Romanian Reports in Physics, Vol. 68, No. 3, P. 979–989, 2016

HOMOGENIZATION RESULTS FOR A DYNAMIC COUPLED THERMOELASTICITY PROBLEM

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Received June 24, 2015

Abstract. The macroscopic behavior of the solution of a dynamic coupled thermoelasticity problem in a periodic composite material made up of two connected constituents with imperfect contact at their interface is analyzed. The homogenized problem, derived *via* the periodic unfolding method, comprises new coupling terms involving the macroscopic displacement and temperature fields, generated by the imperfect bonding at the interface between the two phases of the composite.

Key words: homogenization, coupled thermoelasticity, imperfect interfaces.

1. INTRODUCTION

The prediction of the macroscopic behavior of thermoelastic microstructured materials is a subject of topical interest for a broad category of researchers. The growing interest in such a problem is justified by the increased need of designing advanced composite materials, with useful mechanical and thermodynamical properties. In particular, the problem of multiscale modeling of thermoelastic composites with imperfect interfaces has attracted a lot of interest in the last years, due to the great importance of such heterogeneous materials in many engineering applications. For instance, there are important applications of the interphase effects on the thermoelastic response of polymer nanocomposite materials.

The main purpose of this paper is to analyze, in the framework of the homogenization theory, the macroscopic behavior of the solution of a system of equations for a coupled thermoelasticity problem in a periodic composite material consisting of two connected components with imperfect bonding at their interface. The coupled theory of linear thermoelasticity takes into account the effect of the strain on the temperature field and vice versa. Thus, our dynamic thermomechanical problem is strongly coupled by hyperbolic and parabolic equations and the displacement and the temperature fields must be find simultaneously.

We suppose that the domain Ω occupied by the composite medium is the union of two connected sets, Ω_1^ε and, respectively, Ω_2^ε, separated by an imperfect interface Γ_ε. Here, ε is a small positive parameter that characterizes the scale of the heterogeneity of the material. We assume that both the temperature fluxes and the tractions are continuous across the interface Γ_ε, while the displacement and the temperature fields exhibit jumps of order ε therein, these jumps being proportional to the tractions and, respectively, to the temperature flux across Γ_ε. Using the powerful periodic unfolding method (see [2-4] and, for time-depending problems, [6] and [16]), we derive the homogenized problem, which exhibits some interesting features, such as the appearance of new coupling terms involving the effective displacement and temperature fields and the presence of new homogenized coefficients in the coupled system.

Related problems to the one we address here have been studied, with various methods, over the last years. For a nice presentation of the classical theory of thermoelasticity, the reader is referred to [13]. Also, for some interesting thermoelasticity models, we refer to [1], [7], [8] and [12]. For transmission problems in media with imperfect interfaces, see [4], [5], [10], [14] and [15].

In [11], a similar model was considered, but in a different geometry and with different scalings of the temperature-displacement tensors of the two constituents, leading to different homogenized results. More precisely, the domain Ω was considered to be the union of a connected part Ω_1^ε and a disconnected one Ω_2^ε and the temperature-displacement tensor was supposed to be of order of unity in the connected part of the medium and, respectively, of order ε in the disconnected one. As a consequence, the macroscopic elasticity tensor, the temperature-displacement tensor and the thermic-conductivity tensor corresponding to the disconnected part canceled at the limit. In our case, we keep these tensors in the macroscopic system and, in addition, we get a different specific heat coefficient in the equation for the macroscopic temperature field coming from the disconnected part. Moreover, let us mention the presence of new coupling terms in the macroscopic system and the different functional setting.

The rest of the paper is organized in the following way: in the next section, we write down the coupled system describing our microscopic thermoelasticity problem. Existence and uniqueness results for the solution of the variational formulation of this system are presented in Section 3. Proper estimates of the weak solution are obtained and, *via* the periodic unfolding method, we give some convergence results in suitable Hilbert spaces in Section 4 and we derive the homogenized problem. A few concluding remarks are outlined in the last section.

2. SETTING OF THE MICROSCOPIC THERMOELASTICITY PROBLEM

We consider a material body occupying a bounded domain Ω in $\mathbb{R}^n$ $(n \geq 3)$, with a Lipschitz boundary $\partial\Omega$ made up of a finite number of connected components. The domain Ω is supposed to be a periodic structure formed by two connected parts, Ω_1^ε and Ω_2^ε, separated by an interface Γ^ε. Adopting the geometry in [7], we assume that only the phase Ω_1^ε reaches the outer fixed boundary $\partial\Omega$. Here, ε is considered to be a small real parameter related to the characteristic dimension of our two regions. Let Y_1 be an open connected Lipschitz subset of the unit cube $Y = (0,1)^n$ and $Y_2 = Y \setminus \overline{Y_1}$. We consider that the boundary Γ of Y_2 is locally Lipschitz and that its intersections with the boundary of the reference cell Y are reproduced in an identical manner on the opposite faces of the unit cell. Moreover, if we repeat Y in a periodic way, the union of all the sets $\overline{Y_1}$ is a connected set, with a locally C^2 boundary. Also, we consider that the origin of the coordinate system lies in a ball contained in the above mentioned union (see [7]). For any $\varepsilon \in (0,1)$, let $Z_\varepsilon = \{k \in \mathbb{Z}^n \mid \varepsilon k + \varepsilon Y \subseteq \Omega\}$ and $K_\varepsilon = \{k \in Z_\varepsilon \mid \varepsilon k \pm \varepsilon e_i + \varepsilon Y \subseteq \Omega, \forall i = 1,\ldots,n\}$, where e_i are the vectors of the canonical basis of $\mathbb{R}^n$. We define $\Omega_2^\varepsilon = \mathrm{int}(\bigcup_{k \in K_\varepsilon} (\varepsilon k + \varepsilon \overline{Y_2}))$ and $\Omega_1^\varepsilon = \Omega \setminus \overline{\Omega_2^\varepsilon}$ and we denote by ν the unit normal on Γ_ε, pointing outward to Ω_1^ε.

