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FINITE ELEMENT-HOMOTOPY ANALYSIS FOR NONLINEAR POISSON EQUATION

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Abstract. The purpose of this work is to examine the numerical resolution for a class of nonlinear Poisson equations. They are associated to important problems arising from fluid mechanics, steady reaction-diffusion equations, electrostatics, elasticity. The proposed method combines the powerful technique of homotopy analysis and the reliable finite elements. It gives the possibility to consider complex geometry of the problem's domain both in 2 and in 3 dimensions. The approximate solutions are compared with the exact solutions in applications and the agreement is very good. Also, the computation proves very good accordance between the two convergence regions for absolute and relative errors. This is significant for the case the exact solution is unknown.

Keywords: finite element method, homotopy analysis method, nonlinear Poisson equation, variational methods.

1. INTRODUCTION

We analyze in this paper an efficient way for numerical solving of a class of Poisson nonlinear boundary value problems, in the hypothesis they have a solution. The nonlinearity is surpassed by homotopy analysis method (HAM) which transforms the initial problem in a sequence of linear problems [1, 2].

The geometric complexity of the domain can be overcome by means of a grid of finite elements (triangles, rectangles, isoparametric finite elements, tetrahedrons, parallelepipeds, prisms, etc.). As a part of computational physics, the approximate resolution of nonlinear Poisson equations is related to various problems which come from fluid mechanics, steady reaction-diffusion and heat equations, electrostatics, elasticity, geometry.

The nonlinear differential equations have essential importance in science and engineering. In the last decades, different techniques were developed for their analysis: Adomian decomposition method [3–5], homotopy perturbation method [6–8], the G'/G expansion method [9], the optimal homotopy asymptotic method [10], the variational iteration method [11]. The homotopy analysis method was successfully applied to a wide range of nonlinear problems: viscous flow problems

[12–15], magnetohydrodynamic problems for non-Newtonian fluids [16, 17], to obtain analytical approximate solutions for the Zakharov-Kuznetsov equations [18], in a multiparametric variant to solve integral equations [19], to obtain the analytical solution of the nonlinear cubic-quintic Duffing oscillators [20], to solve the second order random differential equations [21], to obtain the approximate solution of Kadomtsev-Petviashvili equation which appears in nonlinear wave theory [22], to study the system of nonlinear partial differential equations such as the Cauchy problem in the theory of thermoelasticity [23], to determine the exact solutions for the coupled Ramani equations [24].

Applications of the homotopy analysis to nonlinear partial differential equations are investigated in [25–28], where is applied the general boundary element method.

The paper is organized as follows: section 2 is concerning with basic ideas of the homotopy analysis method which is applied to the nonlinear Poisson equation. The associated variational problems by means of finite elements are formulated.

There are described the global approximation spaces of functions using interpolation polynomials. In section 3 we perform numerical tests and comparisons between series solutions and the exact solutions. Although the applications 3.2, 3.3 are manufactured, we selected them because they are various types of nonlinear problems with known exact solutions. The difficulty is not affected in any way. The programs were performed in C/C++ programming language and use Gauss algorithm for band matrix.

2. DESCRIPTION OF THE METHOD

We shall investigate the application of HAM to the nonlinear Poisson equation with Dirichlet boundary conditions:

$$\Delta u = F(x,u,\nabla u)\,,\; x=(x_1,x_2,...x_n)\in D\subset R^n \tag{1}$$

$$u\big|_{FrD} = g\,, \tag{2}$$

where D is an open bounded set with Lipschitz boundary $\Gamma = FrD$.

$$F : D\times H^1(D)\times (L^2(D))^n \to R$$

is a function so that

$$f(x_1,...,x_n) = F(x_1,...,x_n,u(x_1,...,x_n),\nabla u(x_1,...,x_n))$$

$$f\in L^2(D)\,,$$

for any $u\in H^1(D)$, and $g\in H^{1/2}(\Gamma)$ (the usual notations for Sobolev spaces [29]).

We consider the 2-dimensional case, while the 3-d case can be investigated in a similar manner. The modern technique of HAM provides the tools to resolve nonlinear partial equations with boundary conditions

$$\boldsymbol{N}[u(x)] = 0 \tag{3}$$

$$\boldsymbol{N}_1[u(x)]\big|_{\Gamma} = 0 . \tag{4}$$

The method determines a homotopy series

$$\phi(x, p) = \sum_{s=0}^{\infty} u_s(x) p^s \tag{5}$$

so that when p varies from 0 to 1, the series solution ϕ varies from the initial guess $\phi(x,0) = u_0(x)$ to the exact solution $\phi(x,1) = u(x)$. Homotopy technique establishes the zero-order deformation equation [1]:

$$(1 - p)\boldsymbol{L}[\phi(x, p) - u_0(x)] = p\hbar H(x)\boldsymbol{N}[\phi(x, p)] \tag{6}$$

and similar for the boundary conditions (4). The method is flexible in choosing the auxiliary linear operator $\boldsymbol{L}$, the initial approximation u_0 and the auxiliary function $H(x) \neq 0$, $H \in C(\overline{D})$. On the other hand, the convergence-control parameter $\hbar \neq 0$ can always be selected in order to obtain convergent homotopy series for $p = 1$. This will be a solution of the nonlinear problem:

$$u(x) = \phi(x,1) = \sum_{s=0}^{\infty} u_s(x) . \tag{7}$$

Applying the m-th order homotopy derivative $D_m(\phi) = \left.\dfrac{1}{m!}\dfrac{\partial^m \phi}{\partial p^m}\right|_{p=0}$ in eq. (6), we deduce the m-th order deformation equation:

$$\boldsymbol{L}[u_m(x) - \chi_m u_{m-1}(x)] = \hbar H(x) D_{m-1}(\boldsymbol{N}[\phi]) , \tag{8}$$

where $\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m \ge 2 \end{cases}$.

