DYNAMIC CONFORMAL SPHERICALLY SYMMETRIC SOLUTIONS IN AN ACCELERATED BACKGROUND

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Abstract. We consider dynamical spherically symmetric spacetimes, which are conformal to the static spherically symmetric metrics, and find new solutions of Einstein equations by symmetry considerations. Our study help us classify various conformal Black Holes that are embedded within a dynamic background into the one class of solutions with the same conformal symmetry. In addition, Thermodynamics, mathematical and gravitational properties are addressed. These solutions point to have a better resolution of the meaning of the Black Holes in the dynamic background.

Key words: Conformal transformation; perfect fluid; dynamical black holes; black hole thermodynamics.

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1. INTRODUCTION

Exact solutions of the Einstein equations attract more investigations because they allow a global perception [1]. One important issue in the relativistic astrophysics is the evolution of the stars, which needs the interior solutions of Einstein equations and help us find useful signals of the life of the cosmos. Supernova explosions as one possible form of final stage of stellar evolution is an example. Although a star is a dynamical system, because of stellar tangible long time evolution, one can use static solutions for studying the system so, the spherically symmetric static metrics have a particular place in the subject. Some of this significance is due to the symmetry considerations in the stellar formation and evolution. One can write general form of the static spherically symmetric metrics as:

$$ds^2 = -e^\nu d\bar{t}^2 + e^\lambda dr^2 + e^\mu d\Omega^2, \tag{1}$$

where ν, λ and μ are functions of radius (r), only [2–5].

General static solution for isotropic fluid spheres, charged perfect fluid version and certain types of dynamic metrics are presented in the literature [6–8]. Diaz and

Pullin have found solutions for spheres with slow rotation [9]. In the collapsing procedure, density increases and there are various physical phenomena which induce anisotropy [10, 11], Rago has generalized solutions to anisotropic static fluids [12]. The radii at which $e^{\nu(r)} = 0$, points to horizons [5] which obey some special laws. Nowadays these laws, which originally were claimed for static Black Holes (BHs) and were originally proposed by Bekenstein and Hawking [13–15], are recognized as backbone of thermodynamical properties of gravity [16–20]. Exterior solutions of Einstein field equations point us to the effects of the material content of the universe on the background, which is now accepted as an accelerating spacetime. Considering cosmological principle [21], background spacetime can be expressed by conformal form of the so called Friedmann-Robertson-Walker (FRW) model [22]

$$ds^2 = a(\eta)^2[d\eta^2 + dr^2 + r^2 d\Omega^2], \tag{2}$$

where $a(\eta)$ is scale factor in the conformal time (η) notation. The conformal form of the FRW metric enables one to incorporate spatial inhomogeneity via the conformal factor [23].

Considering the above arguments, it is now clear that finding a general form of non static spherically symmetric metrics is desirable and in fact it has attracted some interest. There are four independent approaches for the task. In the first approach, by focusing on symmetry considerations which help us to simplify the Einstein equations, some authors try to find solutions to the Einstein field for static and non-static fluids [24–28]. More solutions including isotropic and anisotropic fluids can be found in the references [29–35].

In the second approach, in order to find the effects of the cosmic expansion on the collapsing systems (specially, spherically symmetric systems) work started by Einstein *et al*. Authors tried to connect Schwarzschild solution to FRW on the boundary by satisfying junction conditions and now, this solution is classified as a more general model named Swiss Cheese model [36–40] attracted more investigations to itself [41–43]. Finally, we should note that the Swiss Cheese models can be classified as a subclass of inhomogeneous Lemaître-Tolman-Bondi models [44, 45].

In the third approach, some authors have embedded spherically symmetric solutions into the FRW background and argued about their surprising corollaries [45–48]. These solutions include the Schwarzschild and Reissner-Nordstöm BHs in various coordinate systems which lay into the FRW spacetime by a conformal factor, which is compatible with the cosmic expansion eras. Since these solutions are conformal to the Schwarzschild and Reissner-Nordstöm spacetimes, their corresponding causal structure remain the same [49]. For these conformal spacetimes, redshift singularities point to the expanding null hypersurfaces which have non-zero confined surface area and cover the BH curvature singularity ($r = 0$) [50]. These hypersurfaces change the causal structure of the metric the same way as in the primary static

