Romanian Reports in Physics, Vol. 68, No. 2, P. 503–514, 2016

INVESTIGATION OF THE BEHAVIOR OF THE FRACTIONAL BAGLEY-TORVIK AND BASSET EQUATIONS *VIA* NUMERICAL INVERSE LAPLACE TRANSFORM

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Received October 16, 2015

Abstract. It is well known that the motion of a rigid plate floated in a Newtonian fluid and the unsteady motion of a sphere immersed in a Stokes fluid are described by equations involving derivative of real order. This work is devoted to the study of the fractional Bagley-Torvik and Basset equations. Because of the difficulty of evaluating the inverse Laplace transforms for complicated non-integer order differential equations, we investigate the validity of applying numerical inverse Laplace transform algorithms in fractional calculus. To this end, we introduce a method based on Gaussian quadrature formulae for numerical integration of the Bromwich's integral. We give a full presentation of the method and eventually show its efficiency and applicability through two illustrative examples.

Key words: Bagley-Torvik equation, Basset equation, Fractional differential equations, Laplace transform, Fractional finite difference method.

1. INTRODUCTION

Fractional order differential equations occur in modeling some physical systems in gravity [1], viscous fluid flows [2], diffusion processes [3–5], thermodynamics [6], porous media [7], control problems [8, 9], and heat conduction [10]. The fractional initial value problems (FIVPs) arise in many application areas, see for instance the linear and nonlinear fractional Bagley-Torvik equations [11, 12], Basset equation [13, 14], oscillation equation [15–17], Riccati equation [18], fractional dynamical systems [19–25], and cosmological and biological population models with fractional order [26, 27].

Due to the fact that the analytical solution of the model containing fractional operator neither always exist nor have a simple form, adequate numerical methods have been developed. Thus the homotopy analysis transform method [26], the predictor-corrector approach [28], the variational iteration method [29], the spectral collocation methods [3, 4, 30–35], and the operational matrix approaches [36, 37] are successfully used for numerical solution of FIVPs.

In recent decades, the Laplace transform technique has been considered as an efficient tool for solving simple linear ordinary and fractional differential equations. The inverse Laplace transform is an important but difficult step in the application of Laplace transform technique. For a complicated differential equation, it is difficult to analytically calculate the inverse Laplace transform. So, the numerical inverse Laplace transform algorithm is often used to obtain reliable results. One of the best ways for numerical inversion of the Laplace transform is to deform the standard contour in the Bromwich integral [38].

The main purpose of this study is to extend the application of the inverse Laplace transform to develop an efficient numerical scheme for solving two physical problems, namely, the Bagley-Torvik and Basset equations. In order to perform the numerical integration of the Bromwich's integral, we suggest a scheme based on Gaussian quadrature formulas.

First, we introduce the Bagley-Torvik equation [11]. In modeling the motion of a rigid plate immersing in a Newtonian fluid, Torvik and Bagley considered the fractional differential equation

$$aD^2y(t)+b\,{}_0^CD_t^{\frac{3}{2}}y(t)+cy(t)=f(t), \quad y(0)=0, \quad y'(0)=0, \quad 0\le t\le T, \quad (1)$$

where $a\neq 0, b, c$ are constants, D^n is the classic differential operator of order $n\in\mathbb{N}$, ${}_0^CD_t^{\alpha}$ denotes the Caputo fractional derivative of order α, and $f(t)$ is a known function. In the literature, Eq. (1) is the so-called the Bagley-Torvik equation. Podlubny gave the analytical solution of the Bagley-Torvik equation in the special case with homogeneous initial conditions by using Green's function; also he proposed a numerical method in his book [15]. Trinks and Ruge modeled the Bagley-Torvik equation again and compared the numerical solution obtained by using the alternative time discretization scheme with the Podlubny's numerical solution [39]. Leszczynski and Ciesielski proposed a numerical solution of Bagley-Torvik equation considering the equation as a system of ordinary differential equations using the Abel integral equations [40]. Arikoglu and Ozkol [41] applied the differential transform method to Bagley-Torvik equation for specified initial conditions and a certain function $f(t)$. Yucel Cenesiz *et al.* solved the Bagley-Torvik with the generalized Taylor collocation method [42].

