

ELASTIC WAVES EQUATION WITH LOCALIZED SOURCES IN ISOTROPIC HALF-SPACE

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Received December 6, 2015

Abstract. The representation of a localized faulting seismic source is reformulated, without resorting to the mechanical torque interpretation of the seismic moment tensor. A volume seismic source is introduced by means of the pressure exerted in a small spherical cavity. The elastic waves equation with localized (point) sources is discussed in an isotropic half space bounded by a plane surface, including the boundary conditions. The near-field approximation is used to get the solution of a quasi-static deformation, and the transient regime of far-field propagating elastic waves is discussed for the (time-pulse) seismic sources introduced here. The far-field approximation (waves region) is also presented.

Key words: seismic sources; elastic waves equation; isotropic half-space; near- and far-field solutions.

PACS: 91.30.Bi; 91.30.Cd; 02.30.Jr; 46.70.Lk; 02.30.Em

1. INTRODUCTION

The main problem in Seismology is the propagation of the seismic waves. It gives information about processes occurring in earthquake focal region, the nature and structure of the Earth's interior, and the effect of the seismic waves on the Earth's surface. The problem originates in the works of Rayleigh, Lamb and Love [1]-[3]. In a simplified model, the Earth is viewed as an isotropic elastic half-space bounded by a plane surface, and the seismic sources are localized beneath the surface. For a sufficiently long distance the localization of the seismic sources may be represented by δ -functions, or their derivatives (point sources). The double-couple representation of point seismic sources by means of the seismic moment tensor emerged gradually in the first half of the 20th century [4]-[17].

The standard way of treating the seismic waves is to employ the (formal) Green function for the elastic waves equation and the Green theorem (the so-called Betti's representation), for a general, anisotropic, elastic body [18]-[21]. In this treatment the seismic sources are located on internal surfaces, either as faulting sources or vol-

ume sources. The faulting seismic sources are related to the discontinuity occurring in the displacement across the faulting surface (fault slip), while the volume sources are related to the dilatational strain (dislocation model) [6, 7]. In both cases equivalent forces are derived for the seismic source representation, and the seismic moment is introduced. The interpretation of the “double-couple” representation of the seismic moment as a mechanical torque (a couple of forces), while ensuring the vanishing of the net force and angular momentum, leaves open the question of the uniqueness of the double-couple distribution. We reformulate here the faulting source and the introduction of the seismic moment tensor, and introduces a volume source related to the pressure exerted in a small spherical cavity.

The elastic waves equation with sources for a half space bounded by a plane surface is discussed, including boundary conditions (usually, for a free surface). This equation is suitable for a stationary regime; we give an approximate near-field solution with the seismic sources introduced here for quasi-static displacements (especially for the surface). The far-field waves generated by these seismic sources (with δ -like time pulses) are given in the transient regime, where the relevant distances are sufficiently long in comparison with the wavelengths. The waves region is also discussed in this context (far-field approximation).

2. SEISMIC SOURCES

The seismic sources are concentrated in small volumes. The linear dimensions of these regions are much smaller than the seismic wavelengths and distances of interest, so we may view them as point sources. In an approximate representation of a faulting source the slip and the associated force occur, during an earthquake, along one direction, lying on the fault surface. Let \mathbf{n} be the unit vector along this direction. The seismic load in the focus consists of two opposing forces, usually at (quasi-) equilibrium, such that the total force and angular momentum are vanishing. During an earthquake, the resistance of the rocks in the focus yields, so that we have a localized, active distribution of forces which is proportional to $\partial\delta(\mathbf{R} - \mathbf{R}_0)/\partial n$, where \mathbf{R}_0 is the position of the focus. This can also be written as $f_i l \partial_i \delta(\mathbf{R} - \mathbf{R}_0) = f l_i \partial_i \delta(\mathbf{R} - \mathbf{R}_0) = f l n_i \partial_i \delta(\mathbf{R} - \mathbf{R}_0)$, where f_i are the components of the force with magnitude f acting along the direction \mathbf{n} and l_i are the components of the spatial extension of the focus along the same direction \mathbf{n} ; the function δ in these expressions should be understood as a function localized over the distance l along the direction \mathbf{n} , and, similarly, along two other transverse directions. The quantity fl is proportional to the seismic moment M (since f can be represented as $f = \mu A$, where μ is the rigidity modulus and A is the area of the fault); we prefer to use the seismic moment

divided by density, $m = M/\rho$; then, the force distribution per unit mass reads

$$\mathbf{F}(\mathbf{R}, t) = m(t)n_i\partial_i\delta(\mathbf{R} - \mathbf{R}_0)\mathbf{n}, \quad F_i(\mathbf{R}, t) = m(t)n_in_j\partial_j\delta(\mathbf{R} - \mathbf{R}_0), \quad (1)$$

where $m(t)$ has a certain time dependence during the earthquake; usually, this function is localized over a finite duration T . The force distribution given by equation (1) represents a point linear dipole; since a strain occurring along a direction \mathbf{n} may generate forces directed both along \mathbf{n} and along the two directions perpendicular to \mathbf{n} , the force distribution given by equation (1) may be generalized by replacing $m(t)n_in_j$ by the tensor $m_{ij}(t)$:

$$F_i(\mathbf{R}, t) = m_{ij}(t)\partial_j\delta(\mathbf{R} - \mathbf{R}_0); \quad (2)$$

in order to have a vanishing angular momentum, the tensor of the seismic moment must be symmetric; Equations (1) and (2) provide the “double-couple” representation of a faulting source (without mechanical-torque interpretation).

Similarly, the force distribution localized in a volume source with a small radius a can be written as

$$\mathbf{F}(\mathbf{R}, t) = p(t)\frac{\mathbf{R} - \mathbf{R}_0}{|\mathbf{R} - \mathbf{R}_0|}\theta(a - |\mathbf{R} - \mathbf{R}_0|), \quad (3)$$

where $p(t) = f(t)/a^3$ is force per unit volume, divided by density (force per unit mass).

3. ELASTIC WAVES EQUATION

The elastic waves equation [22]

$$\ddot{\mathbf{u}} - c_t^2\Delta\mathbf{u} - (c_l^2 - c_t^2)\text{grad}\cdot\text{div}\mathbf{u} = \mathbf{F}, \quad (4)$$

where \mathbf{u} is the displacement, $c_{l,t}$ are the velocities of the “longitudinal” and, respectively, “transverse” waves and \mathbf{F} is force per unit mass, can be decomposed into two equations

$$\ddot{D} - c_l^2\Delta D = \text{div}\mathbf{F}, \quad (5)$$

$$\ddot{\mathbf{u}} - c_t^2\Delta\mathbf{u} = \mathbf{F} + (c_l^2 - c_t^2)\text{grad}D,$$

by taking the *div* in equation (4), and denoting $D = \text{div}\mathbf{u}$. In the second equation (5) the dilatational waves governed by D become a source for the “transverse” waves. For a half space $z < 0$ bounded by the plane surface $z = 0$ it is convenient to use the in-plane position vector \mathbf{r} , $\mathbf{R} = (\mathbf{r}, z)$, and the in-plane (horizontal) displacement \mathbf{u} by changing the notation $\mathbf{u} \rightarrow (\mathbf{u}, v)$; v is the displacement component perpendicular to the surface. Similarly, it is convenient to introduce the time and in-plane Fourier

transformations; the first equation (5) becomes

$$\frac{d^2 D}{dz^2} + \kappa_l^2 D = -\frac{1}{c_l^2} \left(i\mathbf{k}\mathbf{F}_{xy} + \frac{\partial F_z}{\partial z} \right), \quad (6)$$

where $\omega^2 = c_l^2(k^2 + \kappa_l^2)$, \mathbf{k} is the in-plane wavevector and $\mathbf{F} = (\mathbf{F}_{xy}, F_z)$ (for simplicity we use the same notations for functions and their Fourier transforms, the difference being easily seen from the context). Making use of the Green function $e^{i\kappa_l|z|}/2i\kappa_l$ of the one-dimensional Helmholtz equation, the solution of equation (6) is readily obtained as

$$D = \frac{i}{2\kappa_l c_l^2} S_1 + A e^{i\kappa_l|z|}, \quad (7)$$

where

$$S_1 = \int dz' e^{i\kappa_l|z-z'|} \left[i\mathbf{k}\mathbf{F}_{xy}(z') + \frac{\partial F_z(z')}{\partial z'} \right] \quad (8)$$

and A is a constant; the integration is performed from $z = -\infty$ to $z = 0$. Similarly, the solutions of the second equation (5) are

