Romanian Reports in Physics, Vol. 68, No. 4, P. 1335–1348, 2016

COHERENT STATES FOR CONTINUOUS SPECTRUM AS LIMITING CASE OF HYPERGEOMETRIC COHERENT STATES

DUSAN POPOV[1,*], MIODRAG POPOV[2]

[1] Politehnica University Timisoara,
Department of Physical Foundations of Engineering, Timisoara, Romania
[2] Politehnica University of Timisoara, Department of Architecture and Urban Planning, Timisoara, Romania, E-mail: miodrag.popov@up-tim.org, miodrag.popov@upt.ro
Corresponding author: dusan_popov@yahoo.co.uk; dusan.popov@upt.ro

Received December 18, 2015

Abstract. We obtained the coherent states for continuous spectrum by starting from the hypergeometric coherent states for discrete spectrum, and applying a discrete – continuous limit. We extended the diagonal ordering operation technique for discrete spectrum to the coherent states for continuous one, including the thermal states.

Key words: coherent states, operator technique, continuous spectrum.

1. INTRODUCTION

In the last decades the concept of coherent states (CSs) has aroused great scientific interest, due to its applications in condensed matter physics, mathematical physics, signal theory and quantum information [1–4]. Among the various kind of CSs, a privileged place is occupied by the generalized hypergeometric coherent states (GH-CSs), introduced in [5] and applied to the mixed (thermal) states in [6]. Moreover, the calculations involving CSs often involve the use of certain operator ordering rules. For the CSs of one dimensional harmonic oscillator it is used the integration within an ordered product (IWOP) technique [7]. Recently, generalizing the Fan's IWOP technique, we have introduced a new approach to order the operators called the *diagonal ordering operation technique* (DOOT) noted it by the symbol # # [8]. The DOOT can be formulated and used for more general kind of CSs – the HG-CSs. All other CSs for the discrete part of spectrum can be obtained as particular cases of the GH-CSs. On the other hand, a special attention is paid to the construction of CSs for the *continuous part* of spectrum of different quantum systems [9–13]. The aim of present paper is to obtain the CSs for the continuous spectrum as a limiting case of the GH-CSs and to extend the DOOT rules in order to examine the characteristics of these states.

2. CONSTRUCTION OF COHERENT STATES FOR CONTINUOUS SPECTRUM

The generalized hypergeometric coherent states (GH-CSs) *for the discrete spectrum* are labeled by the complex variable $z = |z| \exp(\mathrm{i}\varphi)$, $0 \le |z| \le R \le \infty$, $0 \le \varphi \le 2\pi$ and their expansion in terms of Fock-vectors $|n>$ is [5–6, 8]

$$|z>= \frac{1}{\sqrt{{}_pF_q(\{a_i\}_1^p \ ; \ \{b_j\}_1^q \ ; |z|^2)}} \sum_{n=0}^{n_{\max}} \frac{z^n}{\sqrt{\rho_{p,q}(n)}} \ |n> . \tag{1}$$

The positive constants $\rho_{p,q}(n)$ are assumed to arise as the moments of a probability distribution [9] and for GH-CSs they are defined as follows [6]

$$\rho_{p,q}(n) \equiv n! \frac{\prod_{j=1}^{q}(b_j)_n}{\prod_{i=1}^{p}(a_i)_n} \tag{2}$$

thus, the normalization functions are the generalized hypergeometric functions [14]

$${}_pF_q(\{a\}_1^p ; \{b\}_1^q ; |z|^2) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}(a_i)_n}{\prod_{j=1}^{q}(b_j)_n} \frac{(|z|^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{\rho_{p,q}(n)} (|z|^2)^n \ . \tag{3}$$

Here $(a_i)_n$ are the Pochhammer's symbols, and we will use the abbreviated notations $x_1, x_2, ..., x_m \equiv \{x_i\}_1^m$. The convergence radius R of the GH-CSs is obtained from the limit $R = \lim_{n\to\infty} \sqrt[n]{\rho_{p,q}(n)}$ [9].

Let us we consider a dimensionless Hamiltonian H with a non-degenerate continuous spectrum, and the eigenstates $|E>$ which are formal delta-function normalized, *i.e.* $H|E>=E|E>$, with $<E|E'>=\delta(E-E')$. Then the closure or completeness relation for continuous spectrum (with $0 \le E \le \infty$**)** is

$$\int_0^\infty \mathrm{d}E \ |E><E| = I , \quad \textit{i.e.} \quad \int_0^\infty \mathrm{d}E <E' \ |E><E| \ E''> = \delta\left(E'-E''\right) \tag{4}$$

and this relation performs the resolution of identity operator I acting in the Hilbert space in terms of the projectors $|E><E|$. Due to the property of Dirac delta

function, the relation from the l.h.s. of (4) should be understood as being equivalent to there from r.h.s., for any positive values E' and E''.

