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BRIGHT-DARK LUMP WAVE SOLUTIONS FOR A NEW FORM OF THE $(3+1)$-DIMENSIONAL BKP-BOUSSINESQ EQUATION

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Abstract. In this work, we explore a variety of lump solutions, generated from quadratic functions, for a new form of the $(3+1)$-dimensional BKP-Boussinesq equation, by employing the Hirota's bilinear form. The sufficient and necessary conditions, which demonstrate analyticity, positiveness, and rational localization of the solutions in a systematic manner, are constructed and examined. Moreover, the aforementioned equation has been reduced dimensionally. This will enable us establishing corresponding lump solutions with free parameters, which play indispensable role for influencing and controlling the phase shifts, shapes and energy distributions and propagation directions, for these solutions. The dynamical characteristics of acquired lump solutions have been examined visually, leading to formation of bright-dark lump waves.

Key words: New form of $(3+1)$-dimensional BKP-Boussinesq equation, Bilinear equations, Dimensionally reduced new form of $(3+1)$-dimensional BKP-Boussinesq equation, Bright-dark lump waves.

1. INTRODUCTION

A major thrust of theoretical and experimental studies has been witnessed in the past few decades on nonlinear evolution equations and their applications in diverse areas, such as nonlinear dynamics, optical fibers, matter waves, plasma physics, electro magnetic waves, etc. The nonlinear evolution equations describe a plethora of physical effects in fluid dynamics, condensed matter physics, optics, photonics, and nonlinear fiber optics. The work on these nonlinear equations has been flourishing in recent years to get an insight through qualitative and quantitative features of these nonlinear equations. In the mathematical simulation of innumerable physical phenomena, a distinguished location is held by processes associated with partial differential equations with nonlinear and dispersion terms, such as waves in fluids and ion-acoustic waves in plasmas, vibrations of elastic bars and lattices of coupled oscillators, the dynamics of domain boundaries in magnets and many more. Due to the expansion of domains of physical applications, the study of arising equations makes contributions to the general theory of differential equations and facilitates the development of relevant analytical and numerical methods. Exact solution of nonlinear equation of solitary waves gives an analytic robust understanding of the corresponding physical system especially in places where numerical solutions fail.

In the past few decades, modern theories have been widely equipped with efficient methods and reliable algorithms [1–13] to understand the challenging mysteries of complex natural phenomena. Apart from the physical applications, the close-form solutions of nonlinear partial differential equations (NLPDEs) assist the numerical solvers to compare the accuracy of their results and help them in the stability analysis.

For many years, solitons have been at the forefront of the study of integrable systems. However, in the recent times, the research about lump solutions has become the hot spot, which made a significant appearance in a variety of interesting physically relevant situations in the mathematical physics community. Lump solutions are rational function solutions, localized in all directions in the space with certain potential implications in nonlinear dynamics. By considering important characteristics of lump solutions one can understand the shapes, amplitudes, and velocities of solitons after the collision with another soliton.

Particular examples of lump solutions are found for many integrable equations such as the KPI equation [14], the Davey-Stewartson II equation [15], the BKP equation [16], and the Ishimori I equation [17]. As a matter of fact, authors of Refs. [18–26] have found lump solutions and interaction solutions for some NLPDEs and such solutions could be used to discuss significant properties of nonlinear wave phenomena in different fields of physics and oceanology.

In this study, we would like to furnish lump solutions for a new form of the $(3+1)$-dimensional BKP-Boussinesq equation [27] by taking into consideration the corresponding Hirota bilinear forms. The new form of the $(3+1)$-dimensional BKP-Boussinesq equation is given as:

$$u_{ty} - u_{xxxy} - 3(u_x u_y)_x + u_{tt} + 3u_{xz} = 0, \tag{1}$$

where the term u_{tt} has been added to the generalized $(3+1)$-dimensional BKP equation, which make drastic impact on the dispersion relation and the phase shift. Generalized BKP equation describes the evolution of quasi-one dimensional shallow water-waves, when the effects of surface tension and viscosity are negligible and these are widely used in various physics fields such as the surface and internal oceanic waves, nonlinear optics, plasma physics, ferromagnetics, Bose-Einstein condensation, string theory and so on. Also, one and two-soliton solutions were established [27], where the coefficients of the spatial variables were left as free parameters for Eq. (1). Furthermore, it was found that three-soliton solutions for free parameters do not exist, and this indicates that this equation is not integrable.