$$\begin{aligned} \mathbf{u} &= \frac{i}{2\kappa_t c_t^2} \mathbf{S}_2 - \frac{i(1-\eta)}{4\kappa_t \kappa_l c_t^2} \mathbf{k} S_3 - \\ &- \frac{i(1-\eta)}{2\eta \kappa_t} \mathbf{k} A \left(\frac{2\kappa_t}{\kappa_t^2 - \kappa_l^2} e^{i\kappa_l|z|} - \frac{1}{\kappa_t - \kappa_l} e^{i\kappa_t|z|} \right), \\ v &= \frac{i}{2\kappa_t c_t^2} S_4 - \frac{1-\eta}{4\kappa_t \kappa_l c_t^2} S_5 + \\ &+ \frac{i(1-\eta)}{2\eta \kappa_t} \kappa_l A \left(\frac{2\kappa_t}{\kappa_t^2 - \kappa_l^2} e^{i\kappa_l|z|} - \frac{1}{\kappa_t - \kappa_l} e^{i\kappa_t|z|} \right) + B e^{i\kappa_t|z|}, \end{aligned} \quad (9)$$

where

$$\begin{aligned} \mathbf{S}_2 &= \int dz' e^{i\kappa_t|z-z'|} \mathbf{F}_{xy}(z'), \quad S_3 = \int dz' e^{i\kappa_t|z-z'|} S_1(z'), \\ S_4 &= \int dz' e^{i\kappa_t|z-z'|} F_z(z'), \quad S_5 = \int dz' e^{i\kappa_t|z-z'|} S_1'(z'), \end{aligned} \quad (10)$$

$\omega^2 = c_t^2(k^2 + \kappa_t^2)$ and $\eta = c_t^2/c_l^2$; it is sufficient to include a free solution of the form $B e^{i\kappa_t|z|}$ in v .

The boundary conditions for a pressure \mathbf{P} (divided by density ρ) acting upon the surface $z = 0$ are given by $n_j \sigma_{ij} = -\rho P_i$ for $z = 0$, where $\mathbf{n} = (0, 0, 1)$ is the unit vector normal to the surface (with components n_j) and σ_{ij} is the stress tensor. Making use of $\sigma_{ij} = \frac{E}{1+\sigma} \left(u_{ij} + \frac{\sigma}{1-2\sigma} u_{kk} \delta_{ij} \right)$, where u_{ij} is the strain tensor, E is

Young's modulus and σ is Poisson's ratio, these boundary conditions read

$$\mathbf{k}\mathbf{u}^1 + ik^2v^0 = -\frac{2\rho(1+\sigma)}{E}\mathbf{k}\mathbf{P}_{xy}, \quad (11)$$

$$(1-\sigma)v^1 + i\sigma\mathbf{k}\mathbf{u}^0 = -\frac{\rho(1+\sigma)(1-2\sigma)}{E}P_z,$$

where \mathbf{u}^0, v^0 are the values of the functions for $z = 0$, \mathbf{u}^1, v^1 are the values of the first derivatives of the functions for $z = 0$ and $\mathbf{P} = (\mathbf{P}_{xy}, P_z)$; using the expressions $c_t^2 = E(1-\sigma)/\rho(1+\sigma)(1-2\sigma)$, $c_t^2 = E/2\rho(1+\sigma)$ for velocities [22], equations (11) become

$$\mathbf{k}\mathbf{u}^1 + ik^2v^0 = -\frac{1}{c_t^2}\mathbf{k}\mathbf{P}_{xy}, \quad (12)$$

$$v^1 + i(1-2\eta)ku^0 = -\frac{\eta}{c_t^2}P_z.$$

Using \mathbf{u} and v given by equation (9), the boundary conditions become

$$\begin{aligned} \frac{(1-\eta)(\kappa_t-\kappa_l)}{2\eta\kappa_t(\kappa_t+\kappa_l)}k^2A + ik^2B &= -\frac{i}{2\kappa_t c_t^2}\mathbf{k}\mathbf{S}'_2(0) + \\ &+ \frac{i(1-\eta)}{4\kappa_t\kappa_l c_t^2}k^2S'_3(0) + \frac{k^2}{2\kappa_t c_t^2}S_4(0) + \frac{i(1-\eta)}{4\kappa_t\kappa_l c_t^2}k^2S_5(0) - \frac{1}{c_t^2}\mathbf{k}\mathbf{P}_{xy}, \\ \frac{1-\eta}{2\eta\kappa_t(\kappa_t+\kappa_l)}[(1-2\eta)k^2 - \kappa_t\kappa_l]A - i\kappa_t B &= -\frac{i}{2\kappa_t c_t^2}S'_4(0) + \\ &+ \frac{1-\eta}{4\kappa_t\kappa_l c_t^2}S'_5(0) + \frac{1-2\eta}{2\kappa_t c_t^2}\mathbf{k}\mathbf{S}_2(0) - \frac{(1-\eta)(1-2\eta)}{4\kappa_t\kappa_l c_t^2}k^2S_3(0) - \frac{\eta}{c_t^2}P_z. \end{aligned} \quad (13)$$

With the solution for the coefficients A and B of this system of equations the displacement in the half-space is completely determined. Although pretty unpracticable, the solution of the elastic waves equation is reduced by the above equations to quadratures.

4. NEAR-FIELD APPROXIMATION

In the near-field, or quasi static approximation we may take $\omega^2/c_{t,l}^2 \simeq 0$ and get $\kappa_{t,l} \simeq ik$; in this approximation the z -dependence of the waves is damped. Equations (9) give the displacement

$$\begin{aligned} \mathbf{u} &\simeq \frac{1}{2c^2k}\mathbf{S}_2 + \frac{i(1-\eta)}{4c^2k^2}\mathbf{k}\mathbf{S}_3 + \frac{i(1-\eta)}{4\eta k^2}\mathbf{k}Ae^{-k|z|}, \\ v &\simeq \frac{1}{2c^2k}S_4 + \frac{1-\eta}{4c^2k^2}S_5 + \left[\frac{1-\eta}{4\eta k}A + B\right]e^{-k|z|}, \end{aligned} \quad (14)$$

where we set approximately $c_t = c$ and $\eta \neq 1$; this latter approximation may affect appreciably the solution. Within the same approximation the boundary conditions

given by equations (13) (for a free surface) become

$$\begin{aligned}
 ik^2 B &\simeq -\frac{1}{2c^2 k} \mathbf{kS}'_2(0) - \frac{i(1-\eta)}{4c^2} S'_3(0) - \frac{ik}{2c^2} S_4(0) - \frac{i(1-\eta)}{4c^2} S_5(0) , \\
 -\frac{(1-\eta)^2}{2\eta} A + kB &\simeq -\frac{1}{2c^2 k} S'_4(0) - \frac{1-\eta}{4c^2 k^2} S'_5(0) - \frac{i(1-2\eta)}{2c^2 k} \mathbf{kS}_2(0) + \\
 &\quad + \frac{(1-\eta)(1-2\eta)}{4c^2} S_3(0) .
 \end{aligned} \tag{15}$$

4.1. VERTICAL FAULTING SOURCE

We consider now a vertical faulting source with the force (per unit mass) given by equation (1):

$$F_z = m(t)\delta(\mathbf{r})\partial_z\delta(z - z_0) , \tag{16}$$

where $z_0 < 0$; its Fourier transform is

$$F_z = m(\omega)\partial_z\delta(z - z_0) . \tag{17}$$

Making use of this force we compute the source terms $S_{1\dots 5}$ given by equations (8) and (10); the calculations are straightforward, and we get

$$\begin{aligned}
 S_1 &= m(\omega) [2i\kappa_l\delta(z - z_0) - \kappa_l^2 e^{i\kappa_l|z-z_0|}] , \quad S_2 = 0 , \\
 S_3 &= m(\omega) [2i\kappa_l e^{i\kappa_l|z-z_0|} - \kappa_l^2 I] , \\
 S_4 &= im(\omega)\kappa_t \operatorname{sgn}(z - z_0) e^{i\kappa_t|z-z_0|} , \\
 S_5 &= m(\omega) \left[-2\kappa_t\kappa_l \operatorname{sgn}(z - z_0) e^{i\kappa_t|z-z_0|} + \kappa_l^2 \frac{\partial I}{\partial z_0} \right] ,
 \end{aligned} \tag{18}$$

where

$$\begin{aligned}
 I &= \int dz' e^{i\kappa_t|z-z'|} e^{i\kappa_l|z'-z_0|} = \\
 &= \begin{cases} -\frac{2i\kappa_l}{\kappa_t^2 - \kappa_l^2} [e^{i\kappa_t|z-z_0|} - e^{i\kappa_l|z-z_0|}] - \frac{i}{\kappa_t + \kappa_l} e^{-i\kappa_t z - i\kappa_l z_0} , & z < z_0 , \\ -\frac{2i\kappa_t}{\kappa_t^2 - \kappa_l^2} [e^{i\kappa_t|z-z_0|} - e^{i\kappa_l|z-z_0|}] - \frac{i}{\kappa_t + \kappa_l} e^{-i\kappa_t z - i\kappa_l z_0} , & z > z_0 . \end{cases}
 \end{aligned} \tag{19}$$