For $\alpha \in \{1,2\}$, we define the fourth order elasticity tensors (also called *stiffness tensors*) $A^{\alpha\varepsilon}$ of the two phases of the composite. We take $A^{\alpha\varepsilon}(x) = A^\alpha(x/\varepsilon)$, where A^α are positive definite symmetric tensors, with components $a_{ijkh}^\alpha \in L^\infty(Y)$, for $1 \leq i,j,k,h \leq n$. We suppose that a_{ijkh}^α are Y-periodic real smooth functions. We also consider the second order displacement-temperature tensors $B^{\alpha\varepsilon}(x) = B^\alpha(x/\varepsilon)$ and the thermal-conductivity tensors $K^{\alpha\varepsilon}(x) = K^\alpha(x/\varepsilon)$, with B^α and K^α symmetric tensors with Y-periodic smooth components $b_{ij}^\alpha, k_{ij}^\alpha \in L^\infty(Y)$. Moreover, we consider that K^α are positive definite. In addition, T_0 is the reference temperature, $\rho^{\alpha\varepsilon}$ are the mass densities of the two phases, defined by $\rho^{\alpha\varepsilon}(x) = \rho^\alpha(x/\varepsilon)$, and $c^{\alpha\varepsilon}(x) = c^\alpha(x/\varepsilon)$ represent the specific heats at constant deformation of the two constituents. We define the jump coefficients $h_\varepsilon^u(x) = h^u(x/\varepsilon)$ and $h_\varepsilon^\theta(x) = h^\theta(x/\varepsilon)$ and we suppose that $\rho^\alpha, c^\alpha, h^u, h^\theta \in L^\infty(Y)$ are Y-periodic, smooth and strictly positive functions.

Finally, if $u^{\alpha\varepsilon}$ and $\theta^{\alpha\varepsilon}$ are the displacement and the temperature fields of $\Omega_\alpha^\varepsilon$, the constitutive laws are taken to be of the form

$$\sigma_{ij}^{\alpha\varepsilon} = a_{ijkh}^{\alpha\varepsilon} e_{kh}(u^{\alpha\varepsilon}) - b_{ij}^{\alpha\varepsilon}\theta^{\alpha\varepsilon},$$

with

$$e_{kh}(u^{\alpha\varepsilon}) = \frac{1}{2}\left(\frac{\partial u_k^{\alpha\varepsilon}}{\partial x_h} + \frac{\partial u_h^{\alpha\varepsilon}}{\partial x_k}\right)$$

being the components of the deformation tensor.

Throughout the paper, by C we denote a generic positive constant, which is independent of the scale parameter ε and by an overdot we denote the time derivative.

If $(0,T)$, with $0<T<\infty$, is the time interval under consideration, we shall analyze the macroscopic behavior of the solution of the following microscopic system:

$$-\frac{\partial \sigma_{ij}^{\alpha\varepsilon}}{\partial x_j}+\rho^{\alpha\varepsilon}\ddot{u}_i^{\alpha\varepsilon}=f_i \quad \text{in } (0,T)\times\Omega_\alpha^\varepsilon, \tag{2.1}$$

$$-\frac{\partial}{\partial x_i}\Big(k_{ij}^{\alpha\varepsilon}\frac{\partial \theta^{\alpha\varepsilon}}{\partial x_j}\Big)+T_0 b_{ij}^{\alpha\varepsilon}\dot{e}_{ij}(u^{\alpha\varepsilon})+c^{\alpha\varepsilon}\dot{\theta}^{\alpha\varepsilon}=r \quad \text{in } (0,T)\times\Omega_\alpha^\varepsilon, \tag{2.2}$$

$$\sigma_{ij}^{1\varepsilon}\nu_j=\sigma_{ij}^{2\varepsilon}\nu_j \quad \text{on } (0,T)\times\Gamma_\varepsilon, \tag{2.3}$$

$$k_{ij}^{1\varepsilon}\frac{\partial \theta^{1\varepsilon}}{\partial x_j}\nu_i=k_{ij}^{2\varepsilon}\frac{\partial \theta^{2\varepsilon}}{\partial x_j}\nu_i \quad \text{on } (0,T)\times\Gamma_\varepsilon, \tag{2.4}$$

$$\sigma_{ij}^{1\varepsilon}\nu_j=\varepsilon h_\varepsilon^u(u_i^{2\varepsilon}-u_i^{1\varepsilon}) \quad \text{on } (0,T)\times\Gamma_\varepsilon, \tag{2.5}$$

$$k_{ij}^{1\varepsilon}\frac{\partial \theta^{1\varepsilon}}{\partial x_j}\nu_i=\varepsilon h_\varepsilon^\theta(\theta^{2\varepsilon}-\theta^{1\varepsilon}) \quad \text{on } (0,T)\times\Gamma_\varepsilon, \tag{2.6}$$

$$u^{1\varepsilon}=0, \quad \theta^{1\varepsilon}=0 \quad \text{on } (0,T)\times\partial\Omega, \tag{2.7}$$

$$u^{\alpha\varepsilon}(0,x)=0, \quad \dot{u}^{\alpha\varepsilon}(0,x)=0, \quad \theta^{\alpha\varepsilon}(0,x)=0 \quad \text{in } \Omega_\alpha^\varepsilon, \tag{2.8}$$

where f_i are the components of the body force $f\in L^2(\Omega)^n$ and $r\in L^2(\Omega)$ is the internal heat source.