First of all, we intend to study the Hirota bilinear form of Eq. (1) by using the Bell polynomials and their connections with Hirota's operator. Secondly, we aim to carry out a hunt for positive quadratic function solutions to its corresponding bilinear equations. Moreover, the lump solutions of dimensionally reduced new form of the $(3+1)$-dimensional generalized BKP-Boussinesq equation are produced by taking reduction $z = x$ and $z = y$, respectively. As a consequence, quadratic function solutions with free parameters are found that further leads to a new class of lump solutions along with sufficient and necessary conditions. Also, some graphical representations are explicitly given along with some concluding remarks at the end of the paper.

2. LUMP SOLUTIONS TO A NEW FORM OF THE (3+1)-DIMENSIONAL BKP-BOUSSINESQ EQUATION

The major goal of this study is to explore lump solutions for a new form of the $(3+1)$-dimensional BKP-Boussinesq equation and its dimensionally reduced equations by using corresponding bilinear equations. Considering the Bell polynomials and using the transformation $u(x,y,z,t) = q_x$, Eq. (1) is connected with P-polynomials as follows:

$$P_{t,y}(q) - P_{3x,y}(q) + P_{2t}(q) + 3P_{x,z}(q) = 0. \tag{2}$$

On account of property of the Bell polynomials, Eq. (2) is transformed into Hirota's bilinear form

$$(D_tD_y - D_x^3D_y + D_t^2 + 3D_xD_z)\psi.\psi = 0, \tag{3}$$

where $\psi(x,y,z,t)$ is representing an auxiliary function with

$$q(x,y,z,t) = 2\ln(\psi(x,y,z,t))$$

and $D_x, D_y, D_z,$ and D_t represent Hirota's bilinear operators given as

$$D_{x_1}^{n_1}...D_{x_l}^{n_l}g.f = (\partial_{x_1} - \partial_{x_1'})^{n_1}...(\partial_{x_l} - \partial_{x_l'})^{n_l}g(x_1,...,x_l)f(x_1',...,x_l')|_{x_1'=x_1,...,x_l'=x_l}. \tag{4}$$

Bearing in mind the bilinear form (3), we look for quadratic function solutions to Eq. (1) with the following assumption

$$\begin{aligned}&\psi(x,y,z,t) = G^2 + H^2 + \beta_{11}, \text{ where}\\ &G = \beta_1x + \beta_2y + \beta_3z + \beta_4t + \beta_5, \ \ H = \beta_6x + \beta_7y + \beta_8z + \beta_9t + \beta_{10},\end{aligned} \tag{5}$$

and $\beta_i,\ 1 \le i \le 11$ are real parameters to be determined. The constants β_1, β_6 indicate the wave velocities in the x direction, β_2, β_7 show the wave velocities in the y direction, β_3, β_8 represent the wave velocities in the z direction, and β_4, β_9 specify the frequency. Inserting directly Eq. (5) into bilinear equation (3) results into a system of algebraic equations on the parameters. Symbolic computation of this system generates the following set of constraining equations on the various parameters:

$$\begin{aligned} \beta_3 &= \frac{-\left(\beta_1\beta_2\beta_4+\beta_1{\beta_4}^2-\beta_1\beta_7\beta_9-\beta_1{\beta_9}^2+\beta_2\beta_6\beta_9+\beta_4\beta_6\beta_7+2\,\beta_4\beta_6\beta_9\right)}{3\left({\beta_1}^2+{\beta_6}^2\right)}, \\ \beta_8 &= \frac{-\left(\beta_1\beta_2\beta_9+\beta_1\beta_4\beta_7+2\,\beta_1\beta_4\beta_9-\beta_2\beta_4\beta_6-{\beta_4}^2\beta_6+\beta_6\beta_7\beta_9+\beta_6{\beta_9}^2\right)}{3\left({\beta_1}^2+{\beta_6}^2\right)}, \\ \beta_{11} &= \frac{3\left({\beta_1}^2+{\beta_6}^2\right)^2(\beta_1\beta_2+\beta_6\beta_7)}{(\beta_1\beta_9-\beta_4\beta_6)(\beta_1\beta_7+\beta_1\beta_9-\beta_2\beta_6-\beta_4\beta_6)}, \end{aligned} \tag{6}$$