Using $\kappa_t - \kappa_l \simeq \omega^2(1 - \eta)/2ik\eta c^2$ and setting $\kappa_{t,l} \simeq ik$ we get

$$\begin{aligned} I &\simeq -\frac{1}{2k} e^{-k|z+z_0|}, \\ S_3 &= -m(\omega)k \left[2e^{-k|z-z_0|} + \frac{1}{2}e^{-k|z+z_0|} \right], \\ S_4 &= -m(\omega)k \operatorname{sgn}(z - z_0) e^{-k|z-z_0|}, \end{aligned} \quad (20)$$

$$S_5 = m(\omega)k^2 \left[2\operatorname{sgn}(z - z_0) e^{-k|z-z_0|} - \frac{1}{2}e^{-k|z+z_0|} \right];$$

making use of these source-terms in equations (15) we get the constants A and B ,

$$A = \frac{m(\omega)k}{c^2} a e^{-k|z_0|}, \quad B = \frac{m(\omega)}{c^2} b e^{-k|z_0|}, \quad (21)$$

where a and b are numerical factors,

$$a = \frac{\eta(1 - 2\eta + 5\eta^2)}{2(1 - \eta)^2}, \quad b = \frac{3\eta - 1}{4}. \quad (22)$$

Now we can write the displacement given by equations (14) as

$$\mathbf{u} = -\frac{im(\omega)(1-\eta)}{2c^2} \frac{\mathbf{k}}{k} \left(e^{-k|z-z_0|} + \frac{\eta-2a}{4\eta} e^{-k|z+z_0|} \right), \quad (23)$$

$$v = -\frac{m(\omega)}{2c^2} \left[\eta \operatorname{sgn}(z - z_0) e^{-k|z-z_0|} - \frac{2a(1-\eta) + 8\eta b - \eta(1-\eta)}{4\eta} e^{-k|z+z_0|} \right].$$

In order to compute the Fourier transforms of the displacement given by equations (23) we need the integrals

$$\int d\mathbf{k} \frac{e^{i\mathbf{k}\mathbf{r}}}{k} e^{-k|z|} = \frac{2\pi}{R}, \quad \int d\mathbf{k} e^{i\mathbf{k}\mathbf{r}} e^{-k|z|} = \frac{2\pi|z|}{R^3}, \quad (24)$$

where $R = (r^2 + z^2)^{1/2}$. We get finally

$$\mathbf{u} = \frac{m(t)(1-\eta)}{4\pi c^2} \left(\frac{1}{R_1^3} + \frac{\eta-2a}{4\eta} \frac{1}{R_2^3} \right) \mathbf{r}, \quad (25)$$

$$v = \frac{m(t)}{4\pi c^2} \left[-\eta \frac{z-z_0}{R_1^3} + \frac{2a(1-\eta) + 8\eta b - \eta(1-\eta)}{4\eta} \cdot \frac{|z+z_0|}{R_2^3} \right],$$

where $R_{1,2} = [r^2 + (z \mp z_0)^2]^{1/2}$. On the surface $z = 0$ the displacement is given by

$$\mathbf{u}^0 = \frac{m(t)}{4\pi c^2} \cdot \frac{(1-\eta)(5\eta-2a)}{4\eta} \cdot \frac{\mathbf{r}}{R_0^3}, \quad (26)$$

$$v^0 = \frac{m(t)}{4\pi c^2} \cdot \frac{2a(1-\eta) + 8\eta b - \eta(1-5\eta)}{4\eta} \cdot \frac{|z_0|}{R_0^3},$$

where $R_0 = (r^2 + z_0^2)^{1/2}$; we can see that the maximum values of the displacement are of the order $m/c^2 z_0^2$, reached for $r \simeq |z_0|/\sqrt{2}$ for the horizontal displacement

and for $r = 0$ for the vertical displacement. It is worth noting that we have extended the integration over frequency ω to infinity in equations (25), in contrast with our approximation of small ω .

4.2. HORIZONTAL FAULTING SOURCE

The horizontal faulting force given by equation (1) is

$$\mathbf{F}_{xy} = m(t)n_i\partial_i\delta(\mathbf{r})\delta(z - z_0)\mathbf{n}, \quad (27)$$

with the Fourier transform

$$\mathbf{F}_{xy} = im(\omega)\mathbf{kn}\delta(z - z_0)\mathbf{n}, \quad (28)$$

where \mathbf{n} is the in-plane unit vector which defines the fault direction. The source terms $S_{1...5}$ given by equations (8) and (10) are

$$\begin{aligned} S_1 &= -m(\omega)(\mathbf{kn})^2 e^{i\kappa_l|z-z_0|}, S_2 = im(\omega)(\mathbf{kn})\mathbf{n}e^{i\kappa_l|z-z_0|}, \\ S_3 &= -m(\omega)(\mathbf{kn})^2 I, S_5 = m(\omega)(\mathbf{kn})^2 \frac{\partial}{\partial z_0} I, \end{aligned} \quad (29)$$

where I is given by equation (19) ($S_4 = 0$). Within the near-field approximation the above expressions become

$$\begin{aligned} S_1 &= -m(\omega)(\mathbf{kn})^2 e^{-k|z-z_0|}, S_2 = im(\omega)(\mathbf{kn})\mathbf{n}e^{-k|z-z_0|}, \\ S_3 &= m(\omega)\frac{(\mathbf{kn})^2}{2k} e^{-k|z+z_0|}, S_5 = -\frac{1}{2}m(\omega)(\mathbf{kn})^2 e^{-k|z+z_0|}. \end{aligned} \quad (30)$$

Equations (15) give the coefficients

$$\begin{aligned} A &= -\frac{m(\omega)}{c^2 k} (\mathbf{kn})^2 \frac{\eta(1-6\eta+\eta^2)}{(1-\eta)^2} e^{-k|z_0|}, \\ B &= \frac{m(\omega)}{2c^2 k^2} (\mathbf{kn})^2 e^{-k|z_0|} \end{aligned} \quad (31)$$

and equations (14) give the displacement

$$\begin{aligned} \mathbf{u} &= \frac{im(\omega)}{2c^2} \left[\frac{\mathbf{kn}}{k} \mathbf{n} e^{-k|z-z_0|} - \frac{1-10\eta+\eta^2}{4(1-\eta)} \frac{(\mathbf{kn})^2}{k^3} \mathbf{k} e^{-k|z+z_0|} \right], \\ v &= \frac{m(\omega)}{8c^2} \frac{(\mathbf{kn})^2}{k^2} \frac{1+10\eta-3\eta^2}{1-\eta} e^{-k|z+z_0|}; \end{aligned} \quad (32)$$

the Fourier transforms of the displacement are

$$\begin{aligned} \mathbf{u} &= \frac{im(t)}{2(2\pi)^2 c^2} \left(I_1 \mathbf{n} - \frac{1-10\eta+\eta^2}{4(1-\eta)} \mathbf{I}_2 \right), \\ v &= \frac{m(t)}{8(2\pi)^2 c^2} \frac{1+10\eta-3\eta^2}{1-\eta} I_3, \end{aligned} \quad (33)$$

where

$$\begin{aligned} I_1 &= \int d\mathbf{k} \frac{\mathbf{kn}}{k} e^{i\mathbf{k}\mathbf{r}} e^{-k|z-z_0|}, \\ \mathbf{I}_2 &= \int d\mathbf{k} \frac{(\mathbf{kn})^2}{k^3} \mathbf{k} e^{i\mathbf{k}\mathbf{r}} e^{-k|z+z_0|}, \\ I_3 &= \int d\mathbf{k} \frac{(\mathbf{kn})^2}{k^2} e^{i\mathbf{k}\mathbf{r}} e^{-k|z+z_0|}. \end{aligned} \quad (34)$$

The calculation of these integrals is straightforward, by means of equations (24) and some special manipulations. We get immediately

$$I_1 = -i\mathbf{n} \frac{\partial}{\partial \mathbf{r}} \frac{2\pi}{R_1} = 2\pi \frac{i\mathbf{nr}}{R_1^3}, \quad (35)$$

where $R_1 = [r^2 + (z - z_0)^2]^{1/2}$. Then, we notice that

$$\frac{\partial I_3}{\partial |z + z_0|} = i\mathbf{n} \frac{\partial}{\partial \mathbf{r}} I_1(z_0 \rightarrow -z_0) = -2\pi \left[\frac{1}{R_2^3} - \frac{3(\mathbf{nr})^2}{R_2^5} \right], \quad (36)$$

where $R_2 = [r^2 + (z + z_0)^2]^{1/2}$; the integration can be effected straightforwardly in equation (36) leading to

$$I_3 = -\pi \left[\frac{1}{R_2(R_2 + |z + z_0|)} - (\mathbf{nr})^2 \frac{2R_2 + |z + z_0|}{R_2^3(R_2 + |z + z_0|)^2} \right]; \quad (37)$$

further, we notice that

$$\frac{\partial \mathbf{I}_2}{\partial |z + z_0|} = i \frac{\partial}{\partial \mathbf{r}} I_3, \quad (38)$$

which gives access to \mathbf{I}_2 .