Second, we consider the Basset equation which describes the unsteady motion of a sphere immersed in a Stokes fluid. This equation can be written in terms of fractional derivatives as follow:

$$Dy(t)+\left(\frac{9}{1+2\lambda}\right)^{\frac{1}{2}}\,{}_0^CD_t^{\alpha}y(t)+y(t)=1, \quad y(0)=0, \quad 0\le t\le T. \quad (2)$$

Wu and Yu [43] studied the uniqueness of an inverse Basset problem. Mainardi modeled the Basset force as fractional differential equation and solved it with some values of α, λ, and also compared his solution with asymptotic behaviour of the Basset equation [14]. Edward *et al.* [13] solved Eq. (2) by reducing it to a system of fractional differential equations. Recently, Khosravian-Arab *et al.* proposed a method by the extended Laguerre function for solving the Basset equation [44].

This paper is organized as follows. In Sec. 2, we provide the required notation and the basic concepts of the fractional calculus and Laplace transform. Section 3 is devoted to the numerical and analytical investigations of our method, and the corresponding details are given. In Sec. 4, the numerical finding for both the Bagley-Torvik and the Basset fractional equations are reported. Also, we compare the approximations obtained using our scheme with the solutions obtained using other algorithms. We finally summarize the conclusions in Sec. 5.

2. PRELIMINARIES AND FUNDAMENTAL RELATIONS

In this Section we present some necessary concepts and fundamental relations, which will be required throughout of this paper [15, 45].

2.1. FRACTIONAL CALCULUS

Let $\alpha \in R_+$, for $f(t) \in L_1[0,T]$, the Riemann-Liouville fractional integrals of order α is defined by

$$ {}_0^R I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x) dx, \quad t < T. \tag{3}$$

For $\alpha = 0$, we set ${}_0^R I_t^0 := I$, the identity operator. Also, for $f(t) \in C^n[0,T]$, the Riemann-Liouville and Caputo fractional derivatives of order $n-1 < \alpha \le n$, are defined as

$$ {}_0^R D_t^\alpha f(t) = D^n {}_0^R I_t^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-x)^{n-\alpha-1} f(x) dx, \tag{4}$$

and

$$ {}_0^C D_t^\alpha f(t) = {}_0^R I_t^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-x)^{n-\alpha-1} \frac{d^n}{dx^n} f(x) dx, \tag{5}$$

respectively. Furthermore,

$$ {}_0^G D_t^\alpha f(t) = \lim_{N\to\infty} \frac{1}{h_N^\alpha} \sum_{k=0}^{N} (-1)^k \binom{\alpha}{k} f(t-kh_N), \tag{6}$$

where $h_N = t/N$ is called the Grünwald-Letnikov fractional derivative of order α of the function f.

In the following, we express some properties of fractional operators [15].

Property 1 *For* $\alpha,\beta \geq 0$, $n-1<\alpha\leq n$, $\alpha+\beta\leq m$, $f(t)\in C^m[0,T]$ *and* $\gamma\geq 0$,

i) ${}_0^R I_t^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} t^{\gamma+\alpha}$, ${}_0^R D_t^\alpha t^\gamma = {}_0^C D_t^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} t^{\gamma-\alpha}$

ii) ${}_0^C D_t^\alpha f(t) = {}_0^R D_t^\alpha f(t) - \sum_{k=0}^{n-1} \frac{D^k f(0)}{\Gamma(k-\alpha+1)} t^{k-\alpha}$

iii) *For* $D^k f(0)=0$, $k=0,1,\ldots,m-1$, ${}_0^C D_t^\alpha {}_0^R I_t^\beta f(t) = {}_0^R I_t^\beta {}_0^C D_t^\alpha f(t) = {}_0^C D_t^{\alpha-\beta} f(t) = {}_0^R I_t^{\beta-\alpha} f(t)$

iv) ${}_0^G D_t^\alpha f(t) = \sum_{k=0}^{n-1} \frac{D^k f(0)}{\Gamma(k+1-\alpha)} t^{k-\alpha} + {}_0^C D_t^\alpha f(t) = {}_0^R D_t^\alpha f(t)$.