which need to satisfy following conditions

$$(\beta_1\beta_9-\beta_4\beta_6)\neq 0, (\beta_1\beta_7+\beta_1\beta_9)\neq(\beta_2\beta_6+\beta_4\beta_6) \text{ and } (\beta_1\beta_2+\beta_6\beta_7)>0, \tag{7}$$

to guarantee the well-definedness of ψ, the positiveness of ψ, and the localization of u in all directions in space, respectively. The parameters in the set (7) yield the following class of positive quadratic function solutions to Eq. (3) as

$$\begin{aligned} &\psi(x,y,z,t) = \\ &\left(\beta_1 x+\beta_2 y-\frac{\left(\beta_1\beta_2\beta_4+\beta_1{\beta_4}^2-\beta_1\beta_7\beta_9-\beta_1{\beta_9}^2+\beta_2\beta_6\beta_9+\beta_4\beta_6\beta_7+2\,\beta_4\beta_6\beta_9\right)z}{3({\beta_1}^2+{\beta_6}^2)}+\beta_4 t+\beta_5\right)^2 \\ &+\left(\beta_6 x+\beta_7 y-\frac{\left(\beta_1\beta_2\beta_9+\beta_1\beta_4\beta_7+2\,\beta_1\beta_4\beta_9-\beta_2\beta_4\beta_6-{\beta_4}^2\beta_6+\beta_6\beta_7\beta_9+\beta_6{\beta_9}^2\right)z}{3({\beta_1}^2+{\beta_6}^2)}+\beta_9 t+\beta_{10}\right)^2 \\ &+\beta_{11}. \end{aligned} \tag{8}$$

Reverting back to original variables by means of $u = 2(\ln(\psi))_x$, a class of lump solutions to Eq. (1) is obtained, as follows:

$$u_1 = \frac{4(\beta_1 G+\beta_6 H)}{\psi}, \tag{9}$$

with G and H given as

$$\begin{aligned} G &= \beta_1 x+\beta_2 y-\frac{\left(\beta_1\beta_2\beta_4+\beta_1{\beta_4}^2-\beta_1\beta_7\beta_9-\beta_1{\beta_9}^2+\beta_2\beta_6\beta_9+\beta_4\beta_6\beta_7+2\,\beta_4\beta_6\beta_9\right)z}{3({\beta_1}^2+{\beta_6}^2)}+\beta_4 t+\beta_5, \\ H &= \beta_6 x+\beta_7 y-\frac{\left(\beta_1\beta_2\beta_9+\beta_1\beta_4\beta_7+2\,\beta_1\beta_4\beta_9-\beta_2\beta_4\beta_6-{\beta_4}^2\beta_6+\beta_6\beta_7\beta_9+\beta_6{\beta_9}^2\right)z}{3({\beta_1}^2+{\beta_6}^2)}+\beta_9 t+\beta_{10}. \end{aligned} \tag{10}$$

Note that the involved parameters β_1, β_2, β_4, β_5, β_6, β_7, β_9, and β_{10} are needed to satisfy the conditions (7) for the existence of lump solutions u_1.

3. LUMP SOLUTIONS TO THE REDUCED (3+1)-DIMENSIONAL B-TYPE KP EQUATIONS

3.1. THE REDUCTION WITH $z = x$

By taking $z = x$, the bilinear equation (3) is reduced to

$$(D_t D_y - D_x^3 D_y + D_t^2 + 3D_x^2)\psi.\psi = 0, \tag{11}$$

The reduced bilinear equation (11) corresponds to the following $(2+1)$-dimensional equation through the link between u and ψ :

$$u_{ty} - u_{xxxy} - 3(u_x u_y)_x + u_{tt} + 3u_{xx} = 0 \tag{12}$$

The following quadratic function solution is assumed to find the lump solution of Eq. (12),