Further, we implement the HAM for the problem (1), (2). According to (1), we define the linear operator

$$\boldsymbol{L}[\phi(x, p)] = \Delta[\phi(x, p)] \tag{9}$$

and the nonlinear operator

$$\boldsymbol{N}[\phi(x,p)] = \Delta[\phi(x,p)] - F(x,\phi(x,p),\nabla\phi(x,p)). \tag{10}$$

For the boundary condition, a simple computation by HAM with

$$\boldsymbol{N_1}[\phi(x,p)] = \phi(x,p)\Big|_\Gamma - g(x), \quad \boldsymbol{L_1}[\phi(x,p)] = \phi(x,p)\Big|_\Gamma$$

and $u_0(x)$ so that $u_0\Big|_\Gamma = g$ provides $u_m\Big|_\Gamma = 0,\ m \ge 1$.

For m = 1, eq. (8) yields:

$$\Delta u_1 = \hbar\, H(x_1,x_2)\left[\Delta u_0 - F(x_1,x_2,u_0,\frac{\partial u_0}{\partial x_1},\frac{\partial u_0}{\partial x_2})\right], \text{ in } D \tag{11}$$

$$u_1\Big|_\Gamma = 0. \tag{12}$$

From (8), we obtain for $m \ge 2$

$$\Delta u_m = \left(1+\hbar H(x_1,x_2)\right)\Delta u_{m-1} - \hbar H(x_1,x_2) D_{m-1}[F(x_1,x_2,\phi,\frac{\partial\phi}{\partial x_1},\frac{\partial\phi}{\partial x_2})], \tag{13}$$

$$u_m\Big|_\Gamma = 0. \tag{14}$$

We denote as usual $H_0^1(D) = \{u \in L^2(D) / \frac{\partial u}{\partial x_i} \in L^2(D), i = \overline{1,n}, u\Big|_\Gamma = 0\}$.

Further, we define the approximate variational problem and construct the approximation spaces V_h by means of the finite element method.

For simplicity we suppose $D = (a,b)\times(c,d)$. For domains with more complex geometry, appropriate finite elements will be selected (triangles, rectangles, isoparametric elements in R^2, tetrahedrons, parallelepipeds, prisms in R^3, etc.). We divide the interval $[a,b]$ in r subintervals and $[c,d]$ in n subintervals. We draw parallels to Ox and Oy through the division points and get a grid T_h. It yields

$$D = \bigcup_{K\in \boldsymbol{T_h}} K.$$

We define the function spaces:

$$W_h = \{u_h \in C^0(\overline{D}) / u_h\Big|_K = \sum_{i=1}^{4} u_h(a_i^{(K)})\, p_i^{(K)}, (\forall) K \in T_h\},$$

$$V_h = \{u_h \in W_h \,/\, u_h\Big|_{\Gamma} = 0\}\,.$$

It follows ([30]) $W_h \subset H^1(D)$ and $V_h \subset H_0^1(D)$.

For K the rectangle with the nodes $a_i(a_{i1}, a_{i2}), i=\overline{1,4}$ we obtain ($A(K)$ is the area of K):

$$p_1(x_1,x_2) = \frac{1}{A(K)}(x_1 - a_{31})(x_2 - a_{32}),\ \ p_2(x_1,x_2) = \frac{1}{A(K)}(x_1 - a_{11})(a_{32} - x_2)$$

$$p_3(x_1,x_2) = \frac{1}{A(K)}(x_1 - a_{11})(x_2 - a_{12}),\ \ p_4(x_1,x_2) = \frac{1}{A(K)}(a_{31} - x_1)(x_2 - a_{32}).$$

Suppose E_h is the set of interior nodes of the grid, then a basis in V_h is made of functions $(u_{hM})_{M\in E_h}$ which verify

$$u_{hM}(M) = 1, u_{hM}(P) = 0 \text{ for } P \in E_h, P \neq M\,.$$

Integration on D and Gauss formula lead to the approximate variational problem associated to (11), (12): find $u_{h1} \in V_h$ so that

$$a(u_{h1}, v_h) = \tilde{F}_{h1}(v_h), \quad (\forall) v_h \in V_h\,, \tag{15}$$

where $a: V_h \times V_h \to R$ is the bilinear, continuous, coercive form

$$a(u_h, v_h) = \int_D \sum_{i=1}^{2} \frac{\partial u_h}{\partial x_i}\frac{\partial v_h}{\partial x_i}\mathrm{d}x_1 \mathrm{d}x_2\,. \tag{16}$$