Also, with $\frac{T}{h}=m\in\mathbb{N}$, the finite Grünwald-Letnikov differential operator centred at 0 is given by

$$ {}_F^G D^\alpha f(t) = \frac{1}{h^\alpha}\sum_{k=0}^{m}(-1)^k\binom{\alpha}{k} f(t-kh). \tag{7}$$

This yields a first order approximation for Riemann-Liouville differential operator ${}_0^R D_t^\alpha$ if and only if $f(0)=0$.

2.2. LAPLACE TRANSFORM AND INVERSE LAPLACE TRANSFORM

Let $f(t)$ be a piecewise continuous function for nonnegative real values of t and be of exponential order. The Laplace transform of this function is defined by [15],

$$F(s) = \pounds\{f(t)\} = \int_0^\infty f(t)e^{-st}dt,$$

Also, for $n-1<\alpha\leq n$,

$$\pounds\{{}_0^C D_t^\alpha f(t)\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} D^k f(0). \tag{8}$$

An integral formula for the inverse Laplace transform, called the Bromwich integral or the Mellin's inverse formula, is given by the line complex integral [38],

$$f(t) = \pounds^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)e^{st}ds, \tag{9}$$

where the integration is done along the vertical line $Re(s)=\sigma$ in the complex plane such that σ is greater than the real part of all singularities of $F(s)$.

3. ELEMENTS OF METHODOLOGY

The purpose of this section is to provide a method to get the solution of fractional differential equations using the inversion of Laplace transforms.

It is noteworthy that several approaches have been proposed for numerical inversion of the Laplace transform. One of the interesting and efficient approaches is to use the Gaussian quadrature formula, namely [38],

$$f(t) \approx t^{\gamma-1} \sum_{k=1}^{N} \Big[A_k (\frac{u_k}{t})^{\gamma} F(\frac{u_k}{t}) \Big], \tag{10}$$

where F is the Laplace transform of f, γ is a positive real number, and N is a positive integer that must be chosen for numerical approximations. Also, we use the approach that have been proposed by Piessens to evaluate the nodes u_k and corresponding weights A_k [38]. We emphasize that the nodes u_k and weights A_k are dependent on the value of γ and N. Now, we briefly review how we can find these nodes and weights. To do so, at first we note the polynomials in terms of s^{-1},

$$P_{N,\gamma}(s^{-1}) = -\frac{A}{N(N+\gamma-1)} + \frac{A}{s}\, {}_3F_1(-N+1, N+\gamma, 1; 2; s^{-1}), \tag{11}$$

where $A = (-1)^{N+1} N(N+\gamma-1)$ is the standardization factor, and

$${}_pF_q(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; t),$$

is the hypergeometric function that is defined by

$${}_pF_q(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; t) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \ldots (a_p)_k}{(b_1)_k (b_2)_k \ldots (b_q)_k} \frac{t^k}{k!},$$

with $(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}$. The representation of $P_{N,\gamma}(s^{-1})$ can also be expressed as follows:

$$P_{N,\gamma}(s^{-1}) = (-1)^N {}_2F_0(-N, N+\gamma-1; -; s^{-1}). \tag{12}$$

Property 2 *We mention some interesting properties of the $P_{N,\gamma}(s^{-1})$,*

i) Rodrigues' formula:

$$P_{N,\gamma}(s^{-1}) = (-1)^N e^{-s} s^{N+\gamma-1} \frac{d^N}{ds^N}(e^s s^{-N-\gamma+1})$$

ii) Recurrence relation: The three-term recurrence relation for $P_{N,\gamma}(x)$ is

$$P_{N,\gamma}(x) = (a_{N,\gamma} x + b_{N,\gamma}) P_{N-1,\gamma}(x) + c_{N,\gamma} P_{N-2,\gamma}(x)$$

for $N \geq 2$, where

$$a_{N,\gamma} = \frac{(2N+\gamma-3)(2N+\gamma-2)}{N+\gamma-2}, \; b_{N,\gamma} = \frac{(2N+\gamma-3)(2-\gamma)}{(N+\gamma-2)(2N+\gamma-4)},$$
$$c_{N,\gamma} = \frac{(2N+\gamma-2)(N-1)}{(N+\gamma-2)(2N+\gamma-4)},$$

and $P_{0,\gamma}(x) = 1$, $P_{1,\gamma}(x) = \gamma x - 1$.