$$\begin{aligned} &\psi(x,y,z,t) = G^2 + H^2 + \beta_9, \text{ where} \\ &G = \beta_1 x + \beta_2 y + \beta_3 t + \beta_4, \;\; H = \beta_5 x + \beta_6 y + \beta_7 t + \beta_8, \end{aligned} \tag{13}$$

and $\beta_i,\ 1 \le i \le 9$ are real parameters to be determined. Using symbolic computation, after direct substitution of Eq. (13) into Eq. (11), generates the following set of constraining equations on the parameters:

$$\begin{aligned} \beta_2 &= \frac{-(3\beta_1{}^2\beta_3+6\beta_1\beta_5\beta_7+\beta_3{}^3-3\beta_3\beta_5{}^2+\beta_3\beta_7{}^2)}{\beta_3{}^2+\beta_7{}^2}, \\ \beta_6 &= \frac{3\beta_1{}^2\beta_7-6\beta_1\beta_3\beta_5-\beta_3{}^2\beta_7-3\beta_5{}^2\beta_7-\beta_7{}^3}{\beta_3{}^2+\beta_7{}^2}, \\ \beta_9 &= -\frac{\left(\beta_1{}^2+\beta_5{}^2\right)(\beta_1\beta_3+\beta_5\beta_7)\left(3\beta_1{}^2+\beta_3{}^2+3\beta_5{}^2+\beta_7{}^2\right)}{(\beta_1\beta_7-\beta_3\beta_5)^2}, \end{aligned} \tag{14}$$

which need to satisfy the condition

$$(\beta_1\beta_7 - \beta_3\beta_5) \neq 0, (\beta_1\beta_6 - \beta_2\beta_5) \neq 0 \text{ and } (\beta_1\beta_3 + \beta_5\beta_7) < 0. \tag{15}$$

The parameters in the set (14) yield the following class of positive quadratic function solutions to the reduced Eq. (12) as

$$\begin{aligned} \psi(x,y,t) = &\left(\beta_1 x - \frac{y\left(3\beta_1{}^2\beta_3+6\beta_1\beta_5\beta_7+\beta_3{}^3-3\beta_3\beta_5{}^2+\beta_3\beta_7{}^2\right)}{\beta_3{}^2+\beta_7{}^2} + \beta_3 t + \beta_4\right)^2 \\ &+ \left(\beta_5 x + \frac{y\left(3\beta_1{}^2\beta_7-6\beta_1\beta_3\beta_5-\beta_3{}^2\beta_7-3\beta_5{}^2\beta_7-\beta_7{}^3\right)}{\beta_3{}^2+\beta_7{}^2} + \beta_7 t + \beta_8\right)^2 + \beta_9, \end{aligned} \tag{16}$$

which further leads to furnish a class of lump solutions to Eq. (1), by means of $u = 2(\ln(\psi))_x$, as follows:

$$u_2 = \frac{4(\beta_1 G + \beta_6 H)}{\psi}, \tag{17}$$

with G and H are given as follows:

$$\begin{aligned} G &= \beta_1 x - \frac{y\left(3\beta_1{}^2\beta_3+6\beta_1\beta_5\beta_7+\beta_3{}^3-3\beta_3\beta_5{}^2+\beta_3\beta_7{}^2\right)}{\beta_3{}^2+\beta_7{}^2} + \beta_3 t + \beta_4, \\ H &= \beta_5 x + \frac{y\left(3\beta_1{}^2\beta_7-6\beta_1\beta_3\beta_5-\beta_3{}^2\beta_7-3\beta_5{}^2\beta_7-\beta_7{}^3\right)}{\beta_3{}^2+\beta_7{}^2} + \beta_7 t + \beta_8. \end{aligned} \tag{18}$$

Note that $\beta_1,\ \beta_3,\ \beta_4,\ \beta_5,\ \beta_7,$ and β_8 are demanded to satisfy the conditions (15) to insure the well-definedness of ψ, the positiveness of ψ, and the localization of u_2 in all directions in space, respectively. It can be readily revealed that all the above lump solutions $u_2 \to 0$ at any given time t, if and only if the corresponding sum of squares

$G^2 + H^2 \to \infty$. Also the solution u_2 is analytic in the xy-plane if and only if the parameter $\beta_9 > 0$. The condition (15) guarantees both analyticity and localization of the solutions in (17).