The displacement of the surface $z = 0$ is of particular interest. For $z = 0$ and $r \ll |z_0|$ the integrals $I_{1...3}$ behave as

$$\begin{aligned} I_1 &\simeq 2\pi \frac{i\mathbf{nr}}{|z_0|^3}, \quad I_3 \simeq -\frac{\pi}{2|z_0|^2} \left[1 - \frac{3r^2}{4z_0^2} - \frac{(\mathbf{nr})^2}{2z_0^2} \right], \\ I_2 &\simeq -\frac{i\pi}{4|z_0|^3} \left[\mathbf{r} + \frac{2}{3}(\mathbf{nr})\mathbf{n} \right]; \end{aligned} \quad (39)$$

the corresponding displacement is

$$\begin{aligned} \mathbf{u}^0 &\simeq -\frac{m(t)}{32\pi(1-\eta)c^2|z_0|^3} \left[(9 - 18\eta + \eta^2)(\mathbf{nr})\mathbf{n} + (1 - 10\eta + \eta^2)\mathbf{r} \right], \\ v^0 &\simeq -\frac{m(t)}{64\pi c^2 z_0^2} \frac{1+10\eta-3\eta^2}{1-\eta} \left[1 - \frac{3r^2}{z_0^2} - \frac{(\mathbf{nr})^2}{2z_0^2} \right]; \end{aligned} \quad (40)$$

we can see that the horizontal displacement has a maximum value $\simeq m/c^2 z_0^2$ for r of the order $|z_0|$, while the vertical displacement decreases to zero from the maximum value $\simeq m/c^2 z_0^2$ reached at the origin.

4.3. VOLUME SOURCE

The force generated by a localized volume source is

$$\mathbf{F}_{xy} = p(t) \frac{\mathbf{r}}{R_1} \theta(a - R_1), \quad F_z = p(t) \frac{z - z_0}{R_1} \theta(a - R_1) \quad (41)$$

(equation (2)), where $R_1 = [r^2 + (z - z_0)^2]^{1/2}$; its Fourier transform

$$\begin{aligned} \mathbf{F}_{xy} &= p(\omega) \int d\mathbf{r} e^{-i\mathbf{k}\mathbf{r}} \frac{\mathbf{r}}{R_1} \theta(a - R_1), \\ F_z &= p(\omega) \int d\mathbf{r} e^{-i\mathbf{k}\mathbf{r}} \frac{z - z_0}{R_1} \theta(a - R_1) \end{aligned} \quad (42)$$

can be calculated straightforwardly: we introduce $\mathbf{k} = k(\cos \alpha, \sin \alpha)$, $\mathbf{r} = r(\cos \varphi, \sin \varphi)$ and change the integration over r to an integration over R_1 ; we get

$$\begin{aligned} \mathbf{F}_{xy} &\simeq -i\pi p(\omega) \mathbf{k} \left[\frac{1}{3} a^3 + \frac{2}{3} |z - z_0|^3 - a(z - z_0)^2 \right], \\ F_z &\simeq 2\pi p(\omega) (z - z_0) [a - |z - z_0|], \end{aligned} \quad (43)$$

for $|z - z_0| < a$ and $kr \ll 1$. Within the same approximation we get from equations (8) and (10)

$$\begin{aligned} S_1 &\simeq 4\pi p(\omega) a^2 e^{i\kappa t |z - z_0|}, \quad S_3 \simeq 8\pi p(\omega) a^3 e^{i\kappa t |z - z_0|}, \\ S_2 &\simeq 0, \quad S_4 \simeq 0, \quad S_5 \simeq 0. \end{aligned} \quad (44)$$

The near-field equations (14) and (15) give

$$A \simeq 8\pi \frac{\eta^2}{1 - \eta} \frac{p(\omega) a^3}{c^2} e^{-k|z_0|}, \quad B \simeq 2\pi(1 - \eta) \frac{p(\omega) a^3}{c^2 k} e^{-k|z_0|} \quad (45)$$

and

$$\begin{aligned} \mathbf{u} &\simeq 2\pi i \frac{p(\omega) a^3}{c^2} [(1 - \eta) e^{-k|z - z_0|} + \eta e^{-k|z + z_0|}] \frac{\mathbf{k}}{k^2}, \\ v &\simeq 2\pi \frac{p(\omega) a^3}{c^2 k} e^{-k|z + z_0|}. \end{aligned} \quad (46)$$

We can see that, within this approximation, only dilatation contributes to the displacement, as expected, and, especially, the vertical dilatation; while the horizontal displacement exhibits both a direct and an image source, the vertical displacement includes only the image source. The inverse Fourier transforms of the displacement given by equations (46) lead to

$$\begin{aligned} \mathbf{u} &= i \frac{p(t) a^3}{2\pi c^2} \left[(1 - \eta) \int d\mathbf{k} \frac{\mathbf{k}}{k^2} e^{i\mathbf{k}\mathbf{r}} e^{-k|z - z_0|} + \eta \int d\mathbf{k} \frac{\mathbf{k}}{k^2} e^{i\mathbf{k}\mathbf{r}} e^{-k|z + z_0|} \right], \\ v &= \frac{p(t) a^3}{2\pi c^2} \int d\mathbf{k} \frac{e^{i\mathbf{k}\mathbf{r}}}{k} e^{-k|z + z_0|} = \frac{p(t) a^3}{2\pi c^2} \cdot \frac{1}{R_2}, \end{aligned} \quad (47)$$

where $R_2 = [r^2 + (z + z_0)^2]^{1/2}$. The integral

$$I_4 = \frac{1}{2\pi} \int d\mathbf{k} \frac{\mathbf{k}}{k^2} e^{i\mathbf{k}\mathbf{r}} e^{-k|z|} \quad (48)$$

which appears in equations (47) can be calculated either by using Bessel functions or by using the identity

$$\frac{\partial}{\partial |z|} I_4 = i \frac{\partial}{\partial \mathbf{r}} \int d\mathbf{k} \frac{1}{k} e^{i\mathbf{k}\mathbf{r}} e^{-k|z|} \quad (49)$$

and integrating over $|z|$. The result is

$$I_4 = \frac{i\mathbf{r}}{R(R + |z|)} \quad (50)$$

and

$$\mathbf{u} = -\frac{p(t)a^3}{c^2} \left[(1-\eta) \frac{1}{R_1(R_1 + |z - z_0|)} + \eta \frac{1}{R_2(R_2 + |z + z_0|)} \right] \mathbf{r}, \quad (51)$$

where $R = (r^2 + z^2)^{1/2}$. By setting $z = 0$ in \mathbf{u} and v given here we get the quasi-static surface displacement due to a volume source.