On the one hand, the discrete energy spectrum is characterized by the energy quantum number n, as well as with the maximal number of bound states $n_{\max} \le \infty$. The corresponding GH-CSs for discrete spectrum are expressed through the complex variable z and through the generalized hypergeometric functions of orders p and q, which contain two sequences of coefficients $\{a_i\}_1^p$ and $\{b_j\}_1^q$. One the other hand, the continuous spectrum are labeled continuously by energy E and the eigenvectors $|E>$. The corresponding CSs for continuous spectrum are labeled also by a complex variable Z, but we have used this different notation just to avoid the confusion with discrete case.

We use the notation $X_c = X_c(E)$ for the quantities regarding to the continuous spectrum, respectively $X_d = X_d(n, n_{\max})$ for those regarding to the discrete spectrum.

With the aim to obtain the coherent states referring *separately* to the *continuous spectrum* of a quantum system, we use the following limit (for brevity we will call *the discrete – continuous limit* $d \to c$), defined as (see, also, [13]):

$$X_c(E) = \lim_{\substack{n\to E \\ n_{\max}\to\infty \\ z\to Z \\ p=q \\ \{a_i\}=\{b_j\}}} X_d(n, n_{\max}) \equiv \lim_{d\to c} X_d(n, n_{\max})\,,$$

$$\sum_{n=0}^{n_{\max}} X_d(n,\ n_{\max}) \ \to \ \int_0^\infty \mathrm{d}E X_c(E)\,. \tag{5}$$

Thus, all quantities X_c regarding the continuous spectrum will be obtained as a limiting case of the corresponding quantity X_d of the discrete spectrum, by following operations: *a*) the energy quantum number n must be replaced by the dimensionless energy E; *b*) the maximal number of bound states $n_{\max}$ must tend to infinity; *c*) the complex variable z must be replaced with a new complex variable $Z \equiv |Z| e^{-\mathrm{i}\gamma}$ with $0 \le |Z| \le R_c \le \infty$, $-\infty \le \gamma \le +\infty$, where R_c is the convergence radius of the CSs; *d*) the two indexes of the generalized hypergeometric functions must be equal: $p = q$; *e*) the two sequences of coefficients must be also equal $\{a_i\}_1^p = \{b_j\}_1^q$; *f*) simultaneously, the sum with respect to n must be replaced by the integral with respect to E.

Applying this limit to the quantities appearing in the expression of CSs for the discrete case, we obtain successively

$$\lim_{d\to c}\rho_{p,q}(n)=\lim_{d\to c} n!\frac{\prod_{j=1}^{q}(b_j)_n}{\prod_{i=1}^{p}(a_i)_n}=\Gamma(E+1) \tag{6}$$

$$\lim_{d\to c} {}_pF_q(\{a_i\}_1^p;\ \{b_j\}_1^q;\ |z|^2)=\lim_{d\to c}\sum_{n=0}^{\infty}\frac{1}{\rho_{p,q}(n)}\ (|z|^2)^n=\int_0^{\infty}\mathrm{d}E\ \ \frac{\left(|Z|^2\right)^E}{\Gamma(E+1)}=\nu\left(|Z|^2\right). \tag{7}$$

As we see, the normalization function is the ν-function, defined in [14, Eq. 9.640.1, p. 1043].

Consequently, the expansion of the CSs for the continuous spectrum in terms of energy eigenvectors $|E>$ is expressed through an integral with respect to energy eigenvalue E and which is labeled by a complex variable $Z\equiv|Z|e^{-\mathrm{i}\gamma}$, by applying the the discrete – continuous limit $d\to c$

$$|Z>=\lim_{d\to c}|z>=\frac{1}{\sqrt{\nu\left(|Z|^2\right)}}\int_0^{\infty}\mathrm{d}E\ \ \frac{Z^E}{\sqrt{\Gamma\left(E+1\right)}}\ |E>\ . \tag{8}$$

The expression of CSs for continuous spectrum was firstly obtained, *by another method*, for the Gazeau-Klauder CSs, in [9] and later in [10] and [12].

The structure of the CSs for the continuous spectrum are the same for *all* quantum systems which possesses a continuous spectrum, so, these CSs *are independent of the examined system*. In other words, the CSs for the continuous spectrum "have looked their past history", respectively their original potential.