3.2. THE REDUCTION WITH $z = y$

The bilinear equation (3) is reduced as follows, by using $z = y$:

$$(D_t D_y - D_x^3 D_y + D_t^2 + 3D_x D_y)\psi.\psi = 0, \tag{19}$$

The reduced bilinear equation (19) is connected to the following $(2+1)$-dimensional equation for $u = 2(\ln(\phi))_x$,

$$u_{ty} - u_{xxxy} - 3(u_x u_y)_x + u_{tt} + 3u_{xy} = 0. \tag{20}$$

For bilinear equation (19), a direct substitution of ψ defined by (13), followed by symbolic computation furnishes the following set of constraining equations on the parameters:

$$\begin{aligned}
\beta_2 &= -\frac{3\beta_1\beta_3{}^2 - 3\beta_1\beta_7{}^2 + \beta_3{}^3 + 6\beta_3\beta_5\beta_7 + \beta_3\beta_7{}^2}{(3\beta_1+\beta_3)^2 + (3\beta_5+\beta_7)^2},\\
\beta_6 &= -\frac{6\beta_1\beta_3\beta_7 - 3\beta_3{}^2\beta_5 + \beta_3{}^2\beta_7 + 3\beta_5\beta_7{}^2 + \beta_7{}^3}{(3\beta_1+\beta_3)^2 + (3\beta_5+\beta_7)^2},\\
\beta_9 &= \\
&\frac{-\left(\beta_1{}^2+\beta_5{}^2\right)\left(3\beta_1{}^2\beta_3{}^2 - 3\beta_1{}^2\beta_7{}^2 + \beta_1\beta_3{}^3 + 12\beta_1\beta_3\beta_5\beta_7 + \beta_1\beta_3\beta_7{}^2 - 3\beta_3{}^2\beta_5{}^2 + \beta_3{}^2\beta_5\beta_7 + 3\beta_5{}^2\beta_7{}^2 + \beta_5\beta_7{}^3\right)}{3(\beta_1\beta_7 - \beta_3\beta_5)^2},
\end{aligned} \tag{21}$$

which we need to satisfy the condition

$$(\beta_1\beta_7 - \beta_3\beta_5) \neq 0, \ (\beta_1\beta_6 - \beta_2\beta_5) \neq 0 \text{ and } \beta_9 > 0. \tag{22}$$

The conditions (22) makes the solution ψ, given by (13), to be well defined, positive and rationally localization of u_3 in all directions in the $(x,y)-$ plane. The set of parameters (21) leads to positive quadratic function solutions to reduced Eq. (20) as follows:

$$\begin{aligned}
\psi(x,y,t) = &\left(\beta_1 x - \frac{\left(3\beta_1\beta_3{}^2 - 3\beta_1\beta_7{}^2 + \beta_3{}^3 + 6\beta_3\beta_5\beta_7 + \beta_3\beta_7{}^2\right)y}{(3\beta_1+\beta_3)^2 + (3\beta_5+\beta_7)^2} + \beta_3 t + \beta_4\right)^2\\
&+ \left(\beta_5 x - \frac{\left(6\beta_1\beta_3\beta_7 - 3\beta_3{}^2\beta_5 + \beta_3{}^2\beta_7 + 3\beta_5\beta_7{}^2 + \beta_7{}^3\right)y}{(3\beta_1+\beta_3)^2 + (3\beta_5+\beta_7)^2} + \beta_7 t + \beta_8\right)^2 + \beta_9,
\end{aligned} \tag{23}$$

which further results to obtain a class of lump solutions to Eq. (1), as follows:

$$u_3 = \frac{4(\beta_1 G + \beta_6 H)}{\phi}, \tag{24}$$

with G and H given as follows:

$$\begin{aligned} G &= \beta_1 x - \frac{\left(3\beta_1\beta_3^2 - 3\beta_1\beta_7^2 + \beta_3^3 + 6\beta_3\beta_5\beta_7 + \beta_3\beta_7^2\right)y}{(3\beta_1+\beta_3)^2 + (3\beta_5+\beta_7)^2} + \beta_3 t + \beta_4, \\ H &= \beta_5 x - \frac{\left(6\beta_1\beta_3\beta_7 - 3\beta_3^2\beta_5 + \beta_3^2\beta_7 + 3\beta_5\beta_7^2 + \beta_7^3\right)y}{(3\beta_1+\beta_3)^2 + (3\beta_5+\beta_7)^2} + \beta_7 t + \beta_8. \end{aligned} \tag{25}$$