metric. Among various conformal spacetimes, the curvature scalars do not diverge at redshift singularity only for Sultana and Dyer solution [47]. In addition, the energy conditions is problematic in their solution [45]. From what we have said and the fact that these objects are the conformal transformation of static BHs, it is accepted that some conformal models, such as Thakurta spacetime and solutions by McClure and Dyer, include dynamical BHs [45]. Also, only conformally Schwarzschild solution (Thakurta spacetime) and solution by McClure *et al.* satisfy the energy conditions [45, 50]. In continue, conformally Schwarzschild and Reissner-Nordstöm solutions can be thought as a special group of metrics that have a Ricci scalar that is conformal with the Ricci scalar of the FRW and may include various horizons with different temperatures [50].

In addition, some authors have tried to find dynamical BHs by using the isotropic shape of the FRW metric along as perfect fluid concept [51, 52]. For these solutions, mass and charge will decrease as the functions of the universe expansion [52]. Also, there is a hypersurface that acts like as an event horizon which collapses while the background expands and its radius depends on the curvature of the background. In addition, the curvature scalars diverge on that. Also unlike the Swiss Cheese models, energy conditions are violated by these solutions [45]. These features look unsatisfactory parts of the fourth attempts. Therefore, these solutions do not contain dynamic BHs [45, 53–56]. Considering the prefect fluid concept as well as the dynamic background (2), one can get some solutions which include constant mass, charge and cosmological constant [57]. In addition, the redshift singularity is independent of the background curvature which is in agreement with the FRW background, and points to the horizon-like hypersurfaces [57]. More studies in which the prefect fluid concept is used to derive dynamic spherically symmetric solutions can be found in [58, 59].

In this article, by following symmetry considerations, we want to derive the various possible solutions of non static spherically symmetric metrics. Throughout this paper we set the Einstein gravitational constant (k) to one for simplicity ($k \equiv 8\pi G$=1). The paper is organized as follows: in the next section, by considering a general conformal killing vector and a non static spherically symmetric metric, where metric dynamics comes from a conformal factor which is only a function of time, we try to find new solutions of the Einstein equations and we study their physical and mathematical properties. Sections (3) includes solutions with the BHs that merge into the dynamic background and thermodynamics of these solutions. The last section is devoted to a summary and concluding remarks.

2. CONFORMAL SPHERICALLY SYMMETRIC SPACETIMES

We begin by conformal form of (1), where the conformal factor has only time dependency:

$$ds^2 = a(\eta)^2[-e^\nu d\eta^2 + e^\lambda dr^2 + e^\mu d\Omega^2], \tag{3}$$

where ν, λ and μ have only r dependency, $d\Omega^2 = d\theta^2 + \sin(\theta)^2 d\phi^2$ is the ordinary line element on the unit 2-sphere and η is called conformal time. We define cosmic time t as usual:

$$\eta \to t = \int a(\eta) d\eta. \tag{4}$$

Using the cosmic time coordinate, we obtain

$$ds^2 = -e^\nu dt^2 + a(t)^2[e^\lambda dr^2 + e^\mu d\Omega^2]. \tag{5}$$

Since $n_\alpha = \delta_\alpha^r$ is the normal to the hypersurface $r = const$, we have

$$n_\alpha n^\alpha = g^{rr} = \frac{e^{-\lambda}}{a(t)^2}, \tag{6}$$

which is timelike when $e^{-\lambda} < 0$, null for $e^{-\lambda} = 0$ and spacelike if we have $e^{-\lambda} > 0$. Therefore, it is apparent that the existence of the null horizons is independent of the functional time dependence of a non-zero scale factor $(a(t))$. Indeed, the causal structure of metric (1) is invariant under the conformal transformation [49]. For a co-moving observer, redshift of a radial incoming wave at the point (t, r) when it has been sent from (r_0, t_0) is evaluated as:

$$1 + z = \frac{\lambda(r,t)}{\lambda(r_0,t_0)} = \frac{a(t)}{a(t_0)}\left(\frac{e^{\nu(r)}}{e^{\nu(r_0)}}\right)^{\frac{1}{2}}. \tag{7}$$