3. THE FUNCTIONAL SETTING

In order to write the variational formulation of problem (2.1)-(2.8), we define

$$V_{1\varepsilon}=\left\{v\in C^\infty(0,T;H^1(\Omega_1^\varepsilon))),\ v=0 \text{ on } \partial\Omega \text{ and } v=0 \text{ on } \{0\}\times\Omega\right\},$$

$$V_{2\varepsilon}=\left\{v\in C^\infty(0,T;H^1(\Omega_2^\varepsilon))),\ v=0 \text{ on } \{0\}\times\Omega\right\}$$

and we set $W_\varepsilon=(V_{1\varepsilon}^n\times V_{2\varepsilon}^n)\times(V_{1\varepsilon}\times V_{2\varepsilon})$. An element of the space W_ε is denoted by $V=(v,w)$, with $v=(v^1,v^2)\in V_{1\varepsilon}^n\times V_{2\varepsilon}^n$ and $w=(w^1,w^2)\in V_{1\varepsilon}\times V_{2\varepsilon}$.

Following [13], the weak formulation of problem (2.1)-(2.8) is as follows: find $U^\varepsilon=(u^\varepsilon,\theta^\varepsilon)\in W_\varepsilon$ such that

$$\mathcal{L}_\varepsilon(U^\varepsilon,V)=\mathcal{D}_\varepsilon\left((f,r),V\right),\ \forall V=(v,w)\in W_\varepsilon, \tag{3.1}$$

where $\mathcal{L}_\varepsilon : W_\varepsilon \times W_\varepsilon \to \mathbb{R}$ is the bilinear form defined by

$$\mathcal{L}_\varepsilon(U,V) = \sum_{\alpha=1,2} \int_0^T \int_{\Omega_\alpha^\varepsilon} \Big[(t-T)\Big(\big(-a_{ijkh}^{\alpha\varepsilon} e_{kh}(u^\alpha) + b_{ij}^{\alpha\varepsilon}\theta^\alpha\big)\, e_{ij}(\dot{v}^\alpha)$$

$$+\rho^{\alpha\varepsilon}\dot{u}_i^\alpha \ddot{v}_i^\alpha + + b_{ij}^{\alpha\varepsilon} e_{ij}(u^\alpha)\dot{w}^\alpha + \frac{1}{T_0} c^{\alpha\varepsilon}\theta^\alpha \dot{w}^\alpha\Big) + \rho^{\alpha\varepsilon}\dot{u}_i^\alpha \dot{v}_i^\alpha$$

$$+b_{ij}^{\alpha\varepsilon} e_{ij}(u^\alpha) w^\alpha + \frac{1}{T_0} c^{\alpha\varepsilon}\theta^\alpha w^\alpha + \frac{1}{T_0}\int_0^t k_{ij}^{\alpha\varepsilon} \frac{\partial\theta^\alpha}{\partial x_j}\frac{\partial w^\alpha}{\partial x_i}\,\mathrm{d}s\Big]\,\mathrm{d}x\,\mathrm{d}t$$

$$-\varepsilon \int_0^T \int_{\Gamma_\varepsilon} (t-T) h_\varepsilon^u (u_i^2 - u_i^1)(\dot{v}_i^2 - \dot{v}_i^1)\,\mathrm{d}\sigma\,\mathrm{d}t$$

$$-\frac{\varepsilon}{T_0} \int_0^T \int_{\Gamma_\varepsilon} (t-T) h_\varepsilon^\theta (\theta^2 - \theta^1)(w^2 - w^1)\,\mathrm{d}\sigma\,\mathrm{d}t,$$

for $U=(u,\theta)$, $V=(v,w)$ and $\mathcal{D}_\varepsilon : \big(L^2(\Omega)^N \times L^2(\Omega)\big) \times W_\varepsilon \to \mathbb{R}$ is given by

$$\mathcal{D}_\varepsilon\left((f,r),V\right) = -\sum_{\alpha=1,2}\int_0^T\int_{\Omega_\alpha^\varepsilon} (t-T)\Big(f_i \dot{v}_i^\alpha + \frac{1}{T_0} r w^\alpha\Big)\,\mathrm{d}x\,\mathrm{d}t.$$

We define now the Hilbert space $\mathcal{W}_\varepsilon$ as being the completion of the space W_ε in the norm $\|\cdot\|$ generated by the inner product

$$(U,V)_{W_\varepsilon} = \sum_{\alpha=1,2}\int_0^T\int_{\Omega_\alpha^\varepsilon} [\dot{u}_i^\alpha \dot{v}_i^\alpha + e_{ij}(u^\alpha) e_{ij}(v^\alpha) + \theta^\alpha w^\alpha + \int_0^t \frac{\partial\theta^\alpha}{\partial x_i}\frac{\partial w^\alpha}{\partial x_i}\,\mathrm{d}s]\,\mathrm{d}x\,\mathrm{d}t$$

$$+\varepsilon\int_0^T\int_{\Gamma_\varepsilon} (u_i^2 - u_i^1)(v_i^2 - v_i^1)\,\mathrm{d}\sigma\,\mathrm{d}t + \varepsilon\int_0^T\int_{\Gamma_\varepsilon}\int_0^t (\theta^2-\theta^1)(w^2-w^1)\,\mathrm{d}s\,\mathrm{d}\sigma\,\mathrm{d}t.$$

Following [13], one can see that the space $\mathcal{L}_\varepsilon$ can be continuously extended to $\mathcal{W}_\varepsilon \times \mathcal{W}_\varepsilon$ and $\mathcal{D}_\varepsilon$ can be extended to $\big(L^2(\Omega)^n \times L^2(\Omega)\big) \times \mathcal{W}_\varepsilon$. From the positivity of ρ^α, c^α, h^u and h^θ and the coercivity of A^α and K^α, one easily gets the existence of a constant $C>0$, independent of ε, such that $\|V\|^2 \leqslant C\mathcal{L}_\varepsilon(V,V)$.