$\tilde{F}_{h1}: V_h \to R$ is the linear, continuous form

$$\tilde{F}_{h1}(v_h) = \hbar \int_D H(x)\left[\sum_{i=1}^{2} \frac{\partial u_{h0}}{\partial x_i}\frac{\partial v_h}{\partial x_i} + F(x, u_0, \frac{\partial u_{h0}}{\partial x_1}, \frac{\partial u_{h0}}{\partial x_2}) v_h(x)\right]\mathrm{d}x\,. \tag{17}$$

The approximate variational problem at step $m \geq 2$ and corresponding to eqs. (13), (14) is to find $u_{hm} \in V_h$ such that:

$$a(u_{hm}, v_h) = \tilde{F}_{hm}(v_h), \quad (\forall) v_h \in V_h\,. \tag{18}$$

Here, $\tilde{F}_{hm}: V_h \to R$ is defined by

$$\tilde{F}_{hm}(v_h) = \int_D \left[\left(1+\hbar H(x)\right) \sum_{i=1}^{2} \frac{\partial u_{h,m-1}}{\partial x_i} \frac{\partial v_h}{\partial x_i} + \hbar H(x) D_{m-1} \cdot \right.$$
$$\left. \cdot [F(x, \phi_h, \frac{\partial \phi_h}{\partial x_1}, \frac{\partial \phi_h}{\partial x_2})] v_h(x) \right] \mathrm{d}x. \tag{19}$$

The eq. (15) and the sequence of approximate variational problems (18) with $m \geq 2$ provide the approximate homotopy series solution:

$$\phi(x,p) \cong \phi_h(x,p) = \sum_{m=0}^{\infty} u_{hm}(x) p^m . \tag{20}$$

We construct the initial approximation $u_{h0} \in W_h$ in a flexible and efficient manner by interpolation:

$$u_{h0}(a_i) = \begin{cases} g(a_i), \text{ for } a_i \in \Gamma \\ \tilde{g}(a_i), \text{ for } a_i \in \mathrm{Int}(D) \end{cases}, \tag{21}$$

where a_i are the nodes of the grid; $\tilde{g}$ is a convenient function which provides the values of u_{h0} in the inner nodes of the grid. It will be selected according to the rule of solution expression [1] and takes into consideration the governing differential equation and the boundary conditions. We proceed with applications of the proposed method.

3. APPLICATIONS

APPLICATION 3.1. Consider the nonlinear Poisson equation with boundary conditions:

$$\begin{gathered} \Delta \quad u = 3u^2 \text{ in } D = (0,1)\times(0,1) \\ u\Big|_{x_1=0} = \frac{4}{(3+x_2)^2}, \ u\Big|_{x_1=1} = \frac{4}{(4+x_2)^2} \ . \\ u\Big|_{x_2=0} = \frac{4}{(3+x_1)^2}, \ u\Big|_{x_2=1} = \frac{4}{(4+x_1)^2}. \end{gathered} \tag{22}$$

The exact solution is $u(x_1,x_2) = \dfrac{4}{(3+x_1+x_2)^2}$.

The above problem was investigated in [31] by means of the homotopy method of fundamental solutions. The technique accomplished in the paper works in the case the fundamental solution and the analytical particular solutions of the augmented polyharmonic spline (APS) associated with the considered operator are known.

The nonlinear operator of the equation is

$$\boldsymbol{N}[\phi(x,p)] = \Delta[\phi(x,p)] - 3\phi^2(x,p),$$

where ϕ is given in (5). It follows

$$D_m[\phi^2] = D_m\left[\sum_{s=0}^{\infty}\left(\sum_{i=0}^{s} u_i u_{s-i}\right)p^i\right] = \sum_{i=0}^{m} u_i u_{m-i}. \tag{23}$$

The first order deformation equation will be

$$\Delta u_1 = \hbar\, H(x_1,x_2)\left[\Delta u_0 - 3u_0^2\right], \text{ in } D \text{ and } u_1\Big|_{\Gamma} = 0.$$

For $m \geq 2$, the eqs. (13), (14) provide

$$\Delta u_m = (1+\hbar H(x_1,x_2))\Delta u_{m-1} - \hbar H(x_1,x_2)\sum_{k=0}^{m-1} u_k u_{m-1-k}, \text{ in } D$$

$$u_m\Big|_{\Gamma} = 0.$$

The approximate variational problem corresponding to the eqs. (15), (22) is: find $u_{h1} \in V_h$ so that

$$\int_D \nabla u_{h1}\cdot\nabla v_h \mathrm{d}x = \hbar\int_D H(x)\left[\nabla u_{h0}\cdot\nabla v_h + 3u_{h0}^2(x)v_h(x)\right]\mathrm{d}x,\ (\forall)v_h \in V_h. \tag{24}$$

The approximate variational problem associated to the m-th order $(m \geq 2)$ deformation equation and based on finite element method will be to find $u_{hm} \in V_h$ so that

$$\begin{aligned}\int_D \nabla u_{hm}\cdot\nabla v_h \mathrm{d}x &= \int_D (1+\hbar H(x))\nabla u_{h,m-1}\cdot\nabla v_h \mathrm{d}x + \\ &+3\hbar\sum_{i=0}^{m-1}\int_D H(x)\ u_{hi}(x)\ u_{h,m-1-i}(x)v_h(x)\mathrm{d}x,\quad (\forall)v_h \in V_h.\end{aligned} \tag{25}$$