It is seen that the redshift arises from two factors, one due the cosmic expansion and one due to the local inhomogeneity. One also obtains the following relation for the Ricci scalar:

$$R = \frac{R_{FRW}}{e^\nu} + \frac{R_1}{2a(t)^2}, \tag{8}$$

where, we have

$$R_{FRW} = 6\frac{a(t)\ddot{a}(t) + \dot{a}^2(t)}{a(t)^2}, \tag{9}$$

and

$$R_1 = e^{-\lambda}[2\mu'(\lambda' - \nu') + \nu'(\lambda' - \nu') - 3\mu'^2 - 2\nu'' - 4\mu''] + 4e^{-\mu}. \tag{10}$$

When $(\dot{})$ and $(')$ are derivatives related to t and r, respectively. For $\nu = \lambda = 0$ and $\mu = lnr^2$, we get $R_1 = 0$ and $R_s = R_{FRW}$. It is obvious that, solutions with $R_1 = 0$

have Ricci scalar proportional to FRW's. By defining physical radius ζ as

$$\zeta \equiv a(t)r \tag{11}$$

and introducing apparent horizon as a trapping surface with null tangent from [60], for the apparent horizon radius and its surface gravity we get

$$\partial_\alpha \zeta \partial^\alpha \zeta = 0 \to r_H \tag{12}$$

and

$$\kappa = \frac{1}{2\sqrt{-h}} \partial_a (\sqrt{-h} h^{ab} \partial_b \zeta). \tag{13}$$

Here $h_{ab} = diag(-e^\nu, a^2(t)e^\lambda)$ is induced metric on the two dimensional hypersurface with $d\theta = d\phi = 0$ and temperature on this surface is $T = \frac{\kappa}{4\pi}$. For this spherically symmetric spacetime, the confined Misner-Sharp mass inside radius ζ is [61]:

$$M = \frac{\zeta}{2}(1 - h^{ab} \partial_a \zeta \partial_b \zeta). \tag{14}$$

When the apparent horizon is concerned ($h^{ab}\partial_a\zeta\partial_b\zeta = 0$), this relation reduces to $M = \frac{\zeta}{2}$ which is equal to $M = \rho V$ for the FRW universe [18, 19]. The Einstein equations ($G_{\alpha\beta} = T_{\alpha\beta}$) leads us to an anisotropic fluid ($P_r \neq P_T$), which supports this spacetime. We define the anisotropy function δ as:

$$\delta \equiv a(t)^2(P_r - P_T) = \frac{e^{-\lambda}}{4}[\nu'(\mu' + \lambda' - \nu') + \lambda'\mu' - 2\nu'' - 2\mu'' - \frac{4e^\lambda}{e^\mu}], \tag{15}$$

and therefore, condition $\delta = 0$ yields the isotropic solutions. The only off diagonal elements of the Einstein tensor are

$$G_{rt} = G_{tr} = H(t)\nu'. \tag{16}$$

In the above equation $H(t) \equiv \frac{\dot{a}(t)}{a(t)}$. As previously mentioned, the FRW results can be reproduced by choosing $\nu = \lambda = 0$ and $\mu = 2lnr$. For every observer with four velocity U^α, the stress-energy tensor can be decomposed as [62]

$$T_{\mu\nu} = \rho U_\mu U_\nu + Ph_{\mu\nu} + \Pi_{\mu\nu} + q_\mu U_\nu + q_\nu U_\mu. \tag{17}$$

In this equation, $\rho = T_{\mu\nu}U^\mu U^\nu$ plays the role of the energy density and $h_{\mu\nu} \equiv g_{\mu\nu} + U_\mu U_\nu$ is a projection tensor. In addition, $P = \frac{1}{3}h_{\mu\nu}U^\mu U^\nu$ and $\Pi_{\mu\nu} = T_{\alpha\beta}h^\alpha_{(\mu}h^\beta_{\nu)}$ are the isotropic pressure and the traceless stress tensor respectively. q^μ is the energy flux (the momentum density) relative to U^μ when $q^\mu < 0$ ($q^\mu > 0$) signifies the input (output) flow [62, 63]. For a co-moving observer ($U^\mu = |-g^{00}|^{1/2}\delta^\mu_t$), using the Einstein equations, We get

$$T_{\mu\nu} = \rho U_\mu U_\nu + Ph_{\mu\nu} + q_\mu U_\nu + q_\nu U_\mu. \tag{18}$$

where $P = \frac{1}{3}(G_1^1 + 2G_2^2)$ and

$$q^\mu = -\frac{e^{-\lambda}}{a(t)^2 e^{\frac{\nu}{2}}} H(t)\nu' \delta_r^\mu. \tag{19}$$

Therefore, in order to have an isotropic solution G_1^1 and G_2^2 must meet the $\delta = 0$ condition (Eq. 15) which leads to $P = P_r = P_T = G_1^1 = G_2^2$.