Exactly like in [11], we have the following result.

Theorem 3.1 *The variational problem* (3.1) *possesses a unique solution. Besides, there exists a positive constant C, independent of* ε*, such that:*

$$\|u_i^{\varepsilon\alpha}\|_{L^2((0,T)\times\Omega_\alpha^\varepsilon)} \leqslant C, \quad \|\dot{u}_i^{\varepsilon\alpha}\|_{L^2((0,T)\times\Omega_\alpha^\varepsilon)} \leqslant C, \quad \|\nabla u_i^{\alpha\varepsilon}\|_{L^2((0,T)\times\Omega_\alpha^\varepsilon)} \leqslant C,$$

$$\|\theta^{\varepsilon\alpha}\|_{L^2((0,T)\times\Omega_\alpha^\varepsilon)} \leqslant C, \quad \left\|\int_0^t (\nabla\theta^{\varepsilon\alpha})^2\,\mathrm{d}s\right\|_{L^1((0,T)\times\Omega_\alpha^\varepsilon)} \leqslant C,$$

$$\|u_i^{2\varepsilon} - u_i^{1\varepsilon}\|_{L^2((0,T)\times\Gamma_\varepsilon)} \leqslant C\varepsilon^{-1/2}, \quad \left\|\int_0^t \left(\theta^{2\varepsilon}-\theta^{1\varepsilon}\right)^2\,\mathrm{d}s\right\|_{L^1((0,T)\times\Gamma_\varepsilon)} \leqslant C\varepsilon^{-1/2}.$$

4. THE MACROSCOPIC PROBLEM

We define

$$W_\alpha = L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega))^n \cap W^{1,\infty}(0,T;H^1(\Omega))^n \cap W^{2,\infty}(0,T;L^2(\Omega))^n,$$

$$\overline{W}_\alpha = L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega)) \cap W^{1,\infty}(0,T;H^1(\Omega)),$$

$$\begin{aligned} W = \quad & W_1 \times L^\infty(0,T;L^2(\Omega;H^1_{per}(Y_1))^n \times W_2 \times L^\infty(0,T;L^2(\Omega;H^1_{per}(Y_2))^n \\ & \times \overline{W}_1 \times L^\infty(0,T;L^2(\Omega;H^1_{per}(Y_1)) \times \overline{W}_2 \times L^\infty(0,T;L^2(\Omega;H^1_{per}(Y_2)). \end{aligned}$$

In order to get the main convergence result of this paper, we shall apply two unfolding operators $\mathcal{T}^\varepsilon_\alpha$, with $\alpha = 1,2$ (see [4] and [16]), which transform functions given on the oscillating domains $[0,T]\times\Omega^\varepsilon_\alpha$ into functions defined on the corresponding fixed domains $[0,T]\times\Omega\times Y_\alpha$. Also, by $\widetilde{v}$ we denote the zero extension to $(0,T)\times\Omega$ of a function v defined on $(0,T)\times\Omega^\varepsilon_\alpha$.

Theorem 4.1 *Let $(u^\varepsilon,\theta^\varepsilon)\in\mathcal{W}_\varepsilon$ be the unique solution of the microscopic problem* (2.1)-(2.8)*, with $u^\varepsilon = (u^{1\varepsilon},u^{2\varepsilon})$ and $\theta^\varepsilon = (\theta^{1\varepsilon},\theta^{2\varepsilon})$. Therefore, $\tilde{u}^{\alpha\varepsilon} \rightharpoonup |Y_\alpha|\,u^\alpha$ weakly* in $L^\infty(0,T;L^2(\Omega))^n$ and $\tilde{\theta}^{\alpha\varepsilon} \rightharpoonup |Y_\alpha|\,\theta^\alpha$ weakly* in $L^\infty(0,T;L^2(\Omega))$. Moreover,*

$$\begin{aligned} &\mathcal{T}^\varepsilon_\alpha(u^{\alpha\varepsilon}) \rightharpoonup u^\alpha \;\; \textit{weakly* in} \;\; L^\infty(0,T;L^2(\Omega;H^1(Y_\alpha)))^n, \\ &\mathcal{T}^\varepsilon_\alpha(e_{kh}(u^{\alpha\varepsilon})) \rightharpoonup e_{kh}(u^\alpha) + e^y_{kh}(\widehat{u}^\alpha) \;\; \textit{weakly* in} \;\; L^\infty(0,T;L^2(\Omega\times Y_\alpha)), \\ &\mathcal{T}^\varepsilon_\alpha(\theta^{\alpha\varepsilon}) \rightharpoonup \theta^\alpha \;\; \textit{weakly* in} \;\; L^\infty(0,T;L^2(\Omega;H^1(Y_\alpha))), \\ &\mathcal{T}^\varepsilon_\alpha(\nabla\theta^{\alpha\varepsilon}) \rightharpoonup \nabla\theta^\alpha + \nabla_y\widehat{\theta}^\alpha \;\; \textit{weakly* in} \;\; L^\infty(0,T;L^2(\Omega\times Y_\alpha))^n, \end{aligned} \tag{4.1}$$