In the following remark, we explain how to obtain the nodes and weights in the relation (10).

Remark 1 *Consider the Gaussian quadrature formula* (10). *The abscissas* $u_k, k = 1,2,\ldots,N$, *are the zeros of* $P_{N,\gamma}(s^{-1})$ *and the corresponding weights are obtained by*

$$A_k = (-1)^{N-1}\frac{(N-1)!}{\Gamma(N+\gamma-1)N(u_k)^2}[\frac{2N+\gamma-2}{P_{N-1,\gamma}(u_k{}^{-1})}]^2, \quad k = 1,2,\ldots,N. \quad (13)$$

4. NUMERICAL IMPLEMENTATION

In this Section we present two illustrative examples to demonstrate the efficiency of the proposed method for numerical solution of the Bagley-Torvik and Basset equations. For the sake of comparison, we use the traditional fractional finite difference method (FFDM) based on the Grünwald-Letnikov formula [15, 45].

4.1. BAGLEY-TORVIK EQUATION

Consider the general Bagley-Torvik equation (1), with $a = b = c = 1$ and

$$f(t) = \begin{cases} 8 & \text{for} \quad 0 \le t \le 1, \\ 0 & \text{for} \quad t > 1. \end{cases} \quad (14)$$

By taking the Laplace transform of both sides of (1) and using (8), with the notation $Y(s) = \mathcal{L}\{y(t)\}$, we obtain

$$as^2Y(s) + bs\sqrt{s}Y(s) + cY(s) = 8(\frac{1-e^{-s}}{s}) \Rightarrow Y(s) = \frac{8(1-e^{-s})}{as^3 + bs^2\sqrt{s} + cs}. \quad (15)$$

Now, we calculate u_k and $A_k, k = 1,2,\ldots,N$ for the selected values N and γ by using Remark 1. Then, the numerical solution of the Bagley-Torvik equation is achieved by relation (10).

To apply the FFDM, let us take the time step $h = \frac{T}{m}$. The first order Grünwald-Letnikov approximation of the Bagley-Torvik problem with zero initial conditions is

$$ah^{-2}(y_k - 2y_{k-1} + y_{k-2}) + bh^{-\frac{3}{2}}\sum_{j=0}^{k} w_j^{(\frac{3}{2})} y_{k-j} + cy_k = f_k,$$

$$k = 2,3,\ldots,m, \quad y_0 = y_1 = 0, \quad (16)$$

where $y_k = y(kh), f_k = f(kh), k = 0,1,\ldots,m$, and also

$$w_j^{(\alpha)} = (-1)^j\binom{\alpha}{j} = (-1)^j\frac{\Gamma(\alpha+1)}{\Gamma(j+1)\Gamma(\alpha-j+1)}.$$

So, we derive the following algorithm for obtaining the numerical solution

$$y_k = \frac{h^2 f_k + a(2y_{k-1} - y_{k-2}) - b\sqrt{h}\sum_{j=1}^{k} w_j^{(\frac{3}{2})} y_{k-j}}{a + b\sqrt{h} + ch^2}, \quad k = 2, 3, \ldots, m. \tag{17}$$

The numerical results obtained by our method with $N = 15$ and different values of γ, and the FFDM for $h = 0.1, 0.01$, are plotted in Fig. 1. Also, Table 1 shows a comparison of our method with the FFDM, pseudo-spectral, and differential transform methods, presented in [12, 41]. Clearly, the results obtained by the Laplace transform method are in agreement with other mentioned numerical methods and in total this approach has high accuracy. Also, we achieve a good approximation by using a few terms, and approximation errors are being rapidly reduced when the time of simulation or numbers of nodes are increased.