A four vector ξ which satisfies

$$\mathcal{L}_\xi g_{\gamma\delta} = 2\psi g_{\gamma\delta}, \tag{20}$$

is said to be a conformal killing vector. In the above equation, ψ is called the conformal factor and for the killing vectors it takes zero value [64]. Consider the metric (5), and a Killing vector in the form

$$\xi = \alpha(r,t)\frac{\partial}{\partial t} + \beta(r,t)\frac{\partial}{\partial r}. \tag{21}$$

For the conformal killing vector ξ using (20), we get

$$\psi = \frac{\nu'}{2}\beta + \dot\alpha \tag{22}$$

$$\psi = \frac{\dot a}{a}\alpha + \frac{\lambda'}{2}\beta + \beta'$$

$$\psi = \frac{\dot a}{a}\alpha + \frac{\mu'}{2}\beta$$

$$0 = \alpha' e^\nu - a^2 e^\lambda \dot\beta.$$

From the second and third equations of the set (22) one reaches

$$\beta(r,t) = f(t)e^{\frac{\mu-\lambda}{2}}, \tag{23}$$

Therefore, for a given metric, one can get ξ by considering the conformal factor (ψ) and *vice versa*. In the following subsections we derive three new classes of solutions.

2.1. KILLING VECTORS SOLUTIONS $\psi = 0$

Inserting equation (23) into the second and the first equations of (22) we find:

$$\alpha = -f(t)\frac{a\mu'}{2\dot a}e^{\frac{(\mu-\lambda)}{2}} \tag{24}$$

$$\dot\alpha = -f(t)\frac{\nu'}{2}e^{\frac{(\mu-\lambda)}{2}}$$

respectively. We differentiate with respect to t from the first equation of (24) and comparing the result with the second equation of (24). We get

$$f(t) = c_1\dot a \tag{25}$$

$$\mu = \nu + c$$

and

$$
\begin{aligned}
f(t)(2-q) &= \frac{\dot{f}}{H} \\
-\nu + c &= \mu,
\end{aligned} \tag{26}
$$

where $q \equiv -\frac{a\ddot{a}}{\dot{a}^2}$, c_1 and c are arbitrary constants. Consider conditions (25), after taking the derivative of α with respect to r from the first equation of (24) and comparing the result with the fourth equation of (22), we find

$$
e^{\lambda - \nu} = \mp \frac{1}{2}(\nu'' + \frac{\nu'}{2}(\nu' - \lambda')) \tag{27}
$$

and

$$
df = \pm \frac{c_1 dt}{a(t)}. \tag{28}
$$

Using equation (4), we find:

$$
df = \pm c_1 d\eta \rightarrow f(t) = \pm c_1 \eta(t) + c_2. \tag{29}
$$

Substituting the first set of (25) into (29), one gets;

$$
a(t) = \pm \int \eta(t) dt + \frac{c_2}{c_1} t + c_3. \tag{30}
$$

Let us substitute f from (25) into (28) to obtain

$$
a\ddot{a} = \pm 1, \tag{31}
$$

which yields

$$
\frac{1}{2}\dot{a}^2 = \pm \ln a + C. \tag{32}
$$

In conclusion we find that, ξ is a killing vector of metric (5), when $\mu = \nu + c$, $a(t)$ obeys (31) and the relation between λ and ν comes from (27). Redshift diverges at r_0 if $e^{\nu(r_0)} \rightarrow 0$. Hypersurface located at $r = r_0$ can be timelike, null or even spacelike. It depends on the value of $\frac{e^{-\lambda(r_0)}}{a^2(t)}$. In this radius, all of the curvature scalars diverge and surface area is:

$$
A = \int \sqrt{e^{2\nu(r_0)} e^{2c} a(t)^4 \sin^2(\theta)} d\theta d\phi = 0. \tag{33}
$$