where $(u^1,\widehat{u}^1,u^2,\widehat{u}^2,\theta^1,\widehat{\theta}^1,\theta^2,\widehat{\theta}^2)\in W$ is the unique solution of the macroscopic problem

$$\begin{aligned} &\sum_{\alpha=1,2}\int_0^T\!\!\int_{\Omega\times Y_\alpha}(t-T)\left[a^\alpha_{ijkh}\Big(e_{kh}(u^\alpha)+e^y_{kh}(\widehat{u}^\alpha)\Big)\right. \\ &\qquad\left.-b^\alpha_{ij}\theta^\alpha\right]\Big(\dot{e}_{ij}(\varphi^\alpha)+\dot{e}^y_{ij}(\Phi^\alpha)\Big)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\ &+\sum_{\alpha=1,2}\int_0^T\!\!\int_{\Omega\times Y_\alpha}(t-T)\Big[\rho^\alpha\ddot{u}^\alpha_i\dot{\varphi}^\alpha_i+\frac{1}{T_0}c^\alpha\dot{\theta}^\alpha q^\alpha\Big]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\ &+\frac{1}{T_0}\sum_{\alpha=1,2}\int_0^T\!\!\int_{\Omega\times Y_\alpha}(t-T)k^\alpha_{ij}\Big(\frac{\partial\theta^\alpha}{\partial x_j}+\frac{\partial\widehat{\theta}^\alpha}{\partial y_j}\Big)\Big(\frac{\partial q^\alpha}{\partial x_i}+\frac{\partial Q^\alpha}{\partial y_i}\Big)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \end{aligned} \tag{4.2}$$

$$+\sum_{\alpha=1,2}\int_0^T\int_{\Omega\times Y_\alpha}(t-T)b_{ij}^\alpha\Big(\dot e_{ij}(u^\alpha)+\dot e_{ij}^y(\widehat{u}^\alpha)\Big)q^\alpha\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$$

$$+\int_0^T\int_{\Omega\times\Gamma}(t-T)\Big[h^u(u_i^2-u_i^1)(\dot\varphi_i^2-\dot\varphi_i^1)+\frac{1}{T_0}h^\theta(\theta^2-\theta^1)(q^2-q^1)\Big]\,\mathrm{d}x\,\mathrm{d}\sigma\,\mathrm{d}t$$

$$=\sum_{\alpha=1,2}\int_0^T\int_{\Omega\times Y_\alpha}(t-T)\Big(f_i\dot\varphi_i^\alpha+\frac{1}{T_0}rq^\alpha\Big)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t,$$

for all $(\varphi^1,\Phi^1,\varphi^2,\Phi^2,q^1,Q^1,q^2,Q^2)\in W$*. In addition, for* $\alpha\in\{1,2\}$*, we have*

$$u^\alpha(0,x)=0,\quad \dot u^\alpha(0,x)=0,\quad \theta^\alpha(0,x)=0,\quad \textit{a.e. in }\Omega. \tag{4.3}$$

Proof. The proof will be accomplished in several stages. The convergences (4.1) follow from Theorem 3.1 (see, also, [6], [11] and [16]). For getting the limit problem, we choose in the variational formulation of problem (2.1)-(2.8) as test functions (with no summation of the repeated indices)

$$v_i^\alpha(t,x)=\varphi_i^\alpha(t,x)+\varepsilon\omega_i^\alpha(t,x)\psi_i^{\alpha\varepsilon}(x), \tag{4.4}$$

$$w^\alpha(t,x)=q^\alpha(t,x)+\varepsilon g^\alpha(t,x)p^{\alpha\varepsilon}(x), \tag{4.5}$$

where $\varphi_i^\alpha,\omega_i^\alpha,q^\alpha,g^\alpha\in\mathcal{D}([0,T]\times\Omega)$ and $\psi_i^\alpha,p^\alpha\in H^1_{per}(Y_\alpha)$ and $\psi^{\alpha\varepsilon}(x)=\psi^\alpha(x/\varepsilon)$ and $p^{\alpha\varepsilon}(x)=p^\alpha(x/\varepsilon)$.

