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DYNAMICS OF SHALLOW WATER WAVES WITH LOGARITHMIC NONLINEARITY

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Abstract. The study of shallow water waves, in (1+1)-dimensions, governed by the Korteweg-de Vries (KdV) equation, with logarithmic-law nonlinearity, is conducted in this paper. Exact Gaussian solitary wave solutions are obtained using three integration schemes. The conservation laws are listed. The adiabatic parameter dynamics is also given in presence of perturbation terms. The study is subsequently extended to (2+1)-dimensions where the Kadomtsev-Petviashvili (KP) equation, with log-law nonlinearity, is integrated. Finally, we consider two-layered shallow water waves modeled by coupled KdV equations.

Key words: shallow water waves; exact Gaussian solitary wave solution.

1. INTRODUCTION

The dynamics of shallow water waves is one of the most fascinating areas of research in fluid mechanics, oceanography, and nonlinear dynamics [1]-[43]. There are several models that are studied and several fascinating results have revolutionized this dynamics. These models have been also used in the past few years to address tsunami dynamics. Some of these frequently used models are the KdV equation, the modified KdV (mKdV) equation, the Gardener's equation, the Boussinesq equation, the Peregrine equation, the Kawahara equation, the Benjamin-Bona-Mahoney equation, and the regularized long wave (RLW) models. A plethora of results have been reported with these models over the past few decades. In addition, dispersive solitary waves are also reported with a newly founded model along with its several forms of variations and extensions. These dispersive shallow water wave models are the Rosenau-KdV equation, the Rosenau-Kawahara equation, the Rosenau-RLW equation, the Rosenau-KdV-RLW equation and they have been exhaustively studied both analytically and numerically [19]-[23], [42]-[43].

This dynamics, all along thus far, was analyzed using algebraic forms of nonlinearity only. Wazwaz first introduced the KdV model with log-law nonlinearity during 2014 [32, 35]. This form of KdV equation encompasses the KdV equation, the mKdV equation as well as the Gardener's equation. Thus log-law KdV equation is the most generalized form of KdV equation, so far. Its solution is known as Gaussian solitary waves of simply *Gaussons*. This paper will study this model in further details along with several perturbation terms. The study will also be extended to the KP equation in (2+1)-dimensions as well as to the coupled KdV equations that model two-layered shallow water waves.

2. GOVERNING MODEL

The general form of the KdV equation carries the following structure:

$$q_t + F(q)q_x + bq_{xxx} = 0, \tag{1}$$

where $q(x,t)$ is the dependent variable that represents the shallow water wave profile. The independent variables are the spatial variable x and temporal variable t. The first term is the linear temporal evolution. The second term represents nonlinearity while the real-valued constant parameter b is the dispersion. Solitary waves are the outcome of a delicate balance between dispersion and nonlinearity. This is the balancing principle for formation and sustainment of these waves. The nonlinearity is governed by the functional F from the second term, where

$$F(\cdot) \in \bigcup_{m=1}^{\infty} C((-m,m);R). \tag{2}$$

The special cases of equation (1) are as follows:

1. When $F(q) = aq$, for a real-valued constant a, the KdV equation is given by
$$q_t + aqq_x + bq_{xxx} = 0 \tag{3}$$
2. For $F(q) = aq^2$, the mKdV equation is:
$$q_t + aq^2q_x + bq_{xxx} = 0. \tag{4}$$
3. If $F(q) = aq^n$ for any real number n, this is the KdV equation with power-law nonlinearity:
$$q_t + aq^nq_x + bq_{xxx} = 0. \tag{5}$$
Here, the parameter n represents the power-law nonlinearity and for solitons to exist, one must have $n \neq 4$ [2, 24].

4. With two nonlinear terms, $F(q) = a_1 q + a_2 q^2$, the Gardener's equation takes the form:

$$q_t + \left(a_1 q + a_2 q^2\right) q_x + b q_{xxx} = 0. \tag{6}$$

5. On a generalized note, if $F(q) = a_1 q^n + a_2 q^{2n}$, the Gardener's equation with power-law nonlinearity, that is also known as the KdV equation with dual-power law is

$$q_t + \left(a_1 q^n + a_2 q^{2n}\right) q_x + b q_{xxx} = 0. \tag{7}$$

6. The most generalized form of the KdV equation, recently reported, is with log-law nonlinearity [31, 32, 34]. This means $F(q) = a \ln q$. Therefore, (1) modifies to

$$q_t + a q_x \ln q + b q_{xxx} = 0. \tag{8}$$

Equations (3)-(7) produce solitary waves or solitons. However the solitary wave solution to (8) is referred to as *Gaussons*. The KdV equation with log-law nonlinearity given by (8) incorporates the KdV equation (3), the mKdV equation (4) and the Gardener's equation (6). Upon carrying out Taylor's series expansion of $\ln q$ about $q = 1$ and retaining up to linear or quadratic term leads to the KdV equation or the mKdV equation or to Gardener's equation. Thus, (8) is the most generalized form of the KdV equation that will be studied in this paper.