Therefore, this should be a naked singularity. Now we consider (26) and following the above recipe to obtain

$$
\begin{aligned}
e^{\lambda - \nu} &= \mp \frac{1}{2}(-\nu'' + \frac{\nu'}{2}(\nu' + \lambda')) \\
a\ddot{a} + 2\dot{a}^2 &= \pm 1,
\end{aligned} \tag{34}
$$

where the second equation yields

$$\pm 1 - \frac{1}{Ca^4} = 2\dot{a}^2. \tag{35}$$

One can write (35) in the form of

$$\frac{da}{\sqrt{\pm \frac{1}{2} - \frac{1}{2Ca^4}}} = \pm dt, \tag{36}$$

which leads to

$$\sqrt{1 \mp a^2} \frac{F(x\sqrt{-\sqrt{\pm 1}}, I) - E(x\sqrt{-\sqrt{\pm 1}}, I)}{\sqrt{\frac{\pm a^4 - 1}{2a^4}} a^2 \sqrt{\mp 1}} = \pm t. \tag{37}$$

In the above equation, $F(x\sqrt{-\sqrt{\pm 1}}, I)$ and $E(x\sqrt{-\sqrt{\pm 1}}, I)$ are incomplete elliptic integrals of the first and the second kind, respectively. Eventually, ξ is a killing vector of metric (5), when $\mu = -\nu + c$, λ and ν obey the first equation of (34) and $a(t)$ meets (35). Similar to the previous case, redshift (7) diverges at r_0 when $e^{\nu(r_0)} \to 0$. Also, among curvature scalars, divergence of the Weyl square is not clear. It depends on the behavior of e^λ, ν'', λ' and ν' at this radius. The other curvature scalars will diverge at this radius. For the surface area we get

$$A = \int \sqrt{e^{-2\nu(r_0)} e^{2c} a(t)^4 sin^2(\theta)} d\theta d\phi \to \infty. \tag{38}$$

As the previous case, hypersurface which is located at $r = r_0$ can be timelike, null or even spacelike and it depends on the value of $\frac{e^{-\lambda(r_0)}}{a^2(t)}$. Therefore, it is a surface singularity. Since the redshift singularity either points to the naked or surface singularities, we think that the redshift singularity in the killing vector solutions ($\psi = 0$) does not point to BHs. Briefly, we saw that the naked and surface singularities cannot live in a universe with arbitrary $a(t)$.

2.2. CKV SOLUTIONS ($f(t) = c \neq 0$)

In this case, from the fourth equation of (22) and equation (23), we get $\alpha' = 0$ and $\dot{\beta} = 0$, respectively. Using these results and (22), we find

$$
\begin{aligned}
\mu(r) &= \nu(r) + c \\
\alpha(t) &= a(t) \\
\xi^\alpha &= (a(t), ce^{\frac{\mu - \lambda}{2}}, 0, 0),
\end{aligned} \tag{39}
$$

and for the conformal factor $\psi(r, t)$, we have

$$\psi(r, t) = \frac{c\nu'}{2} e^{\frac{\mu - \lambda}{2}} + \dot{a}(t). \tag{40}$$

Redshift considerations are similar to the case (25). The only major difference is due to the forms of $a(t)$ which are arbitrary in this case, unlike (25) which must meet the special limitations (31), and should be evaluated from cosmological considerations. Briefly, this class of solutions doesn't contain BH.

2.3. SOLUTIONS WITH $f(t) = 0$

Using (22) and (23) we get

$$
\begin{aligned}
\beta &= 0 \\
\alpha &= a(t) \\
\xi^\alpha &= (a(t), 0, 0, 0) \\
\psi &= \dot{a}(t).
\end{aligned}
\tag{41}
$$

Therefore irrespective of ν, λ, and μ, there is a conformal killing vector $\xi^\alpha = (a(t), 0, 0, 0)$ and a conformal factor $\psi(t) = \partial_\gamma \xi^\gamma$. $a(t)$ is an arbitrary function of time and must be evaluated from another parts of physics. We should note that for this class of solutions, from Eq. (22), it is apparent that the functional time dependence of $(a(t))$ does not affect the functional radial dependence of ν, λ and μ. Therefore, when β meets the $\beta = 0$ condition Eq. (41) will be valid for every $a(t)$ independent of ν, λ and μ. For example, one can take it the same as the scale factor $(a(t))$ of the FRW universe.