4.4. FORCE ON THE SURFACE

We give here the displacement caused by a force \mathbf{f} localized on the surface $z = 0$; in equations (13) $\mathbf{P} = \mathbf{f}\delta(\mathbf{r})$ (where \mathbf{f} is divided by density). For a slow time variation ($\omega \simeq 0$) we may use the near-field (quasi-static) approximation $\kappa_{t,l} \simeq ik$; equations (13) give the constants

$$A = \frac{2\eta}{c_t^2(1-\eta)^2} \left(\frac{i\mathbf{k}\mathbf{f}_{xy}}{k} + \eta f_z \right), \quad B = \frac{i}{c_t^2} \frac{\mathbf{k}\mathbf{f}_{xy}}{k^2} \quad (52)$$

and equations (14) give the displacements

$$\begin{aligned} \mathbf{u} &= -\frac{1}{2c_t^2(1-\eta)} \left[\frac{\mathbf{k}(\mathbf{k}\mathbf{f}_{xy})}{k^3} - i\eta f_z \frac{\mathbf{k}}{k^2} \right] e^{-k|z|}, \\ v &= \frac{1}{2c_t^2(1-\eta)} \left(\frac{3-2\eta}{2} \frac{i\mathbf{k}\mathbf{f}_{xy}}{k^2} + \eta f_z \frac{1}{k} \right) e^{-k|z|}. \end{aligned} \quad (53)$$

The inverse Fourier transforms lead to

$$\begin{aligned} \mathbf{u} &= -\frac{1}{4\pi c_t^2(1-\eta)} \left[\frac{1}{2\pi} \int d\mathbf{k} \frac{\mathbf{k}(\mathbf{k}\mathbf{f}_{xy})}{k^3} e^{i\mathbf{k}\mathbf{r}} e^{-k|z|} - \frac{i\eta}{2\pi} f_z \int d\mathbf{k} \frac{\mathbf{k}}{k^2} e^{i\mathbf{k}\mathbf{r}} e^{-k|z|} \right], \\ v &= \frac{3-2\eta}{8\pi c_t^2(1-\eta)} \frac{i}{2\pi} \int d\mathbf{k} \frac{\mathbf{k}\mathbf{f}_{xy}}{k^2} e^{i\mathbf{k}\mathbf{r}} e^{-k|z|} + \frac{\eta}{4\pi c_t^2(1-\eta)} f_z \frac{1}{2\pi} \int d\mathbf{k} \frac{e^{i\mathbf{k}\mathbf{r}}}{k} e^{-k|z|}. \end{aligned} \quad (54)$$

The last integral in the second equation (54) is $1/R$, where $R = (r^2 + z^2)^{1/2}$ (equation (24)); the last integral in the first equation (54) is I_4 given by equation (50). We

denote

$$I_5 = \frac{1}{2\pi} \int d\mathbf{k} \frac{\mathbf{k}\mathbf{f}_{xy}}{k^2} e^{i\mathbf{k}\mathbf{r}} e^{-k|z|} \quad (55)$$

and notice that

$$\frac{\partial I_5}{\partial |z|} = -\frac{1}{2\pi} I_1 = -\frac{i\mathbf{f}_{xy}\mathbf{r}}{R^3}, \quad (56)$$

where I_1 is given by equation (35); we get by integration

$$I_5 = \frac{2i\mathbf{f}_{xy}\mathbf{r}}{R(R+|z|)} \quad (57)$$

and

$$v = -\frac{1}{4\pi c_t^2(1-\eta)R} \left[(3-2\eta) \frac{\mathbf{f}_{xy}\mathbf{r}}{R+|z|} - \eta f_z \right]. \quad (58)$$

The horizontal component from equations (54) is given by

$$\mathbf{u} = -\frac{1}{4\pi c_t^2(1-\eta)} \left[\mathbf{I}_6 + \eta f_z \frac{\mathbf{r}}{R(R+|z|)} \right], \quad (59)$$

where

$$\mathbf{I}_6 = \frac{1}{2\pi} \int d\mathbf{k} \frac{\mathbf{k}(\mathbf{k}\mathbf{f}_{xy})}{k^3} e^{i\mathbf{k}\mathbf{r}} e^{-k|z|}; \quad (60)$$

since

$$\frac{\partial \mathbf{I}_6}{\partial |z|} = i \frac{\partial}{\partial \mathbf{r}} I_5 = -2 \frac{\partial}{\partial \mathbf{r}} \frac{\mathbf{f}_{xy}\mathbf{r}}{R(R+|z|)} \quad (61)$$

we may have access to \mathbf{I}_6 by integration (on the surface $\mathbf{I}_6 \simeq \mathbf{f}_{xy}/r$, $(\mathbf{f}_{xy}\mathbf{r})/r^3$). We can see that the displacement is of the order $f/c_t^2 r$.

The above near-field results are similar with the static displacement problems for an isotropic elastic half-space with an internal or an external point force, up to numerical factors in the amplitudes (the so-called Boussinesq and Mindlin problems), [23]-[34] although the near-field approximation we used is not fully controllable.

5. TRANSIENT REGIME

If the wavelengths are much shorter than the distances of interest we are in the far-field regime; it may happen in this case that the wave source ceases its activity much before the waves reach the surface of the half-space; then we are in a transient regime, where the waves equation can be treated as in an infinite body, and the boundary conditions may be neglected (they are included when the waves are reflected - and refracted - by the surface).

In an infinite body the solution of the wave equation (4) can be decomposed into “longitudinal” (l) and “transverse” (t) waves by means of the Helmholtz potentials Φ and \mathbf{A} as

$$\mathbf{u} = \mathbf{u}_l + \mathbf{u}_t, \quad \mathbf{u}_l = \text{grad}\Phi, \quad \mathbf{u}_t = \text{curl}\mathbf{A} \quad (\text{div}\mathbf{A} = 0); \quad (62)$$

similarly, the force can be decomposed as

$$\mathbf{F} = \mathbf{F}_l + \mathbf{F}_t, \mathbf{F}_l = \text{grad}\phi, \mathbf{F}_t = \text{curl}\mathbf{H} \quad (\text{div}\mathbf{H} = 0); \quad (63)$$

taking the *div* in $\mathbf{F} = \text{grad}\phi + \text{curl}\mathbf{H}$, we get

$$\Delta\phi = \text{div}\mathbf{F}; \quad (64)$$

taking the *curl* in $\mathbf{F} = \text{grad}\phi + \text{curl}\mathbf{H}$ and making use of $\text{curl} \cdot \text{curl} = -\Delta + \text{grad} \cdot \text{div}$ we get

$$\Delta\mathbf{H} = -\text{curl}\mathbf{F}; \quad (65)$$

therefore, the decomposition (63) for the force is attained by solving the two Poisson equations (64) and (65); their solutions (for vanishing boundary conditions at infinity) are

$$\phi = -\frac{1}{4\pi} \int d\mathbf{R}' \frac{1}{|\mathbf{R} - \mathbf{R}'|} \text{div}'\mathbf{F}, \mathbf{H} = \frac{1}{4\pi} \int d\mathbf{R}' \frac{1}{|\mathbf{R} - \mathbf{R}'|} \text{curl}'\mathbf{F} \quad (66)$$

Introducing \mathbf{u} given by equation (62) in the wave equation (4) we get

$$\ddot{\Phi} - c_t^2 \Delta\Phi = \phi, \ddot{\mathbf{A}} - c_t^2 \Delta\mathbf{A} = \mathbf{H} \quad (67)$$

the wave equation (4) is reduced to two standard wave equations. The u_l -waves ($\text{grad}\Phi$) are *P*-waves (compressional waves), while the u_t -waves ($\text{curl}\mathbf{A}$) are *S*-waves (shear waves).

5.1. FAULTING SOURCE

For a faulting source represented by

$$\mathbf{F}(\mathbf{R}, t) = m(t)n_i\partial_i\delta(\mathbf{R})\mathbf{n}, \quad (68)$$

where $m(t)$ is the seismic moment (divided by density) and \mathbf{n} is the direction of the fault slip, we get, from equations (66),

$$\begin{aligned} \phi &= -\frac{m(t)}{4\pi} n_i n_j \int d\mathbf{R}' \frac{\partial'_i \partial'_j \delta(\mathbf{R}')}{|\mathbf{R} - \mathbf{R}'|} = \\ &= -\frac{m(t)}{4\pi} (\mathbf{ngrad})^2 \int d\mathbf{r} \frac{\delta(\mathbf{R} - \mathbf{r})}{r} = -\frac{m(t)}{4\pi} (\mathbf{ngrad})^2 \frac{1}{R} \end{aligned} \quad (69)$$

and

$$H_i = \frac{m(t)}{4\pi} \varepsilon_{ijk} n_k n_l \partial_j \partial_l \frac{1}{R}. \quad (70)$$