The exact solution $u(x)$ in the domain D can be written in the form

$$u(x_1, x_2) = \sum_{i,j=0}^{\infty} a_{ij} (x_1 - \frac{1}{2})^i (x_2 - \frac{1}{2})^j .$$

Due to the boundary conditions and the rule of solution expression, we select the initial approximation in the nodes of the grid to be

$$u_{h0}(a_i) = \begin{cases} g(a_i), & \text{for } a_i \in \Gamma \\ 0, & \text{for } a_i \in \mathrm{Int}(D). \end{cases}$$

We obtain the numeric solution of the problem (22), approximation of the exact solution $u(x)$:

$$u(x) = \phi(x,1) \cong \phi_{hN}(x,1) = \sum_{i=0}^{N} u_{hi}(x) \overset{\text{def}}{=} U_{hN}(x,\hbar). \tag{26}$$

Define now the absolute and relative errors:

$$\begin{aligned} \mathrm{err}_{\mathrm{abs}}(x,\hbar,N) &= \left|U_{hN}(x,\hbar) - u(x)\right|, \\ \mathrm{err}_{\mathrm{rel}}(x,\hbar,N) &= \left|U_{hN}(x,\hbar) - U_{h,N-1}(x,\hbar)\right|, \ x \in D. \end{aligned} \tag{27}$$

The relative error is especially important for the case the exact solution is unknown. We set the auxiliary function $H(x) = 1$. Table 1 contains the absolute errors for two grids and different orders of approximation *N*.

Table 1

The absolute errors for different orders of approximation

(r,n)	N order of approximation	Control convergence parameter	$\sup_{x \in D} \mathrm{err}_{\mathrm{abs}}$
(10,10)	5	–0.99	$4.759 \cdot 10^{-5}$
	10	–0.525	$1.4047 \cdot 10^{-5}$
	15	–1.215	$6.982 \cdot 10^{-6}$
(20,20)	5	–0.975	$3.70533 \cdot 10^{-5}$
	10	–0.994	$3.36983 \cdot 10^{-6}$
	15	–1.232	$3.18456 \cdot 10^{-6}$

Figures 1 and 2 contain the graphs of the functions

$$\hbar \mapsto \sup\{\mathrm{err}_{abs}(x,\hbar,N) / x \in D\},$$
$$\hbar \mapsto \sup\{\mathrm{err}_{rel}(x,\hbar,N) / x \in D\},$$

for $N = 10$, $r = n = 10$ and respectively $N = 15$, $r = n = 20$.

The representations emphasize the appropriate regions for the control-convergence parameter $\hbar$ are horizontal line segments. Also, they prove very good agreement between the two convergence regions for absolute and relative errors.

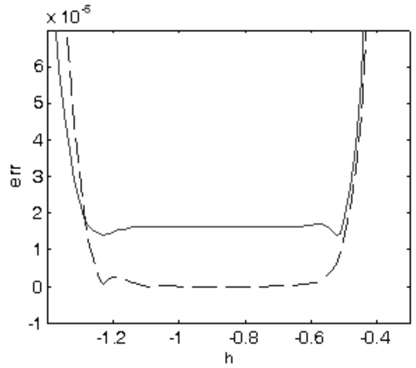

Fig. 1 – $\hbar$ -curve of err_{abs} (solid line) and err_{rel} (dashed line), $N = 10, r = n = 10$.

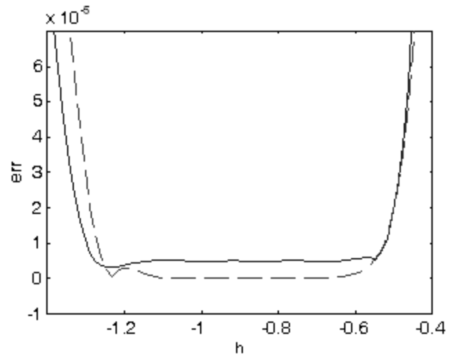

Fig. 2 – $\hbar$ -curve of err_{abs} (solid) and err_{rel} (dashed) for $N = 15, r = n = 20$.

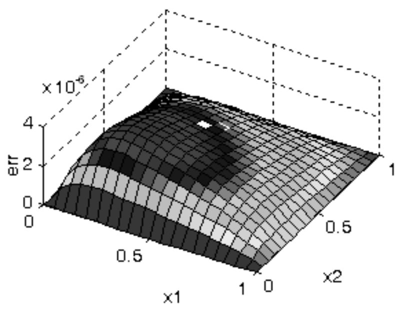

Fig. 3 – Absolute error at 15th order approximation for $r = n = 20$.

Figure 3 is referring to the graph of the function $D \ni x \mapsto \mathrm{err}_{\mathrm{abs}}(x,\ -1.232, 15)$.