Using the relation $<E|E'>=\delta(E-E')$, the overlap of two CSs is

$$<Z|Z'>=\lim_{d\to c}<z|z'>=\frac{\nu\left(Z^* Z'\right)}{\sqrt{\nu\left(|Z|^2\right)}\sqrt{\nu\left(|Z'|^2\right)}}, \tag{9}$$

from where the *continuity in label* Z follows immediately:

$$\lim_{Z'\to Z}\||Z'>-|Z>\|^2=\lim_{Z'\to Z}\left[2\left(1-\mathrm{Re}<Z'|Z>\right)\right]=0\,. \tag{10}$$

Now, we can examine *the resolution of identity* or the completeness relation which is symbolically expressed in terms of the projectors onto the states $|Z>$, as follows

$$\int \mathrm{d}\mu_c(Z)\,|Z><Z|=I \text{ or, equivalent} \int \mathrm{d}\mu_c(Z)\ <Z'|Z><Z|Z''>=<Z'|Z''> \tag{11}$$

integrated over the whole complex space. This requires to obtain the integration measure $\mathrm{d}\mu_c(Z)$ by starting from the corresponding one of the discrete GH-CSs [8]:

$$\mathrm{d}\mu^d_{p,q}(z) = \frac{\mathrm{d}\varphi}{2\pi}\mathrm{d}(|z|^2)\frac{\prod_{i=1}^{p}\Gamma(a_i)}{\prod_{j=1}^{q}\Gamma(b_j)}\,{}_pF_q(\{a_i\}_1^p;\ \{b_j\}_1^q;\ |z|^2)\,G^{q+1,\,0}_{p,\,q+1}\left(|z|^2\,\middle|\,\begin{matrix} / \ ; & \{a_i-1\}_1^p \\ 0\ ,\ \{b_j-1\}_1^q\ ; & / \end{matrix}\right) \tag{12}$$

and then applying the *discrete – continuous limit* $d \to c$:

$$\mathrm{d}\mu_c(Z) = \lim_{d\to c}\mathrm{d}\mu^d_{p,q}(z) = \frac{\mathrm{d}\gamma}{2\pi}\mathrm{d}\left(|Z|^2\right)e^{-|Z|^2}\,\nu\left(|Z|^2\right). \tag{13}$$

Here we have used Eq. (7) and the specialized value of the Meijer's function [15]:

$$G^{1,0}_{0,1}\left(|Z|^2\,\middle|\,0\right) = e^{-|Z|^2}\,. \tag{14}$$

The *temporal stability* of the CSs for continuous spectrum can be showed by using the eigenequation with non-degenerate spectrum $H\,|\,E> = E\,|\,E>$

$$e^{-\mathrm{i}\,\omega\,t\,H}\ |\,Z> = \frac{1}{\sqrt{\nu\left(|Z|^2\right)}}\int_0^\infty \mathrm{d}E\,\frac{\left(|Z|\ e^{-\mathrm{i}\,(\gamma+\omega\,t)}\right)^E}{\sqrt{\Gamma\left(E+1\right)}}\ |\,E> \equiv |\,Z(t)> \tag{15}$$

where $Z(t) \equiv |Z|e^{-\mathrm{i}(\gamma+\omega t)}$ and $\omega = \mathrm{const.}$ So, the CSs are temporally stable.

The expectation value of an operator O_c in the CSs representation $|\,Z>$ is

$$<O_c\ >_Z \equiv <Z\,|\,O_c\,|\,Z> = \frac{1}{\nu\left(|Z|^2\right)}\int_0^\infty \mathrm{d}E\mathrm{d}E'\,\frac{\left(Z^*\right)^E Z^{E'}}{\sqrt{\Gamma\left(E+1\right)\ \ \Gamma\left(E'+1\right)}} <E\,|\,O_c\,|\,E'> \tag{16}$$

When we take $O_c = H$, we can examine the *action identity* of CSs for continuous spectrum

$$<Z\,|\,H\,|\,Z> = |Z|^2\,\frac{\mathrm{d}}{\mathrm{d}\left(|Z|^2\right)}\log\nu\left(|Z|^2\right) \equiv H\left(|Z|^2\right), \tag{17}$$

which is identical with the result obtained by Gazeau and Klauder (Eq. (42) of [9]).

To simplify this expression, we appeal to a property of the ν-function:

$$\frac{\mathrm{d}}{\mathrm{d}|Z|^2}\nu\left(|Z|^2\right)=\int_0^\infty \mathrm{d}EE\frac{\left(|Z|^2\right)^{E-1}}{\Gamma\left(E+1\right)}=\int_0^\infty \mathrm{d}\left(E-1\right)\frac{\left(|Z|^2\right)^{E-1}}{\Gamma\left[\left(E-1\right)+1\right]}=\nu\left(|Z|^2\right)\quad(18)$$

since $\Gamma(E+1)=E\Gamma(E)$ and finally the expression for the action identity becomes

$$<Z|H|Z>\equiv H\left(|Z|^2\right)=|Z|^2 . \quad (19)$$

If we set $H\left(|Z|^2\right)=\omega J$, this relation is uniquely invertible, leading to $|Z|^2=\omega J=f\left(J\right)$, *i.e.* the action identity [9] is accomplished.

So, we have verified that *the CSs for continuous spectrum* $|Z>$ *satisfy all Klauder's minimal requirements* [9]: (a) continuity in the complex variable; (b) resolution of unity; (c) temporal stability and (d) action identity.