It is not difficult to prove that $\varepsilon\omega^\alpha\psi^{\alpha\varepsilon}\longrightarrow 0$ strongly in $L^\infty(0,T;L^2(\Omega))^n$ and $\varepsilon g^\alpha p^{\alpha\varepsilon}\longrightarrow 0$ strongly in $L^\infty(0,T;L^2(\Omega))$. Therefore, using the above mentioned unfolding operators (see [6], [11] or [16]), it follows that $\mathcal{T}_\alpha^\varepsilon(\varepsilon\omega^\alpha\psi^{\alpha\varepsilon})\longrightarrow 0$ strongly in $L^\infty(0,T;L^2(\Omega\times Y_\alpha))^n$ and $\mathcal{T}_\alpha^\varepsilon(\varepsilon g^\alpha p^{\alpha\varepsilon})\longrightarrow 0$ strongly in $L^\infty(0,T;L^2(\Omega\times Y_\alpha))$. In addition, for $\alpha\in\{1,2\}$, it follows that $\mathcal{T}_\alpha^\varepsilon(e_{ij}(\varepsilon\omega^\alpha\psi^{\alpha\varepsilon}))=\varepsilon\psi_i^\alpha\mathcal{T}_\alpha^\varepsilon(e_{ij}(\omega^\alpha))+e_{ij}^y(\psi^\alpha)\mathcal{T}_\alpha^\varepsilon(\omega_i^\alpha)\longrightarrow e_{ij}^y(\Phi^\alpha)$ strongly in $L^\infty(0,T;L^2(\Omega\times Y_\alpha))$, where $\Phi_i^\alpha(t,x,y)=\omega_i^\alpha(t,x)\psi_i^\alpha(y)$. Also, $\mathcal{T}_\alpha^\varepsilon(\nabla(\varepsilon g^\alpha p^{\alpha\varepsilon}))=\varepsilon p^\alpha\mathcal{T}_\alpha^\varepsilon(\nabla g^\alpha)+\nabla_y p^\alpha\mathcal{T}_\alpha^\varepsilon(g^\alpha)\longrightarrow\nabla_y(Q^\alpha)$ strongly in $L^\infty(0,T;L^2(\Omega\times Y_\alpha))$, with $Q^\alpha(t,x,y)=g^\alpha(t,x)p^\alpha(y)$.

The macroscopic problem is obtained by applying to each term of (3.1) the corresponding unfolding operator and passing to limit with $\varepsilon\to 0$. We have:

$$\sum_{\alpha=1,2}\int_0^T\int_{\Omega\times Y_\alpha}(t-T)\Big[-a_{ijkh}^\alpha(e_{kh}(u^\alpha)+e_{kh}^y(\widehat{u}^\alpha))$$

$$+b_{ij}^\alpha\theta^\alpha\Big]\Big(e_{ij}(\dot\varphi^\alpha)+e_{ij}^y(\dot\Phi^\alpha)\Big)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t+\sum_{\alpha=1,2}\int_0^T\int_{\Omega\times Y_\alpha}(t-T)\rho^\alpha\dot u_i^\alpha\ddot\varphi_i^\alpha\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$$

$$+\sum_{\alpha=1,2}\int_0^T\int_{\Omega\times Y_\alpha}(t-T)b_{ij}^\alpha\Big(e_{ij}(u^\alpha)+e_{ij}^y(\widehat{u}^\alpha)\Big)\dot q^\alpha\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$$

$$+\frac{1}{T_0}\sum_{\alpha=1,2}\int_0^T\int_{\Omega\times Y_\alpha}(t-T)c^\alpha\theta^\alpha\dot{q}^\alpha\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t+\sum_{\alpha=1,2}\int_0^T\int_{\Omega\times Y_\alpha}\rho^\alpha\dot{u}_i^\alpha\dot{\varphi}_i^\alpha\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$$

$$+\sum_{\alpha=1,2}\int_0^T\int_{\Omega\times Y_\alpha}b_{ij}^\alpha\Big(e_{ij}(u^\alpha)+e_{ij}^y(\widehat{u}^\alpha)\Big)q^\alpha\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$$

$$+\frac{1}{T_0}\sum_{\alpha=1,2}\int_0^T\int_{\Omega\times Y_\alpha}c^\alpha\theta^\alpha q^\alpha\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \tag{4.6}$$

$$+\frac{1}{T_0}\sum_{\alpha=1,2}\int_0^T\int_{\Omega\times Y_\alpha}\int_0^t k_{ij}^\alpha\Big(\frac{\partial\theta^\alpha}{\partial x_j}+\frac{\partial\widehat{\theta}^\alpha}{\partial y_j}\Big)\Big(\frac{\partial q^\alpha}{\partial x_i}+\frac{\partial Q^\alpha}{\partial y_i}\Big)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$$

$$-\int_0^T\int_{\Omega\times\Gamma}(t-T)h^u(u_i^2-u_i^1)(\dot{\varphi}_i^2-\dot{\varphi}_i^1)\,\mathrm{d}x\,\mathrm{d}\sigma\,\mathrm{d}t$$

$$-\frac{1}{T_0}\int_0^T\int_{\Omega\times\Gamma}(t-T)h^\theta(\theta^2-\theta^1)(q^2-q^1)\,\mathrm{d}x\,\mathrm{d}\sigma\,\mathrm{d}t$$

$$=-\sum_{\alpha=1,2}\int_0^T\int_{\Omega\times Y_\alpha}(t-T)\Big(f_i\dot{\varphi}_i^\alpha+\frac{1}{T_0}rq^\alpha\Big)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t,$$

which leads to (4.2). Following the same techniques as in [6] or [11], we can show that u^α, $\dot{u}^\alpha$ and θ^α vanish at $t=0$. Also, it is not difficult to see that the solution of the limit problem is unique. ▪

Our goal now is to try to decouple the limit problem (4.2) and to write down its strong formulation. To this end, we take as test functions $\varphi_i^1=\varphi_i^2=q^1=q^2=Q^1=Q^2=0$ and, since $\Phi_i^\alpha(t,x,y)=\omega_i^\alpha(t,x)\psi_i^\alpha(y)$, without summation and with $\omega_i^\alpha\in\mathcal{D}([0,T]\times\Omega)$ and $\psi_i^\alpha\in H^1_{per}(Y_\alpha)$, we are lead to

$$\int_{Y_\alpha}a_{ijkh}^\alpha\frac{\partial\widehat{u}_k^\alpha}{\partial y_h}\frac{\partial\psi_i^\alpha}{\partial y_j}\,\mathrm{d}y=-\frac{\partial u_k^\alpha}{\partial x_h}\int_{Y_\alpha}a_{ijkh}^\alpha\frac{\partial\psi_i^\alpha}{\partial y_j}\,\mathrm{d}y+\theta^\alpha\int_{Y_\alpha}b_{ij}^\alpha\frac{\partial\psi_i^\alpha}{\partial y_j}\,\mathrm{d}y. \tag{4.7}$$