3. CONFORMALLY SCHWARZSCHILD-DE SITTER SPACETIME

In this section, we consider solutions respecting Eq. (41) and do our calculations in the FRW background. According to the standard model of cosmology, depending on the equation of state parameter $\omega = \frac{P}{\rho}$, the scale factor either increases as a power law $a(t) = At^{\frac{2}{3(\omega+1)}}$ for $\omega > -1$ or $a(t) = A(t_{br} - t)^{\frac{2}{3(\omega+1)}}$ for $\omega < -1$, where t_{br} is the Big Rip singularity time and will happen if the universe is in the phantom regime. For the dark energy era $(\omega = -1)$, the scale factor is $a(t) = A\, exp\, Ht$. In the phantom regime, the expansion of the universe ends catastrophically and everything will ultimately decompose into its elementary constituents [65]. Simple calculations show that (32) and (35) are not satisfied by the scale factor of the FRW universe. We take

$$
\begin{aligned}
\mu(r) &= 2\ln r \\
\nu(r) &= -\lambda(r) = \ln(1 - 2m(r)/r).
\end{aligned}
\tag{42}
$$

We are looking for isotropic solutions of this spacetime. It means that we want to know the form of $m(r)$ from the isotropy condition $\delta = 0$. Since the anisotropic

function (δ) is independent of time and thus the solutions of the $\delta = 0$ equation, the functional time dependence of $(a(t))$ does not affect the solutions of $\delta = 0$. This yields $m(r) = A + Br^3$. It is apparent that the FRW spacetime is achievable by substituting $A = B = 0$. Consider $B = 0$ and $A > 0$, then we confront the Schwarzschild BH embedded in an accelerating universe, which has been studied by many authors in the literatures for various accelerating regimes [45–48, 50]. $A = 0$ and $B > 0$ yields the de Sitter (dS) spacetime in the static limit $(a(t) \sim c)$. For the dS spacetime unlike the weak energy condition, the strong energy condition is violated [5]. Similar to the dS spacetime, our metric will change its signature at $r_0 = \frac{1}{\sqrt{2B}}$. Also, the divergence of the metric will happen at this radii and for the surface area at this radius we have

$$A = \int a(t)^2 r_0^2 sin(\theta)^2 d\theta d\phi = 4\pi R(t)^2 r_0^2. \tag{43}$$

It is apparent that $\dot{A} \geq 0$. Therefore, the second law of thermodynamics $(\dot{S} \geq 0)$ is satisfied [5]. Using (6), we see that, just same as the dS spacetime, $r = r_0$ is a null hypersurface and $r > r_0$ and $r < r_0$ point to the spacelike and the timelike hypersurfaces respectively. Unlike the Weyl square, the Kretschmann invariant and the Ricci square diverge at this radius as well as the Ricci scalar. Indeed, the Weyl tensor is zero for this spacetime showing that this solution is a conformally flat spacetime [49]. It is due to this fact that solutions with $A = 0$ are conformal to the dS metric which is a conformally flat spacetime [5]. Finally, since this metric is a conformal transformation of the dS spacetime, its causal structure is the same as that of the dS spacetime, and therefore we think that the co-moving radii $r = r_0$ points to a cosmological event horizon like what happen in the similar cases [45]. Consider a co-moving observer, the weak energy condition yields

$$-G_0^0 \geq 0 \Longrightarrow \dot{a}^2(t) \geq -2B(1 - 2Br^2), \tag{44}$$

which is valid when $B \geq 0$. Strong energy condition implies

$$\frac{1}{2}(3T_1^1 - T_0^0) \quad \geq \quad 0 \tag{45}$$
$$\Longrightarrow 2a(t)\ddot{a}(t) + \dot{a}^2(t) \quad \leq \quad -4B(1 - 2Br^2).$$

By combining (44) and (45), we get

$$2a(t)\ddot{a}(t) - \dot{a}^2(t) \leq 0, \tag{46}$$

which is a necessary condition for satisfying (44) and (45) simultaneously. This condition is valid when $\omega \geq -\frac{2}{3}$. So when $\omega \geq -\frac{2}{3}$, strong and weak energy conditions may be satisfied together. In fact as sufficient conditions, (44) and (45) should be satisfied separately. It depends on the values of r and t and can happen when $\omega \geq -\frac{2}{3}$.