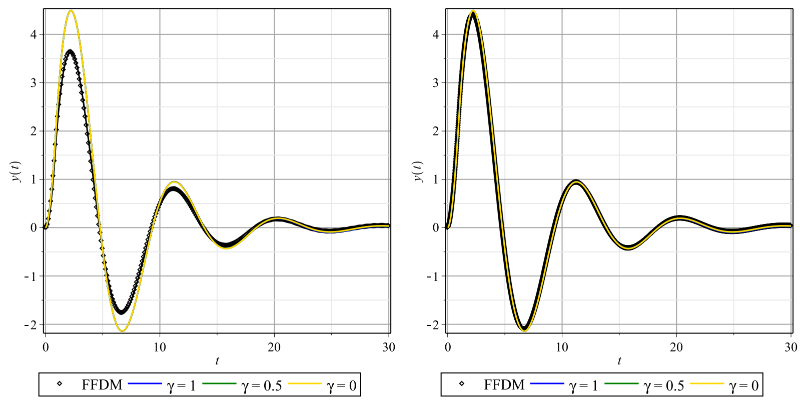

Fig. 1 – Comparison of our method with $N = 15$ and some values of γ, and the FFDM with $h = 0.1$ (left) and $h = 0.01$ (right) for the Bagley-Torvik equation.

Table 1

Comparison of our method for $N = 15, \gamma = 0, 0.5, 1$ and the FFDM for values $h = 0.1, 0.05, 0.01$ and the presented methods in [12, 41] for numerical solution of the Bagley-Torvik equation

	Our Method			FFDM				
t	$\gamma = 0$	$\gamma = 0.5$	$\gamma = 1$	$h = 0.1$	$h = 0.05$	$h = 0.01$	Ref. [12]	Ref. [41]
0.5	0.684335	0.684335	0.684335	0.540554	0.611764	0.669702	0.684029	0.684335
1	2.315148	2.315113	2.315082	2.020297	2.181363	2.291132	2.314565	2.315526
2	4.425789	4.426239	4.426789	3.599140	4.016435	4.344334	4.425495	-
5	-0.456587	-0.456064	-0.454306	-0.437610	-0.450697	-0.455289	-0.435979	-

4.2. BASSET EQUATION

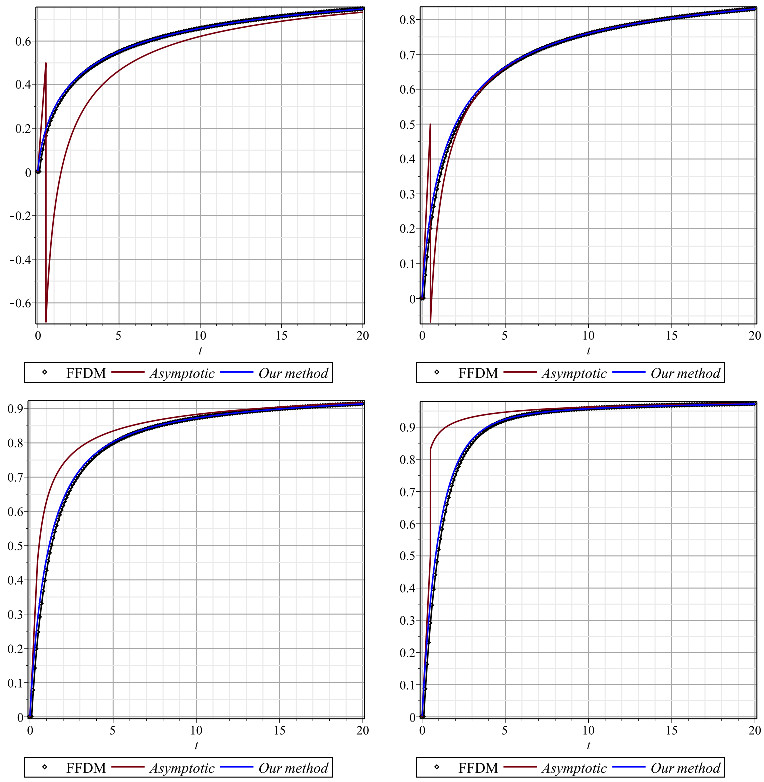

Fig. 2 – Comparison of the FFDM for $h = 0.1$, the asymptotic behaviour of solution of the Basset equation and our method with $N = 15, \gamma = 0$ for $\lambda = 0.25$ (first row left), $\lambda = 2$ (first row right), $\lambda = 10$ (second row left) and $\lambda = 100$ (second row right).