APPLICATION 3.2. Find $u \in C^2(D) \cap C(\overline{D})$ so that

$$\begin{gathered}
\Delta u + 2u = \frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2} + \frac{1}{4}\sin 2x_1 \cdot \sin 2x_2, \quad (x_1, x_2) \in D = (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2}, \frac{1}{2}) \\
u\Big|_{x_1=-\frac{1}{2}} = -\sin\frac{1}{2} \cdot \cos x_2,\ u\Big|_{x_1=\frac{1}{2}} = \sin\frac{1}{2} \cdot \cos x_2 \\
u\Big|_{x_2=-\frac{1}{2}} = \sin x_1 \cdot \cos\frac{1}{2},\ u\Big|_{x_2=\frac{1}{2}} = \sin x_1 \cdot \cos\frac{1}{2}.
\end{gathered} \tag{28}$$

The exact solution of (28) is $u(x_1, x_2) = \sin x_1 \cdot \cos x_2$.

The eqs. (10), (28) provide

$$\boldsymbol{N}[\phi(x,p)] = \Delta[\phi(x,p)] + 2\phi(x,p) - \frac{\partial \phi}{\partial x_1}(x,p) \cdot \frac{\partial \phi}{\partial x_2}(x,p) - \frac{1}{4}\sin 2x_1 \cdot \sin 2x_2 .$$

The exact solution can be expressed as $u(x_1, x_2) = \sum_{i,j=0}^{\infty} a_{ij} x_1^i x_2^j$. The rule of solution expression and the boundary conditions determine the initial approximation $u_{h0} \in W_h$, constructed by finite elements:

$$u_{h0}(a_i) = \begin{cases} g(a_i), \text{ for } & a_i \in \Gamma \\ 0, \quad\quad \text{for } & a_i \text{ inner node}. \end{cases}$$

The approximate variational problem corresponding to the relations (15), (28) is the following: find $u_{h1} \in V_h$ so that

$$\begin{gathered}
\int_D \nabla u_{h1} \cdot \nabla v_h dx = \hbar \int_D H(x) \Big[\nabla u_{h0} \cdot \nabla v_h + \\
+ \left(\frac{\partial u_{h0}}{\partial x_1}\frac{\partial u_{h0}}{\partial x_2} + \frac{1}{4}\sin 2x_1 \cdot \sin 2x_2 - 2u_{h0} \right) v_h(x) \Big] \mathrm{d}x,\ (\forall) v_h \in V_h.
\end{gathered} \tag{29}$$

The approximate variational problem associated to the *m*-th order $(m \geq 2)$ deformation equation and based on finite element method will be: find $u_{hm} \in V_h$ so that

$$\begin{gathered}
\int_D \nabla u_{hm} \cdot \nabla v_h \mathrm{d}x = \int_D (1 + \hbar H(x)) \nabla u_{h,m-1} \cdot \nabla v_h \mathrm{d}x + \\
+ \hbar \int_D H(x) \left[\sum_{i=0}^{m-1} \frac{\partial u_{h,m-1-i}}{\partial x_1}\frac{\partial u_{h,i}}{\partial x_2} - 2u_{h,m-1} \right] v_h(x) \mathrm{d}x,\ (\forall) v_h \in V_h.
\end{gathered} \tag{30}$$

The approximate solution of the problem formulated in (28) is

$$U_{hN}(x,\hbar) = \sum_{i=0}^{N} u_{hi}(x,\hbar) \approx u(x).$$

Table 2 contains the absolute errors for $r = n = 10$, $r = n = 20$ and $N = 5, 10, 15$.

Table 2

The absolute errors for different orders of approximation

(r,n)	N order of approximation	Control convergence parameter	$\sup_{x\in D} \mathrm{err}_{\mathrm{abs}}$
(10,10)	5	–1.02	$2.29217 \cdot 10^{-5}$
	10	–0.735	$1.212146 \cdot 10^{-5}$
	15	–0.730	$1.0163 \cdot 10^{-5}$
(20,20)	5	–1.005	$1.32387 \cdot 10^{-5}$
	10	–0.66	$3.84433 \cdot 10^{-6}$
	15	–0.78	$2.736449 \cdot 10^{-6}$

In Figs. 4 and 5 we present the graphs of the functions

$$\hbar \mapsto \sup\{\mathrm{err}_{\mathrm{abs}}(x,\hbar,N)/(x_1,x_2)\in[-\tfrac{1}{2},\tfrac{1}{2}]\times[-\tfrac{1}{2},\tfrac{1}{2}]\}$$

$$\hbar \mapsto \sup\{\mathrm{err}_{\mathrm{rel}}(x,\hbar,N)/(x_1,x_2)\in[-\tfrac{1}{2},\tfrac{1}{2}]\times[-\tfrac{1}{2},\tfrac{1}{2}]\},$$

for $N = 10$, $r = n = 10$ and respectively $N = 15$, $r = n = 20$.

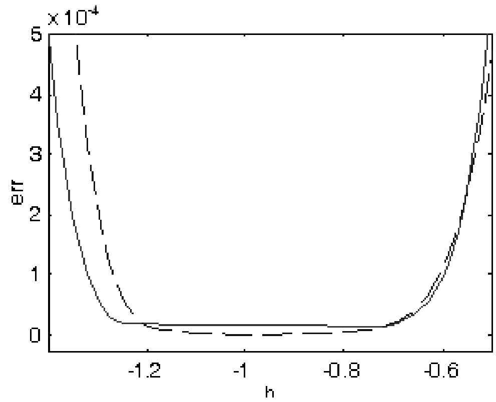

Fig. 4 – $\hbar$-curve of $\mathrm{err}_{\mathrm{abs}}$ (solid) and $\mathrm{err}_{\mathrm{rel}}$ (dashed), $N = 10$, $r = n = 10$.