Making use of these sources we get the solutions of the wave equations (67)

$$\begin{aligned}\Phi &= -\frac{m(t)}{(4\pi c_l)^2} \int d\mathbf{R}' \frac{m(t-|\mathbf{R}-\mathbf{R}'|/c_l)}{|\mathbf{R}-\mathbf{R}'|} (\mathbf{ngrad}')^2 \frac{1}{R'} = \\ &= -\frac{m(t)}{(4\pi c_l)^2} (\mathbf{ngrad})^2 \int d\mathbf{r} \frac{m(t-r/c_l)}{r} \frac{1}{|\mathbf{R}-\mathbf{r}|} = \\ &= -\frac{mT}{(4\pi c_l)^2} (\mathbf{ngrad})^2 \int dr \cdot r \delta(t-r/c_l) \int d\mathbf{r} \frac{1}{|\mathbf{R}-\mathbf{r}|}\end{aligned}\quad (71)$$

and

$$A_i = \frac{mT}{(4\pi c_l)^2} \varepsilon_{ijk} n_k n_l \partial_j \partial_l \int dr \cdot r \delta(t-r/c_l) \int d\mathbf{r} \frac{1}{|\mathbf{R}-\mathbf{r}|}, \quad (72)$$

where we assumed a time δ -impulse $m(t) = mT\delta(t)$ with duration T (much shorter than the times we measure). The angular integral can be calculated straightforwardly; we get

$$\Phi = -\frac{mTt}{4\pi} (\mathbf{ngrad})^2 \left[\frac{1}{R} \theta(R-c_l t) + \frac{1}{c_l t} \theta(c_l t - R) \right], \quad (73)$$

$$A_i = \frac{mTt}{4\pi} \varepsilon_{ijk} n_k n_l \partial_j \partial_l \left[\frac{1}{R} \theta(R-c_l t) + \frac{1}{c_l t} \theta(c_l t - R) \right].$$

The total displacement is readily obtained as

$$\begin{aligned}\mathbf{u} &= \frac{mTt}{4\pi} (\mathbf{ngrad})^2 \mathbf{grad} \cdot \\ &\cdot \left[\frac{1}{R} \theta(R-c_l t) + \frac{1}{c_l t} \theta(c_l t - R) - \frac{1}{R} \theta(R-c_l t) - \frac{1}{c_l t} \theta(c_l t - R) \right] - \\ &- \frac{mTt}{4\pi} \mathbf{n} (\mathbf{ngrad}) \Delta \left[\frac{1}{R} \theta(R-c_l t) + \frac{1}{c_l t} \theta(c_l t - R) \right].\end{aligned}\quad (74)$$

For the near-field contribution we limit ourselves here to the part continuous in time (proportional to functions θ); it is given by

$$\mathbf{u}_n = \frac{mTt}{4\pi} (\mathbf{ngrad})^2 (\mathbf{grad} \frac{1}{R}) [\theta(R-c_l t) - \theta(R-c_l t)] \quad (75)$$

(it is the double-couple result). The far-field displacement is

$$\mathbf{u}_f = \frac{mT}{4\pi} \frac{(\mathbf{nR})^2}{c_l R^4} \mathbf{R} \delta'(R-c_l t) + \frac{mT}{4\pi} \frac{\mathbf{nR}}{c_l R^2} \left[\mathbf{n} - \frac{(\mathbf{nR})\mathbf{R}}{R^2} \right] \delta'(R-c_l t). \quad (76)$$

We note that the far-field displacement given by equation (76) is the far-field displacement given by the double-couple model (note that $\delta'(R-ct) = (1/c^2)\delta'(t-R/c)$). In general, the double-couple model is obtained by replacing $mn_i n_j$ in equation (76) by the parameters m_{ij} . The order of magnitude of the far-field displacement given by the above formulae is $u_f \simeq mT/cRl^2$; for a seismic moment $M = 10^{26} \text{ dyn} \cdot \text{cm}$ (magnitude $M_w = 7$, density $\rho = 5 \text{ g/cm}^3$), velocity $c = 5 \text{ km/s}$,

distance $R = 100\text{km}$, localization length $l = 5\text{km}$ and a duration $T = 10\text{s}$ we get $u_f \simeq 1\text{m}$. The energy flux density for a scalar wave u is $\mathbf{S} = -\rho c^2 \dot{u} \text{grad} u$; for the far-field displacement given here we have $S \simeq \rho c m^2 T^2 / R^2 l^6$; with the same numerical values as above we get $S \simeq 5 \times 10^4 \text{w/cm}^2$.

5.2. VOLUME SOURCE

The force distribution for a volume source is $\mathbf{F} = p(t)(\mathbf{R}/R)\theta(a - R)$ (equation (3)); since $\text{curl} \mathbf{F} = 0$ the potential \mathbf{H} is zero, and so is \mathbf{u}_t ; we have only l -waves, given by $\ddot{\mathbf{u}}_l - c_l^2 \Delta \mathbf{u}_l = \text{grad} \phi$, where $\Delta \phi = \text{div} \mathbf{F}$; since $\Delta = \text{div} \cdot \text{grad}$ we may take $\text{grad} \phi = \mathbf{F}$, such that we have

$$\mathbf{u}_l = \frac{pT}{4\pi c_l^2} \int d\mathbf{R}' \frac{\delta((t - |\mathbf{R} - \mathbf{R}'|)/c_l)}{|\mathbf{R} - \mathbf{R}'|} \frac{\mathbf{R}'}{R'} \theta(a - R') \quad (77)$$

for a time impulse with duration T . In equation (77) we change the variable R' into $\mathbf{r} = \mathbf{R} - \mathbf{R}'$,

$$\mathbf{u}_l = \frac{pT}{4\pi c_l^2} \int d\mathbf{r} \frac{\delta((t - r/c_l))}{r} \frac{\mathbf{R} - \mathbf{r}}{|\mathbf{R} - \mathbf{r}|} \theta(a - |\mathbf{R} - \mathbf{r}|), \quad (78)$$

and notice that $\mathbf{u}_l = u_l \mathbf{R}/R$, as expected; the integral becomes

$$u_l = \frac{pT}{2c_l^2} \int dr \cdot r \delta((t - r/c_l)) \int d\theta \sin \theta \frac{R - r \cos \theta}{(R^2 + r^2 - 2Rr \cos \theta)^{1/2}} \cdot \theta [a - (R^2 + r^2 - 2Rr \cos \theta)^{1/2}]. \quad (79)$$

Introducing $u = \cos \theta$ we get the integration limits from

$$\frac{R^2 + r^2 - a^2}{2Rr} < u < \frac{R^2 + r^2}{2Rr}; \quad (80)$$

making use of the graphical representation of the functions given in equation (80) we get the integration limits $(R^2 + r^2 - a^2)/2Rr < u < 1$ for u , providing $R - a < r < R + a$. The integration with respect to u is performed by using the change of variable $R^2 + r^2 - 2Rru = x^2$; we get

$$\begin{aligned} u_l &= \frac{pT}{4c_l^2 R^2} \int_{R-a}^{R+a} dr \delta(t - r/c_l) \cdot = \\ &\cdot \left\{ (R^2 - r^2 + \frac{1}{3}a^2)a - [R^2 - r^2 + \frac{1}{3}(R-r)^2] |R-r| \right\} = \\ &= \frac{pT}{4c_l R^2} \left[\frac{1}{3}a^3 + (R^2 - c_l^2 t^2)a - (R^2 - c_l^2 t^2) |R - c_l t| - \frac{1}{3} |R - c_l t|^3 \right] \end{aligned} \quad (81)$$

for $R - a < c_l t < R + a$. This displacement has the form of a shock (similar with a function $\delta'(t - R/c_l)$), which, in the (far-field) limit $R, c_l t \gg a$ has two extrema $\simeq \pm \frac{1}{2}(pT/4c_l)c_l t a^2 / R^2$ at $R = c_l t \pm \frac{a}{2}$.

6. WAVES REGION

If the duration T of a seismic source is very short, then it may be approximated by a time δ -impulse and the frequency content extends from zero to large values $\Delta\omega = 2\pi/T$; in this case, the wavelengths extend from short values $2\pi c_{t,l}/\Delta\omega = c_{t,l}/T$ to infinite; for instance, if $T = 1s$ we have wavelengths of the order $5km/s \cdot 1s = 5km$ (for an average velocity $5km/s$). These wavelengths are shorter than our distances of interest, and, for a good portion of the spectrum, we have waves which propagate in the body; we are in the wave region (zone), or far-field approximation.

The elastic waves include dilatation waves, described by the divergence of the displacement $D = \text{div}\mathbf{u}$, governed by the source term S_1 in equations (7) and (8), propagating with velocity c_l ; and displacement waves, described by \mathbf{u} and v , generated both by the localized sources S_2 and S_4 in equations (9) and (10), and by delocalized sources S_3 and S_5 , which involve the derivatives of the dilatation waves; the presence of these delocalized sources complicates appreciably the solution of the elastic waves. The delocalized sources act as if there is an ‘‘interaction’’ between dilatation waves propagating with velocity c_l and waves propagating with c_t , c_l -waves being sources for c_t -waves; a particularly complicate contribution to the solution can be seen in the integral I in equation (19).