Equation (1) has at least three integrals of motion. They are mass (M), linear momentum (P), and Hamiltonian (H). These are respectively represented as [2], [8]-[9]

$$M = \int_{-\infty}^{\infty} q dx \tag{9}$$

$$P = \int_{-\infty}^{\infty} q^2 dx \tag{10}$$

and

$$H = \int_{-\infty}^{\infty} \left[b \left(q_x\right)^2 - \int_0^q \left\{ \int_0^{s_2} F\left(s_1\right) ds_1 \right\} ds_2 \right] dx \tag{11}$$

Next, the center position of the soliton (or Gausson), represented by $\bar{x}$, is defined as

$$\bar{x}(t) = \frac{\int_{-\infty}^{\infty} x q dx}{\int_{-\infty}^{\infty} q dx} = \frac{\int_{-\infty}^{\infty} x q dx}{M}, \tag{12}$$

which leads to the speed (v) of the soliton

$$v = \frac{d\bar{x}}{dt} = \frac{\int_{-\infty}^{\infty} x q_t dx}{\int_{-\infty}^{\infty} q dx} = \frac{\int_{-\infty}^{\infty} x q_t dx}{M} \tag{13}$$

3. GAUSSON SOLUTION

The focus of this section is to carry out the integration of (8) to retrieve Gaussons. There are several methods that can be implemented, see for instance [44]-[52]. A few of the well-known methods are traveling wave hypothesis, semi-inverse variational principle, ansatz method that is also known as the method of undetermined coefficients, G'/G-expansion method, exp-function method, Kudryashov's method, Riccati's equation method, simplest equation approach and several others. This paper will only focus on the first three approaches. These three algorithms will be discussed in the following subsections.

3.1. TRAVELING WAVE HYPOTHESIS

Traveling waves are waves of permanent form that travel with a fixed speed v without any distortion. In order to integrate (8) with traveling wave hypothesis, the starting assumption is [24]

$$q(x,t) = g(x - vt) = g(s), \tag{14}$$

where the waveform is given by the function g with

$$s = x - vt. \tag{15}$$

Substituting this hypothesis into (8) leads to

$$-vg' + ag' \ln g + bg''' = 0, \tag{16}$$

where $g' = dg/ds$, $g'' = d^2g/ds^2$ and so on. Integrating once leads to

$$bg'' = (a+v)\,g - ag\ln g \tag{17}$$

and the integration constant is taken to be zero since the search is for a soliton solution. Multiplying both sides of (17) by g' and integrating gives

$$2b\left(g'\right)^2 = g^2\left(3a + 2v - 2a\ln g\right), \tag{18}$$

where the integration constant is again taken to be zero for the same reason. Separating variables and integrating (18) yields

$$q(x,t) = g(s) = g(x - vt) = Ae^{-B^2(x-vt)^2}, \tag{19}$$

which is the Gausson solution to (8). From (19), the amplitude of the Gausson is

$$A = e^{\frac{3a+2v}{2a}} \tag{20}$$

and its inverse width is

$$B = \frac{1}{2}\sqrt{\frac{a}{b}}, \tag{21}$$

which shows that the inverse width of the Gausson will exist provided

$$ab > 0, \tag{22}$$

or, in other words, the nonlinear term and dispersion term in (8) must both maintain the same sign.

3.2. ANSATZ APPROACH

The second integration scheme applied to (8) is an inverse-problem approach. Here a solution hypothesis is inserted into (8) and this will lead to the relation between the parameters and coefficients after setting the values of undetermined coefficients to zero. It is for this reason, this scheme is alternatively known as method of undetermined coefficients or ansatz method. Therefore the starting hypothesis for the solution hypothesis is [31, 32, 34]

$$q(x,t) = Ae^{-B^2(x-vt)^2}. \tag{23}$$

Substituting this hypothesis into (8) gives

$$v - a\ln A + 6bB^2 + \left(a - 4bB^2\right)\tau^2 = 0, \tag{24}$$

where

$$\tau = B(x - vt). \tag{25}$$

From (24), setting the coefficients of the linearly independent functions τ^m for $m = 0, 2$ to zero yields

$$v = a\ln A - 6bB^2. \tag{26}$$

The width is the same as in (21) along with the condition (22). The velocity of Gausson given by (26) is the same as the expression for the amplitude as in (20). Thus these results are consistent with those given in the previous section.