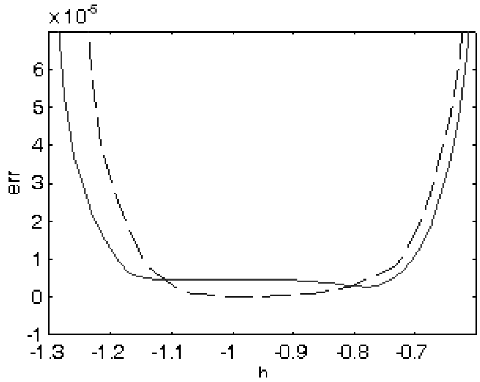

Fig. 5 – $\hbar$ -curve of $\mathrm{err}_{\mathrm{abs}}$ (solid) and $\mathrm{err}_{\mathrm{rel}}$ (dashed) for $N = 15$, $r = n = 20$.

Figure 6 deals with the representation of the function $\bar{D} \ni x \mapsto \mathrm{err}_{\mathrm{abs}}(x, -0.78, 15)$ for $r = n = 20$.

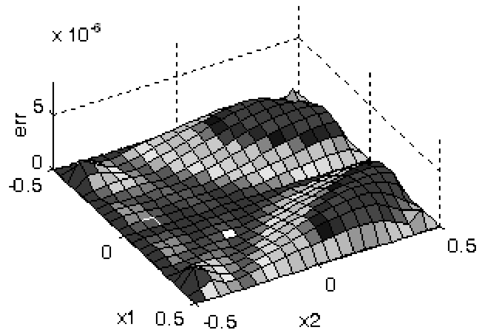

Fig. 6 – Absolute error at 15-th order approximation for $r = n = 20$.

APPLICATION 3.3. Determine $u \in C^2(D) \cap C(\bar{D})$ which verifies

$$\Delta u + u - \cos u = -\cos(x_1 \sin x_2) \quad \text{in } D = (-\frac{\pi}{6}, \frac{\pi}{6}) \times (-\frac{\pi}{6}, \frac{\pi}{6})$$

$$u(-\frac{\pi}{6}, x_2) = -\frac{\pi}{6} \sin x_2, \quad u(\frac{\pi}{6}, x_2) = \frac{\pi}{6} \sin x_2 \tag{31}$$

$$u(x_1, -\frac{\pi}{6}) = -\frac{x_1}{2}, \quad u(x_1, \frac{\pi}{6}) = \frac{x_1}{2}.$$

The eqs. (10), (31) provide

$$\boldsymbol{N}[\phi(x,p)] = \Delta[\phi(x,p)] + \phi(x,p) - \cos\phi(x,p) + \cos(x_1 \sin x_2),$$

where $\phi(x,p) = \sum_{k=0}^{\infty} u_k(x) p^k$.

Taking into consideration the rule of solution expression and the boundary conditions, we select $\tilde{g}(x_1, x_2) = x_1 x_2$ in the relation (21). We deduce the first order deformation equation

$$\Delta u_1 = \hbar\, H(x_1, x_2)\left[\Delta u_0 - \cos u_0 + u_0 + \cos(x_1 \sin x_2)\right],$$

in D

$$u_1\Big|_{\Gamma} = 0.$$

The m-th $(m \geq 2)$ order deformation equation will be

$$\Delta u_m = \left(1 + \hbar H(x_1, x_2)\right)\Delta u_{m-1} + \hbar H(x_1, x_2)\left[u_{m-1} - D_{m-1}(\cos\phi)\right],$$

in D

$$u_m\Big|_{\Gamma} = 0.$$

The variational problem corresponding to the first order deformation equation, formulated by means of finite elements is: find $u_{h1} \in V_h$ so that

$$\begin{aligned}\int_D \nabla u_{h1} \cdot \nabla v_h \mathrm{d}x = \hbar \int_D H(x)\big[\nabla u_{h0} \cdot \nabla v_h + \\ + \left(\cos u_{h0} - u_{h0} - \cos(x_1 \cdot \sin x_2)\right) v_h(x) \big]\mathrm{d}x\end{aligned} \tag{32}$$

for $(\forall) v_h \in V_h$.

The approximate variational problem associated to the m-th order $(m \geq 2)$ deformation equation will be: find $u_{hm} \in V_h$ so that

$$\begin{aligned}\int_D \nabla u_{hm} \cdot \nabla v_h \mathrm{d}x = \int_D (1 + \hbar H(x)) \nabla u_{h,m-1} \cdot \nabla v_h \mathrm{d}x + \\ + \hbar \int_D H(x)\left[D_{m-1}(\cos\phi_h) - u_{h,m-1}\right] v_h(x)\mathrm{d}x,\ (\forall) v_h \in V_h.\end{aligned} \tag{33}$$

We denoted $\phi_h(x,p) = \sum_{k=0}^{\infty} u_{hk}(x) p^k$ the approximate homotopy series.