Let us consider the unique solution $z^\alpha\in\tilde{H}^1_{per}(Y_\alpha)^n$ of the following local problem:

$$\begin{cases} -\dfrac{\partial}{\partial y_j}\Big(a_{ijkh}^\alpha\dfrac{\partial z_k^\alpha}{\partial y_h}-b_{ij}^\alpha\Big)=0 \quad \text{in } Y_\alpha \\ \Big(a_{ijkh}^\alpha\dfrac{\partial z_k^\alpha}{\partial y_h}-b_{ij}^\alpha\Big)\nu_j=0 \quad \text{on } \Gamma. \end{cases} \tag{4.8}$$

and, for $l,m=1,\dots,n$, the unique solutions $w_\alpha^{lm}\in\tilde{H}^1_{per}(Y_\alpha)^n$ of the cell problems

$$\begin{cases} -\dfrac{\partial}{\partial y_j}\Big(a_{ijlm}^\alpha+a_{ijkh}^\alpha\dfrac{\partial w_{\alpha k}^{lm}}{\partial y_h}\Big)=0 \quad \text{in } Y_\alpha \\ \Big(a_{ijlm}^\alpha+a_{ijkh}^\alpha\dfrac{\partial w_{\alpha k}^{lm}}{\partial y_h}\Big)\nu_j=0 \quad \text{on } \Gamma. \end{cases} \tag{4.9}$$

Thus, from (4.7), we obtain

$$\widehat{u}_k^{\alpha}(t,x,y)=\frac{\partial u_l^{\alpha}}{\partial x_m}(t,x)w_{\alpha k}^{lm}(y)+\theta^{\alpha}(t,x)z_k^{\alpha}(y). \tag{4.10}$$

In a similar manner, we can get a problem for $\widehat{\theta}^{\alpha}$:

$$\int_{Y_\alpha} k_{ij}^{\alpha}\frac{\partial\widehat{\theta}^{\alpha}}{\partial y_j}\frac{\partial p^{\alpha}}{\partial y_i}=-\frac{\partial\theta^{\alpha}}{\partial x_j}\int_{Y_\alpha} k_{ij}^{\alpha}\frac{\partial p^{\alpha}}{\partial y_i}. \tag{4.11}$$

We consider, for $k=1,\dots,n$, the solutions $\chi^{\alpha}\in\tilde{H}_{per}^1(Y_\alpha)^n$ of the local problems

$$\begin{cases} -\dfrac{\partial}{\partial y_i}\Big(k_{ik}^{\alpha}+k_{ij}^{\alpha}\dfrac{\partial\chi_k^{\alpha}}{\partial y_j}\Big)=0 \quad \text{in } Y_\alpha \\ \Big(k_{ik}^{\alpha}+k_{ij}^{\alpha}\dfrac{\partial\chi_k^{\alpha}}{\partial y_j}\Big)n_i=0 \quad \text{on } \Gamma. \end{cases} \tag{4.12}$$

The linearity of (4.11) implies that

$$\widehat{\theta}_k^{\alpha}(t,x,y)=\frac{\partial\theta^{\alpha}}{\partial x_k}(t,x)\chi_k^{\alpha}(y). \tag{4.13}$$

The homogenized coefficients are defined as being

$$\begin{aligned} a_{ijlm}^{\alpha*}&=\int_{Y_\alpha}\Big(a_{ijlm}^{\alpha}+a_{ijkh}^{\alpha}\frac{\partial w_{\alpha k}^{lm}}{\partial y_h}\Big)\,\mathrm{d}y,\ b_{lm}^{\alpha*}=\int_{Y_\alpha}\Big(b_{lm}^{\alpha}+b_{ij}^{\alpha}\frac{\partial w_{\alpha i}^{lm}}{\partial y_j}\Big)\,\mathrm{d}y,\\ k_{ik}^{\alpha*}&=\int_{Y_\alpha}\Big(k_{ik}^{\alpha}+k_{ij}^{\alpha}\frac{\partial\chi_k^{\alpha}}{\partial y_j}\Big)\,\mathrm{d}y,\quad \beta_{ij}^{\alpha*}=\int_{Y_\alpha}\Big(a_{ijkh}^{\alpha}\frac{\partial z_k^{\alpha}}{\partial y_h}-b_{ij}^{\alpha}\Big)\,\mathrm{d}y, \end{aligned} \tag{4.14}$$

$$\gamma^{\alpha*}=\int_{Y_\alpha} b_{ij}^{\alpha}\frac{\partial z_i^{\alpha}}{\partial y_j}\,\mathrm{d}y. \tag{4.15}$$

Let us remark that $\beta_{lm}^{\alpha*}=-b_{lm}^{\alpha 1*}$, for any $l,m=1,\dots,n$.