3.3. SEMI-INVERSE VARIATIONAL PRINCIPLE

This is another form of inverse-problem approach that is based on semi-inverse variational principle (SVP) proposed by He a couple of decades earlier. This method has been successfully applied to several problems in the field of nonlinear evolution equations (NLEEs) that are studied in water waves, nonlinear optics, nuclear physics and other areas. This paper will apply SVP to the KdV equation with log-law nonlinearity given by (8). The staring point to integrate (8) is the traveling wave hypothesis given by (14). Following as in Section 3.1, equation (17) is recovered. Next multiplying (17) by g' and integrating gives

$$2b\left(g'\right)^2 = g^2\left(3a + 2v - 2a\ln g\right) + K, \tag{27}$$

where K is the integration constant. The stationary integral is then defined as

$$J = \int_{-\infty}^{\infty} K ds, \tag{28}$$

which by virtue of (27) reduces to

$$J = \int_{-\infty}^{\infty} \left\{ 2b\left(g'\right)^2 - g^2\left(3a + 2v - 2a\ln g\right) \right\} ds. \tag{29}$$

Choosing the hypothesis given by (19) and inserting into (29) leads to

$$J = \frac{A^2}{2B}\left\{a - 4bB^2 + 2(3a+2v)A^2 - 4aA^2\ln A\right\}. \tag{30}$$

SVP states that the Gausson amplitude A and width B are determined from the solution of the equations [7], [11]-[15]

$$\frac{\partial J}{\partial A} = 0 \tag{31}$$

and

$$\frac{\partial J}{\partial B} = 0. \tag{32}$$

From eq. (30), eqs. (31) and (32) simplify to

$$4bB^2 + 4a\ln A = 5a + 4v \tag{33}$$

and

$$4bB^2 - 4a\ln A = -7a - 4v, \tag{34}$$

which, upon uncoupling, gives

$$A = e^{\frac{v+a}{2a}} \tag{35}$$

and

$$B = \frac{1}{2}\sqrt{-\frac{b}{a}}. \tag{36}$$

This shows that Gaussons will exist provided

$$ab < 0, \tag{37}$$

or, in other words, dispersion and nonlinearity must bear opposite signs for retrieving Gaussons with SVP. Therefore the Gausson solution is given by (19) with the definition of parameters in place. Also the speed of the soliton can be obtained from (33) or (34).

4. CONSERVATION LAWS

One of the more important aspects of NLEEs is the conservation laws. Without these conserved quantities (integrals of motion), the study of any NLEE is not complete. This is because the dynamics of the propagating waves, that are governed by these equations, is not completely understood without these laws. The three conserved quantities for the generalized KdV equation are listed in (9)-(11). This leads to the following conserved quantities corresponding to the Gausson (19):

$$M = \int_{-\infty}^{\infty} q dx = \frac{A}{B}\sqrt{\pi}, \tag{38}$$

$$P = \int_{-\infty}^{\infty} q^2 dx = \frac{A^2}{B}\sqrt{\frac{\pi}{2}}, \tag{39}$$

and

$$\begin{aligned} H &= \int_{-\infty}^{\infty} \left[b\left(q_x\right)^2 - \int_0^q \left\{ \int_0^{s_2} F\left(s_1\right) ds_1 \right\} ds_2 \right] dx \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \left\{ bq\left(q_x\right)^2 + 3aq^2 + aq^2 \ln q \right\} dx \\ &= \frac{A^2}{8B}\sqrt{\frac{\pi}{2}}\left(11a + 4a\ln A + 4bA^2B^2\right). \end{aligned} \tag{40}$$

The speed of Gausson is given by

$$v = \frac{\int_{-\infty}^{\infty} x q_t dx}{M} = a\ln A - \frac{3a}{2}, \tag{41}$$

which agrees with the former expression for the speed given by (26) with the usage of (21).

5. PERTURBED MODEL

There are several perturbation terms that tag along with shallow water waves. For the KdV equation model in this paper that is being studied with log-law nonlinearity, the perturbed version will be [2], [8]-[10]

$$q_t + aq_x \ln q + q_{xxx} = \epsilon R, \tag{42}$$

where ϵ is the perturbation parameter and R represents perturbation terms that are

$$\begin{aligned} R &= \alpha q + \beta q_{xx} + \gamma q_x q_{xx} + \delta q^m q_x + \lambda q q_{xxx} + \sigma q_x^3 \\ &+ \xi q_x q_{xxxx} + \eta q_{xx} q_{xxx} + \rho q_{xxxx} + \psi q_{xxxxx} + \kappa q q_{xxxxx}. \end{aligned} \tag{43}$$

From the perturbation terms, the coefficient α accounts for shoaling [2] and the coefficient β is for dissipation [2], [19]-[24]. The coefficient δ is of higher order nonlinear dispersion that comes with a positive integer m and $1 \leq m \leq 4$ [2], [19]-[24]. The coefficient of higher order spatial dispersion is ψ. The term with the coefficient ρ will provide the higher stabilizing term and must therefore be taken into account. The remaining coefficients appear in the context of Whitham hierarchy [2], [19]-[24].

This section will focus on the perturbed KdV equation (42). There are two important aspects that will be studied in this section. These are perturbation theory and the integrability aspect of (42) by the aid of SVP. These are detailed in the next two subsections.