Using the *discrete – continuous limit* $d\to c$ procedure, let us we define the lowering A_- and raising A_+ operators for the continuous spectrum (for brevity we will not write the index *c* – continuous). For the discrete spectrum such operators were defined in [5] and for the GH-CSs were used in [6]:

$$A_-^{(d)}=\sum_{n=0}^{n_{\max}} f_{p,q}(n)\,|n><n+1|,\qquad A_+^{(d)}=\sum_{n=0}^{n_{\max}} f_{p,q}(n)\,|n+1><n|,\quad (20)$$

where the positive functions $f_{p,q}(m)$ are

$$f_{p,q}(m)=\sqrt{(m+1)\frac{\prod_{j=1}^{q}(b_j+m)}{\prod_{i=1}^{p}(a_i+m)}}\,,\quad \prod_{m=0}^{n-1} f_{p,q}(m)=\sqrt{n!\frac{\prod_{j=1}^{q}(b_j)_n}{\prod_{i=1}^{p}(a_i)_n}}\equiv\sqrt{\rho_{p,q}(n)}\,.\quad (21)$$

Applying the *discrete – continuous limit* $d\to c$, we obtain successively

$$A_-=\lim_{d\to c}A_-^{(d)}=\lim_{d\to c}\sum_{n=0}^{n_{\max}} f_{p,q}(n)\,|n><n+1|=\int_0^\infty \mathrm{d}E\sqrt{E+1}\;|E><E+1|\quad(22)$$

$$A_+=\lim_{d\to c}A_+^{(d)}=\lim_{d\to c}\sum_{n=0}^{n_{\max}} f_{p,q}(n)\,|n+1><n|=\int_0^\infty dE\,\sqrt{E+1}\,|E+1><E|\,.\quad (23)$$

The pair operators A_- and A_+ are Hermitian $\left(A_-\right)^+=A_+$ and we have also

$$A_{-}\,|\,E> = \int_0^\infty \mathrm{d}E'\sqrt{E'+1}\ \,|\,E'>\ \delta\left(E'+1-E\right) = \sqrt{E}\ \,|\,E-1> \tag{24}$$

$$A_{+}\,|\,E> = \int_0^\infty \mathrm{d}E'\sqrt{E'+1}\ \,|\,E'+1>\delta\left(E'-E\right) = \sqrt{E+1}\,|\,E+1> \tag{25}$$

$$A_{+}A_{-}\,|\,E> = \sqrt{E}A_{+}\ \,|\,E-1> = E\,|\,E>\,, \qquad A_{+}A_{-} = \int_0^\infty \mathrm{d}E E\,|\,E><E\,|\,. \tag{26}$$

Here we must make a technical observation: *for the continuous spectrum we introduce a real dimensionless energy parameter* $\varepsilon > 0$. This can be interpreted as the “jump unity” in the energy scale of continuous spectra (but *not a quanta*!) and it may be equalized with unity $\varepsilon = 1$. So, if we write, $E \pm m\varepsilon$, with $m = 0,1,2,...$ this may be written simply as $E \pm m$. So, if we successively apply *m*-fold the raising operator A_{+} to the ground or vacuum state $|\,0>$, we obtain

$$\left(A_{+}\right)^m |\,0> = \sqrt{\Gamma(m+1)}\,|\,m>. \tag{27}$$

If we consider $E = m\varepsilon \equiv m$ (which correspond to the m jumps from the ground state $|\,0>$) we can write

$$\left(A_{+}\right)^E |\,0> = \sqrt{\Gamma(E+1)}\,|\,E>. \tag{28}$$

This result can be obtained also if we apply the discrete-continuous limit $d \to c$ to the corresponding relation for the discrete spectrum:

$$\left(A_{+}\right)^E |\,0> = \lim_{d\to c}\left(A_{+}^{(d)}\right)^n |\,0> = \lim_{d\to c}\sqrt{\rho_{p,q}(n)}\,|\,n> = \sqrt{\Gamma(E+1)}\,|\,E>. \tag{29}$$

Let us we examine some expectation values in the $|\,Z>$ representation. For of an operator $O_c = A_{+}A_{-}$ we have, using Eqs. (16) and (18):

$$\begin{aligned} <A_{+}A_{-}>_Z &= \frac{1}{\nu\left(|\,Z\,|^2\right)}\int_0^\infty \mathrm{d}E\mathrm{d}E'\frac{\left(Z^*\right)^E Z^{E'}}{\sqrt{\Gamma\left(E+1\right)\ \ \Gamma\left(E'+1\right)}} <E\,|\,A_{+}A_{-}\,|\,E'> = \\ &= \frac{1}{\nu\left(|\,Z\,|^2\right)}\int_0^\infty \mathrm{d}E\,E\,\frac{\left(|\,Z\,|^2\right)^E}{\Gamma\left(E+1\right)} = \frac{1}{\nu\left(|\,Z\,|^2\right)}|\,Z\,|^2\,\frac{\mathrm{d}}{\mathrm{d}\,|\,Z\,|^2}\nu\left(|\,Z\,|^2\right) = |\,Z\,|^2 \end{aligned} \tag{30}$$