As the second example, we consider the Basset equation (2), with $\alpha = 0.5$ and $\lambda = 0.25, 2, 10, 100$. Applying the Laplace transform to both sides of this equation, we have

$$sY(s) + as^{\frac{1}{2}}Y(s) + Y(s) = \frac{1}{s} \Rightarrow Y(s) = \frac{1}{s^2 + as^{\frac{3}{2}} + s}, \qquad (18)$$

where $a = (\frac{9}{1+2\lambda})^{\frac{1}{2}}$. Therefore we can use Remark 1 and formula (10) to achieve the numerical solution of the Basset equation. Also, the first order Grünwald-Letnikov approximation of the Basset equation with zero condition is

$$h^{-1}(y_k - y_{k-1}) + \frac{a}{\sqrt{h}} \sum_{j=0}^{k} w_j^{(\frac{1}{2})} y_{k-j} + y_k = 1, \quad y_0 = 0, \quad k = 1, 2, \ldots, m. \qquad (19)$$

Table 2

Comparison results of the Laplace transform method for $N=15, \gamma=0$, and the FFDM for $h=0.1, 0.01$ with $\lambda=0.25, 2, 10, 100$ in some points for numerical solution of the Basset equation

t			5	10	15	20
	h	λ				
Our Method	-	0.25	0.516464	0.622844	0.679197	0.715764
FFDM	0.1	0.25	0.514752	0.621963	0.678624	0.715346
FFDM	0.01	0.25	0.516290	0.622748	0.679127	0.715705
Our Method	-	2	0.663762	0.760001	0.803911	0.830228
FFDM	0.1	2	0.661594	0.759121	0.803412	0.829891
FFDM	0.01	2	0.663544	0.759907	0.803851	0.830178
Our Method	-	10	0.803013	0.872679	0.899151	0.913938
FFDM	0.1	10	0.800625	0.872007	0.898827	0.913735
FFDM	0.01	10	0.802775	0.872609	0.899112	0.913908
Our Method	-	100	0.924250	0.957297	0.966791	0.971834
FFDM	0.1	100	0.922054	0.956991	0.966671	0.971763
FFDM	0.01	100	0.924036	0.957266	0.966777	0.971823

So, we have the following relation for calculating the FFDM solutions,

$$y_k = \frac{h + y_{k-1} - a\sqrt{h}\sum_{j=1}^{k} w_j^{(\frac{1}{2})} y_{k-j}}{1 + a\sqrt{h} + h}, \quad k = 1, 2, \ldots, m. \tag{20}$$

Asymptotic formulae for solution of the Basset equation are given in [14]. The asymptotic behaviour of the Basset equation when $t \to 0^+$ is $y(t) = 1 - t$, and for $t \to \infty$, $y(t)$ is given as follows:

$$y(t) \sim a\frac{t^{-\alpha}}{\Gamma(1-\alpha)} = a\frac{\sin\alpha\pi}{\pi}\int_0^\infty e^{-p\,t} p^{\alpha-1} dp.$$

The numerical results for solution of Eq. (2) by Laplace transform method with $N = 15, \gamma = 0$, are provided graphically in Fig. 2. Also in this figure for some values of λ, the results are compared with the FFDM for $h = 0.1$ and the asymptotic behaviour of the Basset equation. Furthermore, with the same set of parameters, comparison results for some values of t are listed in Table 2.

5. CONCLUSION

In this paper, the Laplace transform method as an efficient tool has been adopted for solving fractional differential equations. To avoid the complexity of computing the inverse Laplace transforms for some fractional differential equations, we presented the method based on Gaussian quadrature formulae for numerical integration of the Bromwich's integral. This scheme has been implemented for the Bagley-Torvik and Basset equations with fractional orders, to find the numerical solutions of these equations. Comparisons of our results with the obtained results by other approaches, especially by the FFDM, indicate the efficiency and accuracy of the suggested method in this paper. Also, the introduced method could be easily implemented to solve other fractional differential equations arising in different reasearch areas.

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