5.1. SOLITON PERTURBATION THEORY

When perturbation terms are turned on, the conservation laws are no longer valid. Instead, the Gausson parameters go through adiabatic deformation. This subsection will study this adiabatic dynamics of Gausson parameters by the aid of soliton perturbation theory. The laws of adiabatic changes from the first two conservation laws are

$$\frac{dM}{dt} = \epsilon \int_{-\infty}^{\infty} R dx, \tag{44}$$

$$\frac{dP}{dt} = \epsilon \int_{-\infty}^{\infty} qR dx. \tag{45}$$

The relations (44) and (45) show that the mass and linear momentum experiences adiabatic changes in presence of such perturbation terms. After substituting the Gausson from (19) into (44) and (45) the exact form of these variations will be obtained. This leads to

$$\frac{dM}{dt} = \epsilon\alpha M, \tag{46}$$

which shows that the mass of the wave will be affected with shoaling. Integrating (46) gives

$$M = M_0 e^{\epsilon\alpha t}, \tag{47}$$

where M_0 is the initial mass of the wave. This shows that

$$\lim_{t \longrightarrow \infty} M(t) = 0, \tag{48}$$

provided $\alpha < 0$. This clearly explains the effect of shoaling.

Next, upon substituting the expression for $q(x,t)$ into (45) and carrying out the integrations lead to

$$\frac{dP}{dt} = \frac{\epsilon A^2}{2B}\sqrt{\frac{\pi}{2}}\left(2\alpha - 2\beta B^2 + 3\rho B^4\right). \tag{49}$$

For fixed point analysis of (49) setting the right hand side to zero gives the biquadratic equation for Gausson width B as

$$3\rho B^4 - 2\beta B^2 + 2\alpha = 0, \tag{50}$$

which gives the stable value of the width ($\bar{B}$) of Gausson as

$$\bar{B} = \begin{cases} \left[\frac{\beta+\sqrt{\beta^2-6\rho\alpha}}{3\rho}\right]^{\frac{1}{2}} & : \quad |\beta| > \sqrt{6\alpha\rho} \\ \left[\frac{\beta+\sqrt{6\rho\alpha-\beta^2}}{3\rho}\right]^{\frac{1}{2}} & : \quad |\beta| < \sqrt{6\alpha\rho} \\ \sqrt{\frac{\beta}{3\rho}} & : \quad |\beta| = \sqrt{6\alpha\rho} \end{cases} \tag{51}$$

This means that the perturbed Gausson will travel with a fixed width given by (51) whenever the linear momentum remains constant. From (51), it is possible to conclude that the stable width will exist provided

$$\alpha\rho > 0, \tag{52}$$

and

$$\beta\rho > 0. \tag{53}$$

Now, the slow change in the Gausson speed (v) from (13) using (42) and (43) is

$$v = a\ln A - \frac{3a}{2} + \frac{\epsilon}{2\sqrt{2}}\left[(3\lambda-\gamma)AB^2 + 3(3\xi-\eta-5\kappa)AB^4 - \frac{2\delta\sqrt{2}A^m}{(m+1)^{\frac{3}{2}}}\right]. \tag{54}$$

This shows that when perturbation terms are not present, the soliton speed reduces to (41) with $\epsilon = 0$.

5.2. INTEGRATION OF PERTURBED KDV EQUATION

This section will carry out the integration of the perturbed KdV equation given by (42) after setting $\epsilon = 1$, which makes it into a strong perturbation. Thus the perturbed KdV that will be integrated is

$$\begin{aligned} &q_t + aq_x\ln q + q_{xxx} \\ &= \alpha q + \beta q_{xx} + \gamma q_x q_{xx} + \delta q^m q_x + \lambda q q_{xxx} + \sigma (q_x)^3 \\ &+\xi q_x q_{xxxx} + \eta q_{xx} q_{xxx} + \rho q_{xxxx} + \psi q_{xxxxx} + \kappa q q_{xxxxx}. \end{aligned} \tag{55}$$

The integration tool that will be adopted here is SVP. The other two integration techniques, namely the *ansatz approach* and *traveling wave hypothesis* are not applicable here. However, the starting hypothesis to integrate (55) is the traveling wave hypothesis given by (14). Substituting (14) into (55) and integrating it leads to

$$(3a+2v)g^2 - 2ag^2 \ln g - 2b(g')^2 + \frac{4\delta g^{m+2}}{(m+1)(m+2)} + 2\psi\left\{2g'g''' - (g'')^2\right\} = K, \tag{56}$$

where K is the integration constant. It needs to be noted that only two of the perturbation terms survive this integration process since they are both Hamiltonian terms. These are nonlinear dispersion and higher order spatial dispersion whose coefficients are δ and ψ respectively. The stationary integral is therefore given by

$$\begin{aligned} J &= \int_{-\infty}^{\infty} K ds \\ &= \int_{-\infty}^{\infty} \left[(3a+2v)g^2 - 2ag^2 \ln g - 2b\left(g'\right)^2 + \frac{4\delta g^{m+2}}{(m+1)(m+2)}\right. \\ &+ \left. 2\psi\left\{2g'g''' - \left(g''\right)^2\right\}\right] ds. \end{aligned} \tag{57}$$