In computation we take into account the recurrence relations [32]:

$$D_0(\sin\phi) = \sin u_0,$$
$$D_0(\cos\phi) = \cos\ \ u_0,$$
$$D_m(\sin\phi) = \sum_{k=0}^{m-1}\ (1-\frac{k}{m})D_k(\cos\phi)\ \ u_{m-k},\ m \ge 1, \tag{34}$$
$$D_m(\cos\phi) = -\sum_{k=0}^{m-1}\ (1-\frac{k}{m})D_k(\sin\phi)\ \ u_{m-k},\ m \ge 1.$$

The exact solution of the problem (31) is $u(x_1, x_2) = x_1 \sin x_2$.

Table 3 contains the absolute errors for r = n = 10, r = n = 20 and N = 3, 5, 10.

Tabel 3

The absolute errors for different orders of approximation

(r,n)	N order of approximation	Control convergence parameter	
(10,10)	5	–0.81	$2.071648\cdot 10^{-6}$
	10	–0.44	$1.959482\cdot 10^{-6}$
	15	–0.555	$1.948370\cdot 10^{-6}$
(20,20)	5	–0.885	$4.401830\cdot 10^{-7}$
	10	–0.63	$4.301667\cdot 10^{-7}$
	15	–0.504	$4.024269\cdot 10^{-7}$

Figures 7 and 8 are referring to the representations of the functions

$$\hbar \mapsto \sup\{\mathrm{err}_{\mathrm{abs}}(x,\hbar,N)\,/\,x \in D\},\ \hbar \mapsto \sup\{\mathrm{err}_{\mathrm{rel}}(x,\hbar,N)\,/\,x \in D\},$$

for N = 5, r = n = 10 and respectively N = 15, r = n = 20.

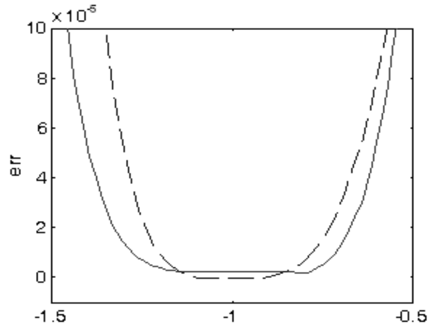

Fig. 7 – $\hbar$-curve of $\mathrm{err}_{\mathrm{abs}}$ (solid) and $\mathrm{err}_{\mathrm{rel}}$ (dashed) for $N = 5, r = n = 10$.

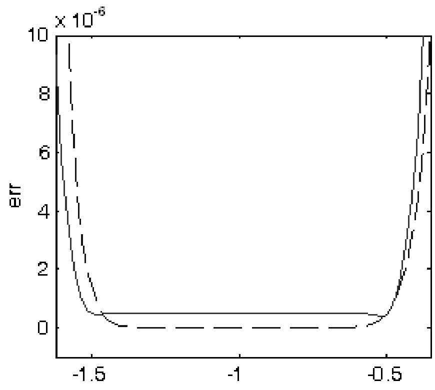

Fig. 8 – $\hbar$ -curve of $\mathrm{err}_{\mathrm{abs}}$ (solid) and (dashed) for $\mathrm{err}_{\mathrm{rel}}$ $N = 15, r = n = 20$.

Figure 9 presents the graph of the function

$$\bar{D} \ni x \mapsto \mathrm{err}_{\mathrm{abs}}(x,\ \hbar = -0.504,\ N = 15).$$

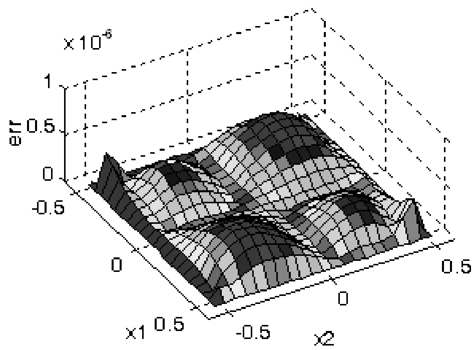

Fig. 9 – Absolute error at 15-th order approximation for $r = n = 20$.

4. CONCLUSIONS

In this work, we investigated a method to compute the numerical solution for a class of nonlinear Poisson equations. The proposed technique combines the homotopy analysis and finite element method. It gives the possibility to consider a complex geometry of the problem's domain both in 2 and 3 dimensions. The examples prove the validity of the method.