Theorem 4.2 *The unique solution* $(u^{\varepsilon},\theta^{\varepsilon})\in\mathcal{W}_{\varepsilon}$ *of problem* (2.1)-(2.8)*, with* $u^{\varepsilon}=(u^{1\varepsilon},u^{2\varepsilon})$ *and* $\theta^{\varepsilon}=(\theta^{1\varepsilon},\theta^{2\varepsilon})$*, converges, in the sense of (4.1), to* (u,θ)*, with* $u=(u^1,u^2)$ *and* $\theta=(\theta^1,\theta^2)$*, the unique solution of the homogenized problem in* $(0,T)\times\Omega$*:*

$$-\frac{\partial}{\partial x_j}\Big(a_{ijkh}^{1*}\frac{\partial u_k^1}{\partial x_h}-b_{ij}^{1*}\theta^1\Big)+\langle\rho^1\rangle_{Y_1}\ddot{u}_i^1-H^u(u_i^2-u_i^1)=|Y_1|\,f_i, \tag{4.16}$$

$$-\frac{\partial}{\partial x_j}\Big(a_{ijkh}^{2*}\frac{\partial u_k^2}{\partial x_h}-b_{ij}^{2*}\theta^2\Big)+\langle\rho^2\rangle_{Y_2}\ddot{u}_i^2+H^u(u_i^2-u_i^1)=|Y_2|\,f_i, \tag{4.17}$$

$$\begin{aligned} -\frac{\partial}{\partial x_i}\Big(k_{ij}^{1*}\frac{\partial\theta^1}{\partial x_j}\Big)+T_0b_{ij}^{1*}\dot{e}_{ij}(u^1)+\Big(T_0\gamma^{1*}+\langle c^1\rangle_{Y_1}\Big)\dot{\theta}^1\\ -H^{\theta}(\theta^2-\theta^1)=|Y_1|\,r, \end{aligned} \tag{4.18}$$

$$-\frac{\partial}{\partial x_i}\Big(k_{ij}^{2*}\frac{\partial\theta^2}{\partial x_j}\Big)+T_0 b_{ij}^{2*}\dot{e}_{ij}(u^2)+\Big(T_0\gamma^{2*}+\left\langle c^2\right\rangle_{Y_2}\Big)\dot{\theta}^2$$
$$+H^\theta(\theta^2-\theta^1)=|Y_2|\,r, \tag{4.19}$$

with the conditions

$$u^\alpha=0,\;\;\theta^\alpha=0\; on\;(0,T)\times\partial\Omega, \tag{4.20}$$

$$u^\alpha(0,x)=0,\;\;\dot{u}^\alpha(0,x)=0,\;\;\theta^\alpha(0,x)=0\;\; in\;\Omega. \tag{4.21}$$

Here, $H^u=\displaystyle\int_\Gamma h^u\,\mathrm{d}\sigma$ *and* $H^\theta=\displaystyle\int_\Gamma h^\theta\,\mathrm{d}\sigma.$

Proof. To prove this theorem, it remains only to show that the limits u^α and θ^α obtained in Theorem 4.1 satisfy the problem (4.16)-(4.19). This can be easily accomplished by introducing the expressions of $\widehat{u}^\alpha$ and $\widehat{\theta}^\alpha$ in (4.2) and using the local problems and the formulas for the effective coefficients. Therefore, (4.2) leads to

$$\sum_{\alpha=1,2}\int_0^T\!\!\int_\Omega(t-T)\Big(a_{ijkh}^{\alpha*}\frac{\partial u_k^\alpha}{\partial x_k}-b_{ij}^{\alpha*}\theta^\alpha\Big)\frac{\partial\dot{\varphi}_i^\alpha}{\partial x_j}\,\mathrm{d}x\,\mathrm{d}t$$
$$+\sum_{\alpha=1,2}\int_0^T\!\!\int_\Omega(t-T)\left[\langle\rho^\alpha\rangle_{Y_\alpha}\ddot{u}_i^\alpha\dot{\varphi}_i^\alpha+\frac{1}{T_0}\langle c^\alpha\rangle_{Y_\alpha}\dot{\theta}^\alpha q^\alpha\right]\,\mathrm{d}x\,\mathrm{d}t$$
$$+\sum_{\alpha=1,2}\int_0^T\!\!\int_\Omega(t-T)\left[\frac{1}{T_0}k_{ij}^{\alpha*}\frac{\partial\theta^\alpha}{\partial x_j}\frac{\partial q^\alpha}{\partial x_i}+\Big(b_{ij}^{\alpha*}\dot{e}_{ij}(u^\alpha)+\gamma^{\alpha*}\dot{\theta}^\alpha\Big)q^\alpha\right]\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^T\!\!\int_\Omega(t-T)\left[H^u(u_i^2-u_i^1)(\dot{\varphi}_i^2-\dot{\varphi}_i^1)+\frac{1}{T_0}H^\theta(\theta^2-\theta^1)(q^2-q^1)\right]\,\mathrm{d}x\,\mathrm{d}t$$
$$=\sum_{\alpha=1,2}\int_0^T\!\!\int_\Omega(t-T)\,|Y_\alpha|\left(f_i\dot{\varphi}_i^\alpha+\frac{1}{T_0}rq^\alpha\right)\,\mathrm{d}x\,\mathrm{d}t, \tag{4.22}$$

which holds for any $\varphi_i^\alpha, q^\alpha\in\mathcal{D}(0,T;H_0^1(\Omega))$. By standard density arguments, it is not difficult to see that (4.22) is exactly the weak formulation of (4.16)-(4.19). ∎

5. CONCLUDING REMARKS

In the framework of homogenization techniques, using different kinds of local problems, we obtained the effective behavior of the solution of a dynamic coupled thermoelasticity problem. The macroscopic problem exhibits some interesting features, such as the appearance of new macroscopic coefficients depending on the corresponding microscopic ones and new coupling terms involving the macroscopic displacement and temperature fields, generated by the imperfect bonding at the interface between the two phases of the composite.

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