i.e. the same value as $< Z \,|\, H \,|\, Z >$. This means the following equality:

$$H = A_+ A_- . \tag{31}$$

Moreover, the number operator N_d for the discrete spectrum, is defined as

$$N_d = \sum_{n=0}^{n_{\max}} n \,|\, n >< n \,| \tag{32}$$

and applying the *limit* $d \to c$ we obtain their continuous counterpart

$$N_c = \lim_{d \to c} N_d = \lim_{d \to c} \sum_{n=0}^{n_{\max}} n \;\; |\, n >< n \,| = \int_0^{\infty} \mathrm{d}E E \,|\, E >< E \,| = A_+ A_- . \tag{33}$$

The expectation value of an integer power s of number operator N_c is

$$< E \,|\, \left(N_c\right)^s \,|\, E > \equiv < \left(N_c\right)^s >_E = < E \,|\, \left(A_+ A_-\right)^s \,|\, E > = E^s \tag{34}$$

so that the Mandel parameter [16] in the $|\, E >$ representation is

$$Q_E^{(c)} = \frac{< N_c^2 >_E - \left(< N_c >_E\right)^2}{< N_c >_E} - 1 = -1 \tag{35}$$

and this shows that the energy eigenstates $|\, E >$ have the sub-Poissonian behavior, like the Fock-vectors $|\, n >$.

3. DIAGONAL ORDERING OPERATION TECHNIQUE FOR CONTINUOUS SPECTRUM

We briefly recall some basic elements of the DOOT introduced in [8] for the CSs of the discrete spectrum and we adapt these rules to operators corresponding to the continuous spectrum. The operators A_- and A_+ act as *lowering* and *raising* operators, so their normal ordered product $A_+ A_-$ is a *diagonal* operator in the energy eigenvectors basis $|\, E >$. Consequently, we will examine *only general functions depending on the normal ordered operator product* $A_+ A_-$, say $F(A_+ A_-)$. We adopt the following rules for the DOOT calculus for continuous spectrum:

a) Inside the symbol $\#\;\;\#$ the order of the operators A_- and A_+ can be permuted like commutable operators, so that finally we obtain an operator function depending only on the powers of normally ordered operator product A_+A_-, *i.e.*

$$\#(A_-)^E(A_+)^E\# = \#\,(A_+)^E(A_-)^E\# = \#(A_+A_-)^E\#\,; \tag{36}$$

b) A symbol $\#\;\;\#$ inside another symbol $\#\;\;\#$ can be deleted;

c) If the integration is convergent, a normally ordered product of operators can be integrated or differentiated with respect to *c*-numbers according to the usual rules and the *c*-numbers can be taken out from the symbol $\#\;\;\#$.

d) The projector $|\,0><0\,|$ of the normalized vacuum state $|\,0>$, in the frame of DOOT is (this assertion will be demonstrated below)

$$|\,0><0| = \lim_{d\to c}\#\frac{1}{{}_pF_q(\{a_i\}_1^p\,;\,\{b_j\}_1^q\,;A_+A_-)}\# = \#\frac{1}{\nu(A_+A_-)}\#, \tag{37}$$

where $\nu(A_+A_-)$ is the ν-function depending on the operatorial "variable" A_+A_-.

A function $\#F(A_+A_-)\#$, has the following spectral decomposition

$$\#F(A_+A_-)\# = \int_0^\infty \mathrm{d}E\,F(E)\,|\,E><E\,|. \tag{38}$$

Using Eq. (29) and their counterpart, we have

$$|\,E> = \frac{1}{\sqrt{\Gamma(E+1)}}(A_+)^E\,|\,0>, < E\,| = \frac{1}{\sqrt{\Gamma(E+1)}} < 0\,|\,(A_-)^E \tag{39}$$

and the closure relation for continuous spectrum (4) becomes successively

$$\begin{aligned}\int_0^\infty \mathrm{d}E\,|\,E><E\,| &= \int_0^\infty \mathrm{d}E\,\frac{1}{\Gamma(E+1)}\#\left(A_+\right)^E|\,0><0\,|\left(A_-\right)^E\# = \\ = |\,0><0\,|\int_0^\infty \mathrm{d}E\,\frac{1}{\Gamma(E+1)}\#\left(A_+A_-\right)^E\# &= |\,0><0\,|\,\#\,\nu\left(A_+A_-\right)\# = I,\end{aligned} \tag{40}$$

which proves the expression (37) for the vacuum projector.