Now SVP states that the solution of the perturbed KdV equation (55) will be the same as the solution of the unperturbed KdV equation, namely (19) [9, 19, 20, 24]. However the amplitude and width will be determined from the solution of the coupled system (31) and (32). Now substituting the solution (19) into (57) and performing the integration leads to

$$J = \frac{A^2}{B}\left\{3a + 2v - 2aA^2 \ln A - 21\psi B^4 + \frac{4\delta\sqrt{2}A^m}{(m+1)(m+2)^{\frac{3}{2}}}\right\}. \tag{58}$$

The coupled system (31) and (32) now reduces to

$$2(a+v) - 2a\ln A - 21\psi B^4 + \frac{2\delta\sqrt{2}A^m}{(m+1)\sqrt{m+2}} = 0 \tag{59}$$

and

$$3a + 2v + 2a\ln A + 63\psi B^4 + \frac{4\delta\sqrt{2}A^m}{(m+1)(m+2)^{\frac{3}{2}}} = 0, \tag{60}$$

respectively. Upon uncoupling (59) and (60) leads to the transcendental equation for the amplitude A as

$$\begin{aligned} &(3a+2v)A^2 + 6(a+v)A + 2aA^2 \ln A - 6a\ln A \\ &+ \frac{2\delta\sqrt{2}A^{m+1}}{(m+1)(m+2)^{\frac{3}{2}}}\{2A + 3(m+1)\} = 0. \end{aligned} \tag{61}$$

The speed v can be obtained from (59) or (60) while the width B in terms of the amplitude and speed is

$$B = \left[\frac{1}{21\psi}\left\{2(a+v) - 2a\ln A + \frac{2\delta\sqrt{2}A^m}{(m+1)\sqrt{m+2}}\right\}\right]^{\frac{1}{4}}. \tag{62}$$

This gives a complete picture of the solution structure of the perturbed KdV equation by SVP. It must be noted that from (61) that the exact solution for the amplitude A is not available for any general m. It is only numerical simulation that will be possible. However, for special cases when $m = 0$, 1 or 2, an exact solution for the amplitude A is retrievable.

6. (2+1)-DIMENSIONAL MODEL

In real life situation, waves are two-dimensional. Therefore it is practical to consider nonlinear waves in (2+1)-dimensions. This is known as the KP equation that is the two-dimensional analog of the KdV equation, which is (1+1)-dimensional. Thus, the KP equation, with log-law nonlinearity, in dimensionless form is given by

$$(q_t + aq_x \ln q + bq_{xxx})_x + cq_{yy} = 0, \tag{63}$$

where c is a real-valued constant that represents dispersion in the y-direction. This model was first introduced by Wazwaz during 2015 [34]. This section will integrate equation (63) using two integration algorithms and thus secure Gaussian soliton solutions. These are traveling wave hypothesis and SVP that will be discussed in details in the following subsections.

6.1. TRAVELING WAVES

The starting hypothesis for the wave of permanent form in this case is

$$q(x,y,t) = g(B_1 x + B_2 y - vt) = g(s), \tag{64}$$

where, in this case

$$s = B_1 x + B_2 y - vt. \tag{65}$$

The parameters B_1 and B_2 are widths along x and y directions, respectively and conform to direction ratios. Substituting (64) into (63) implies

$$vB_1 g'' + aB_1^2 \left(g' \ln g\right)' + bB_1^4 g^{(iv)} + cB_2^2 g'' = 0. \tag{66}$$

Integrating (66) twice and choosing the integration constants, both times, to zero lead to

$$2bB_1^4 \left(g'\right)^2 = \left(2vB_1 + 3aB_1^2 - cB_2^2\right) g^2 - 2aB_1^2 g^2 \ln g. \tag{67}$$

Separating variables and integrating gives

$$q(x,y,t) = g(s) = Ae^{-B^2(B_1x+B_2y-vt)^2}. \tag{68}$$

This is the Gausson in (2+1)-dimensions, where the amplitude A is given by

$$A = \exp\left[\frac{2vB_1 + 3aB_1^2 - cB_2^2}{2aB_1^2}\right] \tag{69}$$

and the parameter B is defined as

$$B = \frac{1}{B_1}\sqrt{\frac{a}{2b}}. \tag{70}$$

These relations pose the constraints

$$B_1 \neq 0, \tag{71}$$

and

$$ab > 0. \tag{72}$$

This again shows that the nonlinear term and the dispersion along x direction must carry the same sign.