We can see that the major complication in the solution of the elastic waves arises from the presence of two velocities $c_{t,l}$; a crude approximation would be to put $c_t = c_l$ in equations (5), which would transform them in standard wave equations; unfortunately, such an approximation fails to satisfy the boundary conditions. We adopt here a different approximation, based on the fact that the two wavevectors $\kappa_{t,l}$ are close to each other for small values of the in-plane wavevector \mathbf{k} . Indeed,

$$\Delta\kappa = \kappa_t - \kappa_l = \frac{\omega^2}{c_t^2} \frac{1-\eta}{\kappa_t + \kappa_l} \ll \frac{1}{2}(\kappa_t + \kappa_l) = \kappa \quad (82)$$

for $k^2 \ll \frac{\omega^2}{2c^2}(1+\eta)$; for large values of ω/c there is still enough room for the variation of the wavevector \mathbf{k} . This amounts to leave aside the quasi-static contributions. Consequently, we set $\kappa_l = \kappa_t = \kappa$, $c_t = c$ and preserve $\eta \neq 1$ in equations (9) and (13), which become

$$\begin{aligned} \mathbf{u} &\simeq \frac{i}{2c^2\kappa} \mathbf{S}_2 - \frac{i(1-\eta)}{4c^2\kappa^2} \mathbf{k} S_3 - \frac{i(1-\eta)}{4\eta\kappa^2} \mathbf{k} A e^{i\kappa|z|}, \\ v &\simeq \frac{i}{2c^2\kappa} S_4 - \frac{1-\eta}{4c^2\kappa^2} S_5 + \left[\frac{i(1-\eta)}{4\eta\kappa} A + B \right] e^{i\kappa|z|} \end{aligned} \quad (83)$$

and

$$\begin{aligned}
 ik^2 B &\simeq -\frac{i}{2c^2\kappa} \mathbf{k} \mathbf{S}'_2(0) + \frac{i(1-\eta)}{4c^2\kappa^2} k^2 S'_3(0) + \\
 &\quad + \frac{k^2}{2c^2\kappa} S_4(0) + \frac{i(1-\eta)}{4c^2\kappa^2} k^2 S_5(0), \\
 -\frac{1-\eta}{4\eta} A - i\kappa B &\simeq -\frac{i}{2c^2\kappa} S'_4(0) + \frac{1-\eta}{4c^2\kappa^2} S'_5(0) + \\
 &\quad + \frac{1-2\eta}{2c^2\kappa} \mathbf{k} \mathbf{S}_2(0) - \frac{(1-\eta)(1-2\eta)}{4c^2\kappa^2} k^2 S_3(0);
 \end{aligned} \tag{84}$$

$\Delta\kappa$ is neglected in these equations in comparison with κ , according to equation (82). The approximation described here gives only one type of wave ($\kappa_t = \kappa_l = \kappa$) and affects the numerical coefficients. It is worth noting that the approximation $\kappa_l = \kappa_t = \kappa$ while preserving $\eta \neq 1$ makes difficult to assess the validity of the results.

6.1. WAVES GENERATED BY A VERTICAL FAULTING SOURCE

The source terms for a vertical fault are taken from equations (16) and (17); within the wave-zone approximation the integral I from equation (19) reduces to

$$I \simeq -\frac{i}{2\kappa} e^{i\kappa|z+z_0|} \tag{85}$$

and the coefficients A and B given by equations (84) are

$$A \simeq \frac{im(\omega)}{c^2} \frac{\eta(5+3\eta)}{2(1-\eta)} \kappa e^{i\kappa|z_0|}, \quad B \simeq -\frac{m(\omega)}{c^2} \frac{1-2\eta}{2} e^{i\kappa|z_0|}; \tag{86}$$

equations (83) lead immediately to

$$\mathbf{u} \simeq \frac{m(\omega)(1-\eta)}{2c^2} \frac{\mathbf{k}}{\kappa} e^{i\kappa|z-z_0|} + \frac{m(\omega)(3+\eta)}{4c^2} \frac{\mathbf{k}}{\kappa} e^{i\kappa|z+z_0|}, \tag{87}$$

$$v \simeq -\frac{m(\omega)\eta}{2c^2} \operatorname{sgn}(z-z_0) e^{i\kappa|z-z_0|} - \frac{m(\omega)(2-\eta)}{2c^2} e^{i\kappa|z+z_0|}.$$

The inverse Fourier transforms of these functions make use of the integral

$$I_0 = \frac{i}{2\pi} \int d\mathbf{k} \frac{e^{i\mathbf{k}\mathbf{r}}}{\kappa} e^{i\kappa|z|} = \frac{e^{i\frac{\omega}{c}R}}{R}, \tag{88}$$

where $R = (r^2 + z^2)^{1/2}$. We assume a time δ -impulse $m(t) = mT\delta(t)$ ($m(\omega) = mT$) and get

$$\begin{aligned}
 \mathbf{u} &\simeq \frac{mT(1-\eta)}{4\pi c^3} \frac{\mathbf{r}}{R_1^2} \delta'(t - R_1/c) + \frac{mT(3+\eta)}{8\pi c^3} \frac{\mathbf{r}}{R_2^2} \delta'(t - R_2/c), \\
 v &\simeq -\frac{mT\eta}{4\pi c^3} \frac{z-z_0}{R_1^2} \delta'(t - R_1/c) - \frac{mT(2-\eta)}{4\pi c^3} \frac{|z+z_0|}{R_2^2} \delta'(t - R_2/c)
 \end{aligned} \tag{89}$$

where $R_{1,2} = [r^2 + (z \mp z_0)^2]^{1/2}$ (in these equations we omit the derivatives of $1/R_{1,2}$ and keep only the derivatives of the δ -functions, in accordance with our wave-zone assumption $R_{1,2} \gg c/T$).

We can see that the displacement consists of the derivative of two spherical waves, one generated by the direct source, another generated by the image source, with magnitude of the order $(mT/cl^2)\mathbf{R}/R^2$, where l is the localization length of the δ -function; on the surface $z = 0$ these waves are

$$\mathbf{u}^0 \simeq \frac{mT}{8\pi c^3} (5 - \eta) \frac{\mathbf{r}}{R_0^2} \delta'(t - R_0/c), \quad v^0 \simeq -\frac{mT}{2\pi c^3} \frac{|z_0|}{R_0^2} \delta'(t - R_0/c), \quad (90)$$

where $R_0 = (r^2 + z_0^2)^{1/2}$. It is also worth noting that the horizontal displacement has a maximum value for distances r of the order $|z_0|$. For a seismic moment $M = 10^{26} \text{ dyn} \cdot \text{cm}$ (corresponding to magnitude 7), density 5 g/cm^3 , an average velocity $c = 5 \text{ km/s}$, duration $T = 10 \text{ s}$, depth $|z_0| = 100 \text{ km}$ and $l = 5 \text{ km}$ (localization of the δ -function) we get a displacement of the order 1 m . The occurrence of the derivatives of the δ -function in the displacement is a consequence of the presence of the surface.

We note that the wave component including $|z - z_0|$ in the exponent in equations (87) can be viewed as a wave incident on the surface $z = 0$, while the wave component with $|z + z_0|$ in the exponent is a reflected wave; however, for $z < z_0$ the $|z - z_0|$ -wave propagates in the opposite direction, as a reflected wave. Actually, both waves are spherical waves (or spatial derivatives of the spherical waves), and a separation in ‘‘longitudinal’’ and ‘‘transverse’’ waves has only a limited character and is not convenient. This is a general situation, caused both by the presence of the surface at $z = 0$ and the directional character of the wave source.