Consequently, the projector in the energy eigenvectors space $|\,E><E\,|$ is

$$|\,E><E\,| = \frac{1}{\Gamma(E+1)}\#\frac{1}{\nu(A_+A_-)}(A_+A_-)^E\#. \tag{41}$$

Using the above relations and the DOOT rules, the CSs for the continuous spectrum can be written in the following operatorial manner:

$$| Z >= \frac{1}{\sqrt{\nu\left(| Z |^2\right)}} \int_0^\infty \mathrm{d}E \frac{\left(Z \quad A_+\right)^E}{\Gamma\left(E+1\right)} | 0 > = \frac{1}{\sqrt{\nu\left(| Z |^2\right)}} \nu\left(ZA_+\right) | 0 > \tag{42}$$

and similarly their counterpart, so that the projector on the CSs state $| Z >$ is

$$| Z >< Z | = \lim_{d \to c} | z >< z | = \frac{1}{\nu\left(| Z |^2\right)} \# \frac{\nu\left(Z A_+\right) \nu\left(Z^* A_-\right)}{\nu\left(A_+ A_-\right)} \# . \tag{43}$$

The expectation values for integer s -power of the number operator is

$$< N_c^s >_Z = \frac{1}{\nu\left(| Z |^2\right)} \int_0^\infty \mathrm{d}E \, E^s \frac{\left(| Z |^2\right)^E}{\Gamma\left(E+1\right)} = \frac{1}{\nu\left(| Z |^2\right)} \left(| Z |^2 \frac{\mathrm{d}}{\mathrm{d} | Z |^2} \right)^s \nu\left(| Z |^2\right) . \tag{44}$$

which is useful to compute the Mandel parameter [16, 17]

$$Q_{|Z|}^{(c)} = \frac{< (\Delta N_c)^2 >_Z}{< N_c >_Z} - 1 = \frac{< N_c^2 >_Z - \left(< N_c >_Z\right)^2}{< N_c >_Z} - 1 . \tag{45}$$

The Mandel parameter can be expressed through the derivatives of the ν-function (prime and second signify the derivatives with respect to the variable $| Z |^2$):

$$Q_{|Z|}^{(c)} = | Z |^2 \left[\frac{\nu''\left(| Z |^2\right)}{\nu'\left(| Z |^2\right)} - \frac{\nu'\left(| Z |^2\right)}{\nu\left(| Z |^2\right)} \right] = 0 , \tag{46}$$

where we used that $\nu''(| Z |^2) = \nu'(| Z |^2) = \nu(| Z |^2)$. Zero value of the Mandel parameter show that the CSs $| Z >$ have the Poissonian behavior or statistics [16, 17].

The probability density of the transition from state $| E >$ to state $| Z >$ is

$$P_E\left(| Z |^2\right) = |< Z | E >|^2 = \frac{1}{\nu\left(| Z |^2\right)} \frac{\left(| Z |^2\right)^E}{\Gamma\left(E+1\right)} , \tag{47}$$

which can be regarded as the continuous generalized Poisson probability density function. This interpretation is justified because, if we apply the *inverse* limit, *i.e.* the *continuous – discrete limit* $c \to d$, we obtain just the Poisson probability density function for the discrete case:

$$\lim_{c \to d} P_E\left(|Z|^2\right) \equiv \lim_{\substack{E \to n \\ \int \to \sum}} P_E\left(|Z|^2\right) = e^{-|Z|^2} \frac{\left(|Z|^2\right)^n}{n!} \equiv P_n\left(|Z|^2\right). \tag{48}$$

4. MIXED (THERMAL) STATES

If a quantum system with the *dimensionless* Hamiltonian H and the continuous energy spectrum E is in the thermodynamic equilibrium with the environment at temperature $T = (k_B \beta)^{-1}$ then it is characterized by a mixed state with a normalized density operator obeying the canonical distribution

$$\rho_c = \frac{1}{Z_c} e^{-\beta \hbar\omega H} = \frac{1}{Z_c} \# e^{-\beta \hbar\omega A_+ A_-} \# = \frac{1}{Z_c} \int_0^\infty \mathrm{d}E e^{-\beta \hbar\omega E} \, |E\rangle\langle E|. \tag{49}$$

The partition function is determined by normalizing the density operator:

$$\mathrm{Tr}\rho_c = \int_0^\infty \mathrm{d}E' \langle E'|\rho_c|E'\rangle = \frac{1}{Z_c} \frac{1}{\beta \ \ \hbar\omega} = 1, \qquad Z_c = \frac{1}{\beta\hbar\omega}. \tag{50}$$

Using the DOOT rules, and Eq. (49) the density operator becomes

$$\rho_c = \frac{1}{Z_c} \# \frac{1}{\nu\left(A_+ A_-\right)} \int_0^\infty \mathrm{d}E \frac{1}{\Gamma(E+1)} \left(e^{-\beta \hbar\omega} A_+ A_-\right)^E \# = \frac{1}{Z_c} \# \frac{\nu\left(e^{-\beta \hbar\omega} A_+ A_-\right)}{\nu\left(A_+ A_-\right)} \#. \tag{51}$$