In this class of lump solutions, parameters $\beta_1,\ \beta_3,\ \beta_4,\ \beta_5,\ \beta_7$, and β_8 are arbitrary provided conditions (22) are satisfied to guarantee analyticity, positiveness, and rational localization of u_3 in every direction.

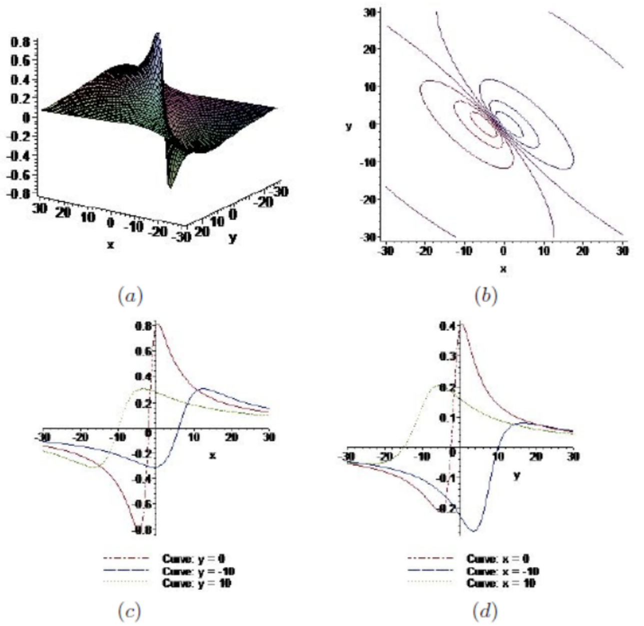

Fig. 1 – Lump dynamic characteristics of u_1 *via* Eq. (8) with $t=0, z=0, \beta_1=1, \beta_2=0.5, \beta_4=1, \beta_5=2, \beta_6=0.5, \beta_7=1, \beta_9=1, \beta_{10}=1$: (a) 3-dimensional plot; (b) density plots; (c) the wave along x-axis with $y=0, y=-10, y=10$; (d) the wave along y-axis with $x=0, x=-10, x=10$.

4. DISCUSSION AND REMARKS

The current study offers a useful finding that enhances the variety of the dynamical features of solutions for higher-dimensional nonlinear evolution equations. For this reason, the results and plots are presented for versions that most fully portray the features of the acquired solutions. Some graphical representations of lump solu-