6.2. WAVES GENERATED BY A HORIZONTAL FAULTING SOURCE

For a horizontal faulting source we use the source terms given by equations (27); the coefficients A and B from equations (84) are

$$A \simeq -\frac{2im(\omega)}{c^2} \frac{\eta}{1 - \eta} \frac{(\mathbf{kn})^2}{k^2} \kappa e^{i\kappa|z_0|}, \quad B \simeq \frac{m(\omega)}{2c^2} \frac{(\mathbf{kn})^2}{k^2} \kappa e^{i\kappa|z_0|}, \quad (91)$$

where we keep only the leading contributions, in accordance with our assumption of small $\Delta\kappa$ (equation (82)). From equations (83) we get the displacement

$$\begin{aligned} \mathbf{u} &\simeq -\frac{m(\omega)}{4\pi c^2} \mathbf{n} \cdot \frac{1}{2\pi} \int d\mathbf{k} e^{i\mathbf{k}\mathbf{r}} \frac{\mathbf{kn}}{\kappa} e^{i\kappa|z-z_0|} - \frac{m(\omega)}{4\pi c^2} \cdot \frac{1}{2\pi} \int d\mathbf{k} e^{i\mathbf{k}\mathbf{r}} \frac{(\mathbf{kn})^2}{k^2} \frac{\mathbf{k}}{\kappa} e^{i\kappa|z+z_0|}, \\ v &\simeq \frac{m(\omega)}{4\pi c^2} \cdot \frac{1}{2\pi} \int d\mathbf{k} e^{i\mathbf{k}\mathbf{r}} \frac{(\mathbf{kn})^2}{k^2} e^{i\kappa|z+z_0|}, \end{aligned} \quad (92)$$

or

$$\begin{aligned} \mathbf{u} &\simeq -\frac{m(\omega)}{4\pi c^2} \mathbf{n} \left(-i \mathbf{n} \frac{\partial}{\partial \mathbf{r}}\right) \frac{1}{2\pi} \int d\mathbf{k} \frac{e^{i\mathbf{k}\mathbf{r}}}{\kappa} e^{i\kappa|z-z_0|} - \\ &\quad -\frac{m(\omega)}{4\pi c^2} \left(-i \frac{\partial}{\partial \mathbf{r}}\right) \frac{1}{2\pi} \int d\mathbf{k} \frac{e^{i\mathbf{k}\mathbf{r}}}{\kappa} \frac{(\mathbf{k}\mathbf{n})^2}{k^2} e^{i\kappa|z+z_0|}, \quad (93) \\ v &\simeq \frac{m(\omega)}{4\pi c^2} \left(-i \frac{\partial}{\partial |z+z_0|}\right) \frac{1}{2\pi} \int d\mathbf{k} \frac{e^{i\mathbf{k}\mathbf{r}}}{\kappa} \frac{(\mathbf{k}\mathbf{n})^2}{k^2} e^{i\kappa|z+z_0|}. \end{aligned}$$

The first integral in the expression of \mathbf{u} is I_0 given by equation (88) for $z \rightarrow z - z_0$. The second integral occurring in equations (93) can be calculated by using Bessel functions and an angular parametrization of the vectors $\mathbf{k} = k(\cos \varphi, \sin \varphi)$, $\mathbf{r} = r(\cos \alpha, \sin \alpha)$ and $\mathbf{n} = (\cos \beta, \sin \beta)$; we get

$$\begin{aligned} I_5 &= \frac{1}{2\pi} \int d\mathbf{k} \frac{e^{i\mathbf{k}\mathbf{r}}}{\kappa} \frac{(\mathbf{k}\mathbf{n})^2}{k^2} e^{i\kappa|z|} = \\ &= I_0 \sin^2(\alpha - \beta) + \cos 2(\alpha - \beta) \int dk k \frac{\partial}{\partial (kr)} J_1(kr) \frac{1}{\kappa} e^{i\kappa|z|}, \quad (94) \end{aligned}$$

where J_1 is the Bessel function of the first kind and the first order. It is easy to see, by integrating by parts and using the approximation of small $\Delta\kappa$, that the integral including the Bessel function J_1 in equation (94) may be neglected. We get

$$I_5 = i \left[1 - \frac{(\mathbf{r}\mathbf{n})^2}{r^2} \right] \frac{e^{i\frac{\omega}{c}R}}{R}. \quad (95)$$

and

$$\begin{aligned} \mathbf{u} &\simeq -\frac{mT}{4\pi c^3} \mathbf{n} \frac{\mathbf{r}}{R_1^2} \delta'(t - R_1/c) + \frac{mT}{4\pi c^3} \left[1 - \frac{(\mathbf{r}\mathbf{n})^2}{r^2} \right] \frac{\mathbf{r}}{R_2^2} \delta'(t - R_2/c), \\ v &\simeq -\frac{mT}{4\pi c^3} \left[1 - \frac{(\mathbf{r}\mathbf{n})^2}{r^2} \right] \frac{|z+z_0|}{R_2^2} \delta'(t - R_2/c). \quad (96) \end{aligned}$$

We can see that the displacement caused by a horizontal faulting source consists of the derivatives of two spherical waves, one coming from the direct source (except for the vertical component of the displacement), the other coming from the image source, with a particular angular dependence generated by the directional character of the horizontal faulting source.

6.3. WAVES GENERATED BY VOLUME SOURCES

A volume source is given by equation (41). Within our approximation of small $\Delta\kappa$, from equations (83) and (84) we get

$$A \simeq \frac{8\pi p(\omega)a^3}{c^2} \eta e^{i\kappa|z_0|}, \quad B \simeq \frac{2\pi ip(\omega)a^3}{c^2} (1 - \eta) \frac{1}{\kappa} e^{i\kappa|z_0|} \quad (97)$$

and

$$\mathbf{u} \simeq -\frac{2\pi ip(\omega)a^3}{c^2}(1-\eta)\frac{\mathbf{k}}{\kappa^2} [e^{i\kappa|z-z_0|} + e^{i\kappa|z+z_0|}], \quad (98)$$

$$v \simeq \frac{4\pi ip(\omega)a^3}{c^2}(1-\eta)\frac{1}{\kappa} e^{i\kappa|z+z_0|}.$$

For a time δ -impulse $p(t) = pT\delta(t)$ the inverse Fourier transform of the vertical component can readily be obtained as a spherical wave

$$v \simeq \frac{2pTa^3}{c^2}(1-\eta)\frac{\delta(t-R_2/c)}{R_2} \quad (99)$$

generated by the image source, where $R_2 = [r^2 + (z+z_0)^2]^{1/2}$. For the horizontal component, apart from using $-i\frac{\partial}{\partial \mathbf{r}}$ to account for \mathbf{k} , we insert also $\int d|z \mp z_0| \frac{\partial}{\partial |z \mp z_0|}$ in the inverse Fourier transform; we get two-dimensional waves

$$\mathbf{u} \simeq -\frac{pTa^3}{c}(1-\eta) \left[\frac{\mathbf{r}_-}{(c^2t^2 - r_-^2)^{3/2}} + \frac{\mathbf{r}_+}{(c^2t^2 - r_+^2)^{3/2}} \right], \quad (100)$$

where $r_{\pm} = [c^2t^2 - (z \pm z_0)^2]^{1/2}$ for $r_{\pm} < ct$ and zero otherwise. On the surface $z = 0$ the horizontal component of the displacement is

$$\mathbf{u}^0 \simeq -\frac{2pTa^3}{c}(1-\eta)\frac{\mathbf{r}}{(c^2t^2 - r^2)^{3/2}}; \quad (101)$$

such a two-dimensional wave (or rather a conical wave), which is non-vanishing behind the wavefront (in contrast with the spherical waves, which are non-vanishing only on a spherical surface) distinguishes the volume sources from the faulting sources. It is worth noting that the wavefront singularity exhibited by equation (101) for $r = ct$ is in fact smoothed out by the uncertainty a in the linear dimensions of the ‘‘point’’ source, so that the denominator $(c^2t^2 - r^2)^{3/2}$ should be read there $(ra)^{3/2}$.

We note that the generalization to the seismic moment m_{ij} is limited for the treatment of the elastic waves given above, because of the separation in horizontal and vertical components of the solution.

7. CONCLUDING REMARKS

The double-couple seismic source is reformulated here without resorting to the mechanical torque representation; a volume source is also put forward, based on the pressure exerted on the internal surface of a small spherical cavity. The elastic waves equations and the boundary conditions are conveniently written down for an isotropic elastic half-space by using in-plane (horizontal) Fourier transformations; it is shown that the dilatational waves act a sources for ‘‘transverse’’ waves. The problem of the elastic waves in a half-space is thereby reduced to quadratures, at

least in principle. Near-field (quasi-static) approximations are solved explicitly for the sources introduced here, including the displacement caused by a localized force acting upon the surface. The transient regime of far-field waves is also solved for both types of sources. The far-field approximation (waves region) is also discussed.

Acknowledgements. The author is indebted to the members of the Institute for Earth's Physics at Magurele-Bucharest for useful discussions. This work has been supported partially by Romanian Government Research Grant PN 16 35 01 07/11.03.2016.

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