6.2. SEMI-INVERSE VARIATIONAL PRINCIPLE

In order to proceed with SVP for the KP equation, the starting hypothesis stays the same as given by (64). Next substituting (64) into (63) and integrating twice leads to

$$2bB_1^4\left(g'\right)^2 + 2aB_1^2g^2\ln g - \left(2vB_1 + 3aB_1^2 - cB_2^2\right)g^2 = K, \tag{73}$$

where K is the constant of second integration. Therefore, the stationary integral is given by

$$J = \int_{-\infty}^{\infty}\left\{2bB_1^4\left(g'\right)^2 + 2aB_1^2g^2\ln g - \left(2vB_1 + 3aB_1^2 - cB_2^2\right)g^2\right\}ds. \tag{74}$$

The solution hypothesis is taken to be given by (68). Substituting this hypothesis into the stationary integral (74), the integral leads to

$$J = \frac{A^2}{2B}\sqrt{\frac{\pi}{2}}\left\{4bB_1^4B^2 + 4aB_1^2\ln A - \left(4vB_1 + 7aB_1^2 - 2cB_2^2\right)\right\}. \tag{75}$$

The coupled system of equations (31) and (32) in this case are

$$4bB_1^4B^2 + 2aB_1^2(1 + 2\ln A) = 4vB_1 + 7aB_1^2 - 2cB_2^2 \tag{76}$$

and

$$4bB_1^4B^2 - 4aB_1^2\ln A = -\left(4vB_1 + 7aB_1^2 - 2cB_2^2\right), \tag{77}$$

respectively. Uncoupling the system (76) and (77) yields (69) and

$$B = \frac{1}{2B_1}\sqrt{-\frac{a}{b}}, \tag{78}$$

which implies

$$ab < 0. \tag{79}$$

Therefore SVP applied to the KP equation (63) leads to the Gausson given by (68) with the amplitude as in (69). The parameter B is given by (78) together with the condition (79).

7. TWO-LAYERED FLOW

This section will discuss two-layered shallow water wave dynamics. This situation arises when there is a shallow water wave flow with another layer of a different fluid on top. Such a situation is applicable during oil spill, for example, from Exxon-Valdez in the coast of Alaska during the year 1989. Another viable example is the BP oil spill in the Gulf coast of Louisiana during the year 2010 although this one is more of a deep water oil spill. In such cases, the double-layered shallow water wave dynamics is governed by the following coupled KdV equations. There are two models that can describe the shallow water dynamics. These are detailed in the next two subsections.

7.1. ZAREAMOGHADDAM MODEL

This model for the dynamics of two-layered shallow water waves is described in a dimensionless form by [38]

$$q_t + a_1 q_x \ln q + b_1 q_x \ln r + c_1 q_{xxx} = 0, \tag{80}$$

$$r_t + a_2 r_x \ln r + b_2 r_x \ln q + c_2 r_{xxx} = 0. \tag{81}$$

Here $q(x,t)$ and $r(x,t)$ represent the wave variables of the two layers. There are special cases of (80) and (81) with regular nonlinearity that were studied in the context of coupled shallow water waves [25].

This section will focus on the integrability of this coupled system of equations (80) and (81). Being a coupled system, it is only possible to integrate them by ansatz method. Therefore a judicious choice for Gaussians are

$$q(x,t) = A_1 e^{-B^2(x-vt)^2} = A_1 e^{-\tau^2} \tag{82}$$

and

$$r(x,t) = A_2 e^{-B^2(x-vt)^2} = A_2 e^{-\tau^2}. \tag{83}$$

Here A_1 and A_2 are the amplitudes of the two components. The widths and the speeds of the components are assumed to remain the same. The definition of τ is given by (26). Now, substituting (82) and (83) into (80) and (81) and simplifying gives

$$v - a_1 \ln A_1 - b_1 \ln A_2 + 6c_1 B^2 + \left(a_1 + b_1 - 4c_1 B^2\right)\tau^2 = 0 \tag{84}$$

and

$$v - a_2 \ln A_2 - b_2 \ln A_1 + 6c_2 B^2 + \left(a_2 + b_2 - 4c_2 B^2\right)\tau^2 = 0. \tag{85}$$

From (84) and (85), setting the coefficients of τ^m, for $m = 0, 2$ to zero gives

$$B = \frac{1}{2}\sqrt{\frac{a_1 + b_1}{c_1}} \tag{86}$$

$$v = \frac{1}{2}\left\{2a_1 \ln A_1 + 2b_1 \ln A_2 - 3(a_1 + b_1)\right\} \tag{87}$$

and

$$B = \frac{1}{2}\sqrt{\frac{a_2 + b_2}{c_2}} \tag{88}$$

$$v = \frac{1}{2}\left\{2a_2 \ln A_2 + 2b_2 \ln A_1 - 3(a_2 + b_2)\right\}, \tag{89}$$

which come with the constraints

$$c_1(a_1 + b_1) > 0 \tag{90}$$

and

$$c_2(a_2 + b_2) > 0. \tag{91}$$

Now equating the width of the Gaussons from (86) and (88) gives the condition

$$c_1(a_2 + b_2) = c_2(a_1 + b_1). \tag{92}$$

Again equating the speed of the Gaussons from (87) and (89) leads to yet another constraint on the parameters

$$2(a_1 - b_2)\ln A_1 - 2(a_2 - b_1)\ln A_2 = 3(a_1 - a_2 + b_1 - b_2). \tag{93}$$

Thus, the Gausson solutions to the coupled system are given by (82) and (83) with all the necessary parameters and constraints as listed.