REFERENCES

1. S.J. Liao, *Beyond perturbation: introduction to the homotopy analysis method*, Boca Raton, Chapman & Hall/CRC Press, 2003.
2. S.J. Liao, Y. Tan, *A general approach to obtain series solutions of nonlinear differential equations,* Stud. Appl. Math. **119**, 297–355 (2007).
3. G. Adomian, *Solving Frontier Problems of Physics: The Decomposition Method,* Kluwer, Academic publishers, Boston, 1994.
4. A.M. Wazwaz, *Solving Schlomilch's integral equation by the regularization – Adomian method,* Romanian Journal of Physics **60**, 56–71 (2015).
5. I.A. Cristescu, *Decomposition method for neutron transport equation*, Romanian Journal of Physics **60**, 179–189 (2015).
6. J.H. He, H*omotopy perturbation technique*, Computer Methods in Applied Mechanics and Engineering **178**, 257–262 (1999).
7. Alireza K. Golmankhaneh, Ali K. Golmankhaneh, D. Baleanu, *Homotopy perturbation method for solving a system of Schrodinger-Korteweg-de Vries equations*, Romanian Reports in Physics **63**, 609–623 (2011).
8. H. Noshad, S.S. Bahador, *Numerical solution of Fokker-Planck equation for energy straggling of protons*, Romanian Reports in Physics **6**, 99–108 (2014).
9. R. Abazari, *General solution of a special class of nonlinear BBM-B equations by using the* (G'/G) *– expansion method*, Romanian Reports in Physics **66**, 286–295 (2014).
10. N. Herisanu, V. Marinca, T. Dordea, Gh. Madescu, *A new analytical approach to nonlinear vibration of an electrical machine*, Proceedings of the Romanian Academy Series A **9** (2008).
11. H. Jafari, A. Kadem, D. Baleanu, T. Yilmaz, *Solutions of the fractional Davey-Stewartson equations with variational iteration method*, Romanian Reports in Physics **64**, 337–346 (2012).
12. S.J. Liao, *A uniformly valid analytic solution of two-dimensional viscous flow over a semi-infinite flat plate*, Journal Fluid Mechanics **385**, 101–128 (1999).
13. T. Hayat, T. Javed, M. Sajid, *Analytic solution for rotating flow and heat transfer analysis of a third-grade fluid*, Acta Mechanica **191**, 219–229 (2007).
14. T. Hayat, M. Sajid, *On analytic solution for thin film flow of a forth grade fluid down a vertical cylinder*, Physics Letters A **361**, 316–322 (2007).
15. R. Ellahi, M. Raza, K. Vafai, *Series solutions of non-Newtonian nanofluids with Reynolds' model and Vogel's model by means of the homotopy analysis method*, Mathematical and Computer Modelling **55**, 1876–1891 (2012).
16. T. Hayat, Z. Abbas, M. Sajid, S. Asghar, *The influence of thermal radiation on MHD flow of a second grade fluid,* Int Journal Heat Mass Transfer **50**, 931–941 (2007).
17. Hayat T., Sajid M., *Homotopy analysis of MHD boundary layer flow of an upper-convected Maxwell fluid*, Int. Journal Engineering Sciences **45**, 393–401 (2007).
18. A. Jafarian, P. Ghaderi, Alireza K. Golmankhaneh, D. Baleanu, *Analytical approximate solutions of the Zakharov-Kuznetsov equations,* Romanian Reports in Physics **66**, 296–306 (2014).
19. K. Sayevand, D. Baleanu, M. Fardi, *Development of a multi-convergence controller parametric perturbation approach for some linear and nonlinear integral equations,* Romanian Reports in Physics **67**, 359–374 (2015).
20. K. Sayevand, D. Baleanu, M. Fardi, *A perturbative analysis of nonlinear cubic-quintic Duffing oscillators*, Proceedings Romanian Academy Series A **15**, 228–234 (2014).
21. Alireza K.Golmankhaneh, Neda A. Porghoveh, D. Baleanu, *Mean square solutions of second-order random differential equations by using homotopy analysis method,* Romanian Reports in Physics **65**, 350–362 (2013).
22. A. Jafarian, P. Ghaderi, Alireza K. Golmankhaneh, *Construction of soliton solution to the Kadomtsev-Petviashvili – II equation using homotopy analysis method*, Romanian Reports in Physics **65**, *1*, 76–83 (2013).

23. A. Jafarian, P. Ghaderi, Alireza K. Golmankhaneh, D. Baleanu, *On a one-dimensional nonlinear coupled system of equations in the theory of thermoelasticity*, Romanian Journal of Physics **58**, *7–8*, 694–702 (2013).
24. A. Jafarian, P. Ghaderi, Alireza K. Golmankhaneh, *Homotopy analysis method for solving coupled Ramani equations*, Romanian Journal of Physics **59**, 26–35 (2014).
25. S.J. Liao, *Boundary element method for general nonlinear differential operators*, Engineering Analysis with Boundary Elements **20**, 91–99 (1997).
26. S.J. Liao, *On the general boundary element method*, Engineering Analysis with Boundary Elements, **21**, *1*, 39–51 (1998).
27. S.J. Liao, *The general boundary element method and its further generalizations*, Int. J. Numer. Methods Fluids **31**, *3*, 627–655 (1999).
28. Y.Y. Wu, SJ. Liao, XY. Zhao, *Some notes on the general boundary element method for highly nonlinear problem*, Commun. Nonlin. Sci. Numer. Simulat. **10**, *7*, 725–735 (2005).
29. R. Adams, *Sobolev Spaces*, Academic Press, 1975.
30. P.G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North Holland, 1978.
31. C.C. Tsai, *Homotopy method of fundamental solutions for solving certain nonlinear partial differential equations*, Engineering Analysis with Boundary Elements **36**, 1226–1234 (2012).
32. S.J. Liao, *Notes on the homotopy analysis method: Some definitions and theorems*, Commun. Nonlin. Sci. Numer. Simulat. **14**, 983–997 (2009).