For the energy flux we get

$$q^\mu = \frac{2Br\dot{a}(t)}{a(t)^3(1-2Br^2)^{1/2}}\delta_r^\mu.$$

(47)

We see that in the expanding universe $\dot{a}(t) > 0$ satisfying the $B > 0$ condition, the energy flux meets the $q^\mu > 0$ condition. This fact tells us that everything is ejecting from the radii r_0 during the expansion called the backreaction effect [63]. The similar result is valid in the dS spacetime ($a(t) = 1$) [5]. Since our approach doesn't constrain the value of B, the anti de Sitter (AdS) solution is allowed by our scheme. In conclusion, solutions with $B \neq 0$ and $A = 0$, include the dS and the AdS BHs in a dynamic universe.

Now taking account the condition $A, B \neq 0$, we get the Schwarzschild-de Sitter (SdS) spacetime embedded in the FRW background, by substitutions $A = m$ and $B = \frac{\Lambda}{6}$. For the Weyl square we get

$$W = \frac{48A^2}{r^6 a(t)^4}.$$

(48)

Since $A \neq 0$ for these solutions, they are not conformal to the FRW spacetime and therefore, apart of a radial singularity at $r = 0$ which is due to this fact that our spacetime is conformal to the SdS spacetime, the Weyl square suffers from another singularity at the bing bang time ($t = 0$) produced by the functional time dependence of the scale factor ($a(t)$). In addition, Eq. (48) predicts that this metric is an asymptotically conformally flat spacetime confirmed by the asymptotic behavior of the metric and the Weyl tensor in the $r \gg 1$ limit. From Eqs. (7) and (6), we see that there are two redshift singularities pointing to the null hypersurfaces and located at $r_c^2 = \frac{1}{2\Lambda}(3(1 + \sqrt{1 - \frac{4m^2\Lambda}{3}}))$ and $r_e^2 = \frac{1}{2\Lambda}(3(1 - \sqrt{1 - \frac{4m^2\Lambda}{3}}))$ which are the same as that of the SdS spacetime. Since our spacetime is conformal to the SdS spacetime, its causal structure is the same as that of SdS spacetime. The off diagonal elements of the Einstein tensor (G_{tr}) will vanish for large values of r. Therefore, the perfect fluid solution is attainable in this limit. By evaluating the Ricci scalar we find

$$R = \frac{R_{FRW}}{1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}} + \frac{4\Lambda}{a^2(t)}.$$

(49)

From the Einstein equations, the density and the pressure in this model are

$$\rho(r,t) = -T_0^0 = \frac{\rho_{FRW}}{1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}} + \frac{\Lambda}{a^2(t)}$$

(50)

$$P(r,t) = T_i^i = \frac{P_{FRW}}{1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}} - \frac{\Lambda}{a^2(t)},$$

where ρ_{FRW} and P_{FRW} are the density and the pressure of the FRW universe, re-

spectively. By considering $u^\mu = |-g^{00}|^{1/2}\delta_t^\mu$ and using (19), we get

$$q^\mu = -\frac{2(m - \frac{\Lambda}{3}r^3)\dot{a}(t)}{r^2 a(t)^3(1 - \frac{2m(r)}{r})^{1/2}}\delta_r^\mu, \tag{51}$$

as the radial energy flux which is induced by the background fluid. Positive (negative) values of q^μ lead to a mass decrease (increase) for the BH and at the radius r, which dependents on the values of m and Λ [63]. In fact, $q^\mu < 0$ is satisfied when the condition $r < (\frac{m}{m_0})^{1/3}$ in which $m_0 = \frac{\Lambda}{3}$ holds. The surface area at the radii $r_i\epsilon\{r_e, r_c\}$ is given by

$$A = \int a(t)^2 r_i^2 sin(\theta)^2 d\theta d\phi = 4\pi a(t)^2 r_i^2, \tag{52}$$

