi) = \sum_{i=0}^{s-1}(1-\frac{i}{s})D_i(\cos\psi)\psi_{s-i} = \sum_{i=0}^{s-1}(1-\frac{i}{s})D_i(\cos\psi)\left(\sum_{j=0}^{s-i}\frac{\partial u_j}{\partial x_1}u_{s-i-j}\right)\\
&D_s(\cos\psi) = -\sum_{i=0}^{s-1}(1-\frac{i}{s})D_i(\sin\psi)\psi_{s-i} = -\sum_{i=0}^{s-1}(1-\frac{i}{s})D_i(\sin\psi)\left(\sum_{j=0}^{s-i}\frac{\partial u_j}{\partial x_1}u_{s-i-j}\right).
\end{aligned} \tag{29}$$

The eqs. (13), (15) yield the s^{th} $(s \geq 2)$ order deformation equation

$$\Delta u_s = \left(1+\hbar H(x)\right)\Delta u_{s-1} + \hbar H(x)\left[(x_1^2+x_2^2)u_{s-1} - D_{s-1}(\sin\psi)\right] \tag{30}$$

$$u_s\Big|_{\Gamma} = 0\,.$$

The equation (25) is highly nonlinear. The usual difference scheme of order $O(h^2)$ $v'(x_i) \cong \dfrac{v(x_i+h)-v(x_i-h)}{2h}$ to compute $\dfrac{\partial u_j}{\partial x_1}$ in (29) is not sufficiently accurate. We apply the usual way to deduce difference schemes of order $O(h^p)$:

$$\frac{\mathrm{d}^k v}{\mathrm{d}x^k} = \frac{1}{h^k}\sum_{i=-i_1}^{i_2} a_i v(x+ih) + O(h^p),$$

where $i_1, i_2 \geq 0, i_1 + i_2 = k + p - 1$. We use Taylor's formula of order $k + p - 1$ to compute $v(x + ih)$ and identify the same powers of h in the left and right terms. This provides the linear system

$$\sum_{i=-i_1}^{i_2} i^s a_i = \delta_{sk}, \quad s = \overline{0, k + p - 1}.$$

In particular, we obtain the following formulas of order $O(h^4)$, where $v_i \cong v(x_i)$:

$$v'(x_i) \cong \frac{1}{12h}\left[-3v_{i-1} - 10v_i + 18v_{i+1} - 6v_{i+2} + v_{i+3}\right]$$

$$v'(x_i) \cong \frac{1}{12h}\left[v_{i-2} - 8v_{i-1} + 8v_{i+1} - v_{i+2}\right]$$

$$v'(x_i) \cong \frac{1}{12h}\left[-v_{i-3} + 6v_{i-2} - 18v_{i-1} + 10v_i + 3v_{i+1}\right].$$

The rule of solution expression and the boundary conditions determines the initial approximation $u_{0ij} = 0, i = \overline{1, n-1}, j = \overline{1, m-1}$.

The approximate solution of the problem (25) is

$$U_{hkN}(x, \hbar) = \sum_{i=0}^{N} u_i(x, \hbar) \approx u(x).$$

We performed the following numerical tests:

Table 2

The absolute errors for different orders of approximation

(m, n)	N order of approximation	Control convergence parameter	$\sup\limits_{x \in D} err_{abs}$
(10,10)	10	–0.448	$2.549467 \cdot 10^{-3}$
	20	–0.185	$3.071450 \cdot 10^{-4}$
	40	–0.17	$2.270969 \cdot 10^{-4}$
(20,20)	10	–0.272	$1.381562 \cdot 10^{-3}$
	20	–0.316	$2.027479 \cdot 10^{-4}$
	40	–0.53	$1.639574 \cdot 10^{-4}$

Figures 4 and 5 present the dependence of absolute and relative errors on parameter $\hbar$.

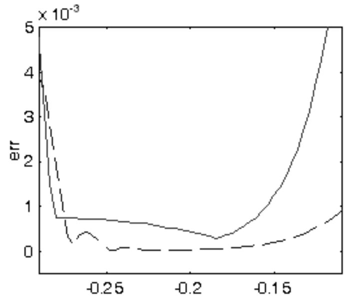

Fig. 4 – $\hbar$ -curve of err_{abs} (solid) and err_{rel} (dashed)
$N = 20, m = n = 10$.

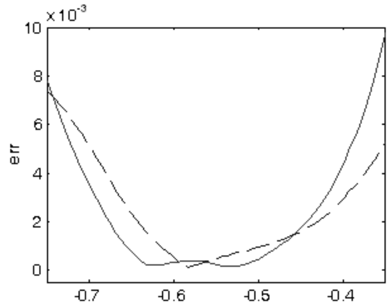

Fig. 5 – $\hbar$ -curve of err_{abs} (solid) and err_{rel} (dashed)
for $N = 40, m = n = 20$.

Figure 6 is referring to the graph of the function

$$\bar{D} \ni x \mapsto \mathrm{err}_{\mathrm{abs}}(x; \hbar = -0.53; N = 40)$$

for $m = n = 20$.

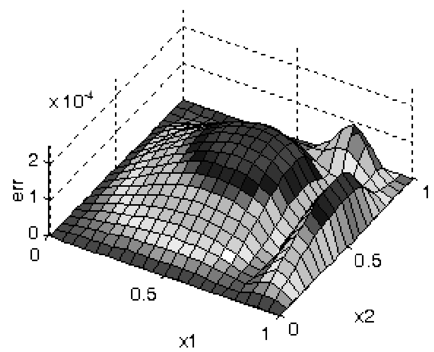

Fig. 6 – Absolute error at 40th order approximation for $m = n = 20$.

4. CONCLUSIONS

We accomplished an analysis of a technique to obtain the approximate solution for a class of nonlinear partial differential equations. The method combines the finite differences and homotopy analysis. The numerical tests prove the technique works.

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