Consequently, the Q-distribution function, defined as the diagonal elements of the normalized density operator in the CSs representation [13], is

$$\begin{aligned} Q_c\left(|Z|^2\right) &\equiv \langle \rho_c \rangle_Z = \\ &= \frac{1}{Z_c} \frac{1}{\nu\left(|Z|^2\right)} \int_0^\infty \mathrm{d}E \mathrm{d}E' \frac{\left(Z^*\right)^E Z^{E'}}{\sqrt{\Gamma\left(E+1\right) \ \Gamma\left(E'+1\right)}} \langle E| \# e^{-\beta \hbar\omega A_+ A_-} \# |E'\rangle \end{aligned} \tag{52}$$

so that finally, we obtain the Q-distribution function normalized to unity:

$$Q_c\left(|Z|^2\right) = \frac{1}{Z_c} \frac{\nu\left(e^{-\beta\hbar\omega} \ |Z|^2\right)}{\nu\left(|Z|^2\right)}. \tag{53}$$

The diagonal expansion of the ρ_c in terms of the $|Z\rangle$ – projectors is:

$$\rho_c = \int \mathrm{d}\mu_c(Z) P_c(|Z|^2)\,|Z><Z|\,, \tag{54}$$

where the P-quasi-distribution must be determined using Eqs. (13) and (8), the result of angular integration being $\left(|Z|^2\right)^E \delta(E-E')$.

Comparing with Eq. (49), we must solve the Stieltjes moment problem

$$\int_0^\infty \mathrm{d}\left(|Z|^2\right) e^{-|Z|^2} P_c(|Z|^2)(|Z|^2)^E = \frac{1}{Z_c(\beta)}\left(e^{-\beta\hbar\omega}\right)^E \Gamma(E+1). \tag{55}$$

The substitutions $E = s-1$ and $\widetilde{P}_c(|Z|^2) = e^{-|Z|^2} P_c(|Z|^2)$ leads to

$$\int_0^\infty \mathrm{d}\left(|Z|^2\right)(|Z|^2)^{s-1}\tilde{P}_c(|Z|^2) = \frac{1}{Z_c(\beta)} e^{\beta\hbar\omega}\left(e^{\beta\hbar\omega}\right)^{-s}\Gamma(s). \tag{56}$$

The solution of this problem is [15]

$$\tilde{P}_c(|Z|^2) = \frac{1}{Z_c(\beta)} e^{\beta\hbar\omega} G_{0,\,1}^{1,\,0}\left(e^{\beta\hbar\omega}\ |Z|^2 \middle| 0\right) = \frac{1}{Z_c(\beta)} e^{\beta\hbar\omega} e^{-e^{\beta\hbar\omega}|Z|^2} \tag{57}$$

and finally, the normalized to unity P -quasi-distribution function becomes

$$P_c(|Z|^2) = \frac{1}{Z_c(\beta)} e^{\beta\hbar\omega} e^{-(e^{\beta\hbar\omega}-1)\ |Z|^2}. \tag{58}$$

The thermal expectation values (thermal averages) of the operators $\#O_c\#$ which characterize the quantum system finally are expressed as

$$<\#O_c\#> = \mathrm{Tr}(\#\rho_c O_c\#) = \frac{1}{Z_c(\beta)}\int_0^\infty \mathrm{d}E e^{-\beta\hbar\omega E} <E|\#O_c\#|E>. \tag{59}$$

Particularly, for integer s-power of the number operator, we have

$$<O_c^s> \equiv <\#\left(A_+A_-\right)^s\#> = \frac{1}{Z_c(\beta)}\int_0^\infty \mathrm{d}E E^s e^{-\beta\hbar\omega E} = \frac{s!}{\left(\beta\hbar\omega\right)^s}. \tag{60}$$

The thermal counterpart of the Mandel parameter is defined similarly to that for the pure states [18]. Finally, after straightforward calculations, we obtain:

$$Q_c(\beta) = \frac{<O_c^2> - \left(<O_c>\right)^2}{<O_c>} - 1 = \frac{k_B}{\hbar\omega}T - 1 \equiv \frac{T}{T_D} - 1, \tag{61}$$

where we have used the expression for the Debye temperature $T_D = \hbar\omega / k_B$. The behavior of the thermal Mandel parameter $Q_c(\beta)$ as function of the equilibrium temperature T is: for $T < T_D$, $Q_c(\beta) < 0$ is sub-Poissonian, for $T = T_D$, $Q_c(\beta) = 0$ is Poissonian, and for $T > T_D$, $Q_c(\beta) > 0$ is super-Poissonian.