which increases by the expansion. Also, it is apparent that the second law of the thermodynamics ($\dot{S} \geq 0$) is satisfied. Based on the properties of the radii $r_i\epsilon\{r_e, r_c\}$ and since the changes in the metric signature at the radii r_i are the same as those of the SdS spacetime, we think that there is an event horizon at co-moving radius r_e where its physical radius is $\tilde{r}_e = a(t)r_e$ and a cosmological horizon with co-moving radii r_c and the physical radii $\tilde{r}_c = a(t)r_c$. This conclusion is supported by this fact that our solution is conformal to the SdS spacetime [45, 49]. If we define the function $f(r) = 1 - 2\frac{m}{r} - \frac{\Lambda}{3}r^2$ and using (12), we have $\frac{r_H}{f(r_H)} = \pm\frac{1}{\dot{a}(t)}$ for the apparent horizon radius, which is a fourth order equation of r and its solutions depend on the values of m and Λ and are not straightforward when for the physical radius we get $\tilde{r}_H = a(t)r_H$. By the slow expansion approximation ($a(t) \sim C$), we obtain

$$ds^2 \approx -f(\rho)dt^2 + \frac{dr^2}{f(\rho)} + \rho^2 d\Omega^2, \tag{53}$$

where $\rho \equiv Cr$ and $f(\rho) \equiv 1 - \frac{2mC}{\rho} - \frac{\Lambda'\rho^2}{3}$. Λ' in the definition of $f(\rho)$ is $\frac{\Lambda}{C^2}$. By following [50], temperature on the event and cosmological horizons can be calculated by:

$$T_i \simeq \frac{\frac{\partial f}{\partial\rho}|_{r_i}}{4\pi}, \tag{54}$$

which is compatible with the previous studies [50, 57, 66, 67]. Temperature on the apparent horizon can also be evaluated by (13). In the limit of zero cosmological constant ($\Lambda \to 0$), the results of previous studies are reproduced [45–48, 50, 66, 67]. In fact, the slow expansion approximation helps us to get an intuitive interpretation of the BH in the expanding background [50, 57].

As an another example, we consider a special subclass of solutions which has the Ricci scalar conformal to the FRW one. For this class, the condition $R_1 = 0$ is valid. By choosing (42) and inserting into $R_1 = 0$, we get $m(r) = A + \frac{B}{r}$ as general solution. $A = 0$ in the static limit ($a(t) = C$), points to the charged massless

BHs. Although this solution looks un-interesting, but it is allowed in the framework of the Yang-Mills theory [68]. $B = 0$ and $A, B \neq 0$ are nothing but the conformally Schwarzschild and conformally Reissner-Nordstöm BHs in the FRW background, respectively [69, 70]. Physical and thermodynamical properties of the general solution $m(r) = A + \frac{B}{r}$, can be found in [45, 50].

4. CONCLUDING REMARKS

In order to find BHs in a dynamic background, we started by the general form of the static spherically symmetric metrics which merges smoothly to the dynamic background by a scale factor $a(t)^2$ and respects certain symmetries. Since we have used the conformal transformation, the causal structure of the transformed metric (5) is the same as those of the primary metric (1). We found out that some solutions with the naked and surface singularities cannot be embedded in an arbitrary dynamical background. In continue, we could find and classify some special solutions which include various kinds of BHs, within the same class with the conformal killing vector $\xi^\alpha = a(t)\partial^t \delta_t^\alpha$ and the conformal factor $\psi(t) = \dot{a}(t)$, where $a(t)$ is an arbitrary function of time. For this class of solutions, since $a(t)$ does not affect the functional radial dependence of the metric (ν, λ and μ), the general properties of the metric, such as the validity of Eq. (51) are independent of the functional time dependence of $a(t)$. We should note that the quantitative behavior of the spacetime properties and their rates of changes, such as the energy flux, depend on $a(t)$. In continue, without loss of generality, we took $a(t)$ the same that of the FRW spacetime. Among these solutions, the conformally Schwarzschild BH has special properties. This solution points to the isotropic fluid and has the Ricci scalar conformal to the FRW's. The temperature on the redshift singularity surfaces, that act like horizons, and the apparent horizon have been addressed. Although the definition of a BH in an expanding universe is vague [39], but our analysis can help us to clarify this subject. In the early universe, the slow expansion approximation obviously breaks down and a non-equilibrium analysis will be needed. Astrophysical motivations for $a(t)$ was not our aim in this paper. This title could be interesting problem for future works.

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