7.1.1. CONSERVATION LAWS

In order to evaluate conserved quantities for the Zareamoghaddam model, we resort to the invariance and multiplier approach based on the well known result that the Euler-Lagrange operator annihilates a total divergence. First, if (T^t, T^x) is a conserved vector corresponding to a conservation law, then

$$D_t T^t + D_x T^x = 0 \tag{94}$$

(divergence equation) along the solutions of the differential equation

$$F(x,t,u,u_t,u_x,\ldots) = 0. \tag{95}$$

Moreover, if there exists a nontrivial differential function Q, called a 'multiplier', such that

$$E_u[QF] = 0, \tag{96}$$

then

$$QF = D_t T^t + D_x T^x, \tag{97}$$

where E_u is the Euler-Lagrange operator. Thus, a knowledge of each multiplier Q leads to a conserved vector determined by, inter alia, a *homotopy operator* [11, 15].

Let as consider a system

$$F^1(x,t,q,q_t,q_x,r,r_t,r_x,\ldots) = 0 \tag{98}$$

$$F^1(x,t,q,q_t,q_x,r,r_t,r_x,\ldots) = 0, \tag{99}$$

$Q = (f,g)$, say, so that

$$fF^1 + gF^2 = D_t T^t + D_x T^x, \tag{100}$$

and

$$E_{(q,r)}[D_t T^t + D_x T^x] = 0. \tag{101}$$

Here, T^t is the *conserved density*, while T^x is the flux.

It turns out that the only interesting case for the Zareamoghaddam model (80) and (81) that yields nontrivial conserved vectors are $a_j = b_j = c_j = 1$ for $j = 1,2$. Therefore, the conserved density and flux are given by

$$T_1^x = q\{r(\log q + \log r - 1) + r_{xx}\} + q_{xx} r - q_x r_x \tag{102}$$

$$T_1^t = qr \tag{103}$$

$$\begin{aligned} T_2^x &= \frac{1}{2}[-r\{q_t(\log q + \log r - 1) + q_{xxt}\} + q\{r_t(\log q + \log r - 1) + r_{xxt}\} \\ &+ r_x q_{xt} - q_x r_{xt} + r_t q_{xx} - q_t r_{xx}] \end{aligned} \tag{104}$$

$$T_2^t = r\{q_x(\ln q + \ln r - 1) + q_{xxx}\} + q\{r_x(\ln q + \ln r - 1) + r_{xxx}\} \tag{105}$$

$$T_3^x = \frac{1}{2}\left(-r_t q + q_t r + 2q_{xx} r_x - 2q_x r_{xx}\right) \tag{106}$$

$$T_3^t = \frac{1}{2}\left(r_x q - q_x r\right) \tag{107}$$

The corresponding conserved quantities are computed as follows:

$$I_1 = \int_{-\infty}^{\infty} T_1^t dx = \int_{-\infty}^{\infty} qr dx = \frac{A_1 A_2}{B}\sqrt{\frac{\pi}{2}} \tag{108}$$

$$\begin{aligned} I_2 &= \int_{-\infty}^{\infty} T_2^t dx \\ &= \int_{-\infty}^{\infty} \left[r\left\{q_x\left(\ln q + \ln r - 1\right) + q_{xxx}\right\} + q\left\{r_x\left(\ln q + \ln r - 1\right) + r_{xxx}\right\}\right] dx = 0 \end{aligned} \tag{109}$$

$$I_3 = \int_{-\infty}^{\infty} T_3^t dx = \int_{-\infty}^{\infty} \left(r_x q - q_x r\right) dx = 0 \tag{110}$$

These conserved quantities are obtained with Gaussons given by (82) and (83) from the previous sub-section.