5. CONCLUDING REMARKS

The CSs for continuous spectrum of a given system (*e.g.* radial Coulomb [11], free particle and the inverted harmonic oscillator [19], Morse oscillator [13]), previously introduced by Gazeau and Klauder [9], can and obtained also by beginning from the discrete CSs of respectively quantum systems. We have presented a more general procedure to obtain the CSs for the continuous spectrum, by starting from the generalized hypergeometric coherent states (GH-CSs) which contain, as particular cases, all other particularly kind of CSs for the discrete part of spectrum [5, 6]. We have shown that, beginning from the GH-CSs (for the discrete spectrum), and applying a suitable limit (the discrete – continuous limit $d \to c$), we can obtain the CSs for the continuous spectrum of the Hamiltonian operator, as well as all their characteristics. A first conclusion of this procedure is that, regardless of the type CSs for the discrete part of the spectrum of a gyven quantum system, by applying an appropriate limits (which obviously varies from case to case), we obtain *the same kind* of CSs for the continuous spectrum, *i.e.* having the same mathematical structure. Things happen as the system would “forget” their own history, *i.e.* as though the particle was liberated to “grip” of potential which has been subjected in the discrete spectrum. We have used a new ordering procedure – the diagonal ordering operation technique (DOOT) [8] which allows us to find, relatively easy and fast, a series of characteristics and properties of these states (integration measure, the probability density, the density operator, $Q-$and $P-$distribution functions, Mandel parameter and so on). These results are in agreement with the literature [9–10, 12, 18–19] which proves that the proposed method is correct. The CSs for continuous spectrum $|Z>$ obey the Poissonian statistics, their Mandel parameter being zero. Their probability density function tends, at the inverse (continuous – discrete $c \to d$) limit $\lim_{c \to d}$ to the corresponding discrete Poissonian probability density function. The values of thermal counterpart of the Mandel parameter $Q_c(\beta)$ shows that the thermal states for continuous spectrum obeys either sub-Poissonian, Poissonian or super-Poissonian statistics, depending on the value of equilibrium temperature T.

Acknowledgements. The authors are grateful to the referee for pertinent observations that have helped us to improve the manuscript.

REFERENCES

1. J.R. Klauder, B. Skagerstam, *Coherent States – Applications in Mathematical Physics*, World Scientific, Singapore, 1985.
2. A. Perelomov, *Generalized coherent states and their applications*, Springer, Berlin, 1986.
3. W. M. Zhang, D.H. Feng, R. Gilmore, Rev. Mod. Phys. **62**, 867-927 (1990).
4. J.P. Gazeau, *Coherent States in Quantum Physics*, Wiley-VCH, Berlin, 2009.
5. T. Appl and D.H. Schiller, J. Phys. A: Math. Gen., **37**, *7*, 2731-2750 (2004).
6. D. Popov, Electronic Journal of Theoretical Physics (EJTP) **3**, *11*, 123-132 (2006).
7. H. Fan, Commun. Theor. Phys. (Beijing, China) **31**, *2*, 285-290 (1999).
8. D. Popov, M. Popov, Phys. Scr. **90**, 035101, 1-16 (2015).
9. J.P. Gazeau, J.R. Klauder, J. Phys. A: Math. Gen. **32**, 123-132 (1999).
10. A. Inomata, M. Sadiq, *Modification of Klauder's coherent states*, 8th International Conference *Path Integrals. From Quantum Information to Cosmology*, http://www1.jinr.ru/Proceedings/Burdik-2005/pdf/inomata.pdf
11. M. Thaik, A. Inomata, J. Phys. A: Math. Theor. **38**, 1767-1775 (2005).
12. J. Ben Geloun, J.R. Klauder, J. Phys. A: Math. Theor. **42**, 375209 (2009).
13. D. Popov, S.-H. Dong, M. Popov: Ann. Phys. **362**, 449-472 (2015).
14. I.S. Gradshteyn, I.M. Ryshik, *Table of Integrals, Series and Products*, Seventh edition, Academic Press, Amsterdam, 2007; see also: http://mathworld.wolfram.com/NuFunction.html.
15. A.M. Mathai, R.K. Saxena, *Generalized Hypergeometric Functions with Applications in Statistics and Physical Sciences*, Lect. Notes Math. Vol. (Springer-Verlag, Berlin) **348**, 1973.
16. L. Mandel, E. Wolf, *Optical Coherence and Quantum Optics*, Cambridge University Press, Cambridge, 1994.
17. D.F. Walls, G.J. Milburn, *Quantum Optics*, Springer-Verlag, Berlin, 1995.
18. D. Popov, J. Phys. A: Math. Gen. **35**, 7205-7223 (2002).
19. J. Ben Geloun, J. Hnybida, J.R. Klauder, J. Phys. A: Math. Theor, **45**, *8*, 085301 (2012).