7.2. GEAR-GRIMSHAW MODEL

This model, in its dimensionless form, is given by [3, 26]

$$q_t + a_1 q_x \ln q + b_1 q_x \ln r + c_1 \left(r \ln q\right)_x + \alpha_1 q_{xxx} + \beta_1 r_{xxx} = 0, \tag{111}$$

$$r_t + a_2 r_x \ln r + b_2 r_x \ln q + c_2 \left(q \ln r\right)_x + \alpha_2 r_{xxx} + \beta_2 q_{xxx} = 0. \tag{112}$$

This model has been studied earlier with power-law KdV equation [3, 26]. Moreover the previous model given by (80) and (81) is a special case of the Gear-Grimshaw model. In order to solve this system, the starting hypothesis is the same as given by (82) and (83). Substituting this hypothesis into (111) and (112) gives

$$\begin{aligned} & vA_1 - a_1 A_1 \ln A_1 - b_1 A_1 \ln A_2 - c_1 A_2 - c_1 A_2 \ln A_1 + 6B^2\left(\alpha_1 A_1 + \beta_1 A_2\right) \\ + \quad & \tau^2\left\{c_1 A_2 - 4B^2\left(\alpha_1 A_1 + \beta_1 A_2\right)\right\} = 0, \end{aligned} \tag{113}$$

and

$$\begin{aligned} & vA_2 - a_2 A_2 \ln A_2 - b_2 A_2 \ln A_1 - c_2 A_1 - c_2 A_1 \ln A_2 + 6B^2\left(\alpha_2 A_2 + \beta_2 A_1\right) \\ + \quad & \tau^2\left\{b_2 A_2 - 4B^2\left(\alpha_2 A_2 + \beta_2 A_1\right)\right\} = 0. \end{aligned} \tag{114}$$

Setting the coefficients of the linearly independent functions, τ^m for $m = 0, 2$, from (113) and (114), to zero gives

$$v = \frac{1}{A_1}\left\{a_1 A_1 \ln A_1 + b_1 A_1 \ln A_2 + c_1 A_2 + c_1 A_2 \ln A_1 - 6B^2\left(\alpha_1 A_1 + \beta_1 A_2\right)\right\}, \tag{115}$$

$$B = \frac{1}{2}\sqrt{\frac{c_1 A_2}{\alpha_1 A_1 + \beta_1 A_2}}, \tag{116}$$

and

$$v = \frac{1}{A_2}\left\{a_2 A_2 \ln A_2 + b_2 A_2 \ln A_1 + c_2 A_1 + c_2 A_1 \ln A_2 - 6B^2\left(\alpha_2 A_2 + \beta_2 A_1\right)\right\}, \tag{117}$$

$$B = \frac{1}{2}\sqrt{\frac{b_2 A_2}{\alpha_2 A_2 + \beta_2 A_1}}. \tag{118}$$

The widths of the Gaussons introduce the conditions

$$c_1 A_2\left(\alpha_1 A_1 + \beta_1 A_2\right) > 0 \tag{119}$$

and

$$b_2 A_2\left(\alpha_2 A_2 + \beta_2 A_1\right) > 0, \tag{120}$$

respectively.

Next, equating the speed of the Gaussons from (115) and (117) gives the relation between the parameter B and the amplitudes of the two components of Gaussons:

$$B = \left[\frac{A_1 A_2\left\{\left(a_1 \ln A_1 - a_2 \ln A_2\right) + \left(b_1 \ln A_2 - b_2 \ln A_1\right)\right\} + \left(c_1 A_2^2 - c_2 A_1^2\right) + c_2\left(A_2^2 \ln A_1 - A_1^2 \ln A_2\right)}{6\left\{\left(\alpha_1 - \alpha_2\right) A_1 A_2 + \beta_1 A_2^2 - \beta_2 A_1^2\right\}}\right]^{\frac{1}{2}}. \tag{121}$$

Finally, equating the widths of Gaussons from (116) and (118) leads to the ratio of the widths of the amplitudes being

$$\frac{A_1}{A_2} = \frac{\beta_1 b_2 - c_1 \alpha_2}{c_1 \beta_2 - \alpha_1 b_2}. \tag{122}$$

Thus, Gaussons of the coupled system (111) and (112) are given by (82) and (83), where the ratio of the amplitudes is seen in (122) and the width is given by (121).

8. CONCLUSIONS

A detailed study of shallow water waves with logarithmic nonlinearity is conducted in this paper. This nonlinearity is applied to the KdV equation and the corresponding model is analyzed in detail. The traveling wave hypothesis retrieved Gausson solitary waves. Two other integration schemes also retrieved the soliton solution with corresponding integrability criteria. The soliton perturbation theory formulated the adiabatic parameter dynamics of the soliton parameters. Subsequently, the model is extended to the case of two-dimensional waves where the KP equation is integrated with log-law nonlinearity. Finally, the study is generalized to two-layered shallow water wave dynamics that are modeled by the Zareamoghaddam model as well as by the Gear-Grimshaw model. The corresponding solitary waves for the coupled system are also obtained.

The results of this paper carry a lot of scope with further investigations in this direction. Later, this model will be applied to secure the quasi-stationary soliton solution of the perturbed KdV equation with log-law nonlinearity. The same is applicable for the KP equation with log-law nonlinearity. Additional integration tools will be implemented to secure Gausson solutions to such models studied in this paper. A few of these algorithms are exp-function method, simplest equation approach, Hirota's bilinear formulation, Adomian decomposition scheme as well as the variational iteration technique. These results will published elsewhere.

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