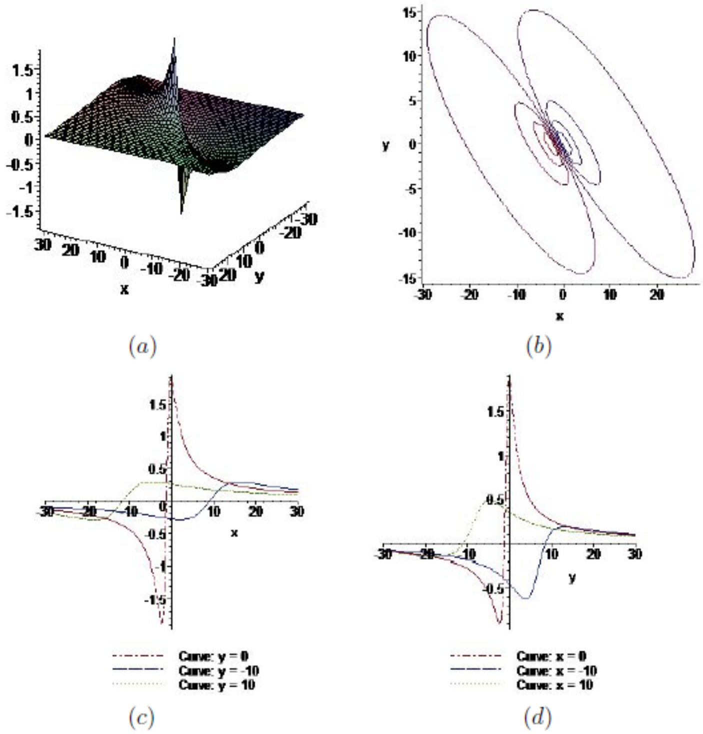

Fig. 2 – Lump dynamic characteristics of u_2 *via* (17) with $t = 0, \beta_1 = 1.5, \beta_3 = -4, \beta_4 = 2, \beta_5 = 2.5, \beta_7 = 1, \beta_8 = 3$: (a) 3-dimensional plot; (b) density plots; (c) the wave along x-axis with $y = 0, y = -10, y = 10$; (d) the wave along y-axis with $x = 0, x = -10, x = 10$.

tions u_1, u_2 and u_3 are depicted to distinctly illustrate the dynamic process of lump solution wave solutions by taking fixed values to free parameters as shown in Figs. 1, 2, and 3. The analysis for peak and valley of the lump waves are executed with some numerical values of involved parameters as mentioned below:

Solution	Peak Point	Valley Point	Amplitudes
u_1	$(0.4494, 0.0000)$	$(-4.4494, 0.0000)$	0.8164 and -0.8164
u_2	$(-0.2642, 0.0869)$	$(-2.3853, 0.0869)$	1.8858 and -1.8858
u_3	$(0.5410, -2.0277)$	$(-1.4299, -2.0277)$	2.0294 and -2.0294

It can be remarked from Figs. 1, 2, and 3 that the bright-dark lump wave are produced in a very uniform manner because the height of the peak is equal to the depth of the valley bottom for all u_1, u_2 and u_3. We assert that the obtained results have potential applications for studying different attributes of $(3+1)$-dimensional BKP-

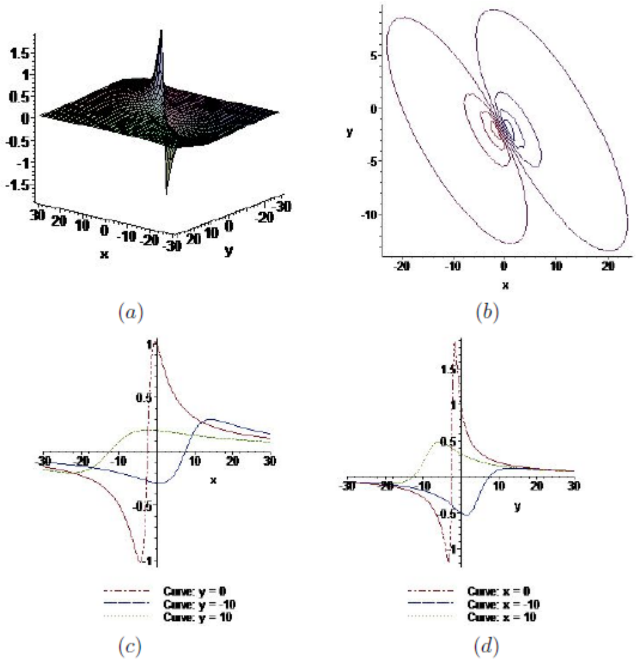

Fig. 3 – Lump dynamic characteristics of u_3 *via* Eq. (24) with $t=0, \beta_1=1.5, \beta_3=-3, \beta_4=2, \beta_5=1, \beta_7=1, \beta_8=5$: (a) 3-dimensional plot; (b) density plots; (c) the wave along x-axis with $y=0, y=-10, y=10$; (d) the wave along y-axis with $x=0, x=-10, x=10$.

Boussinesq equation such as rational function amplitudes efficiently describe nonlinear wave phenomena in both oceanography and nonlinear optics.

5. CONCLUSIONS

In the current study, a new form of the $(3+1)$-dimensional BKP-Boussinesq equation and its dimensionally reduced equations are investigated for exploiting lump solutions with the help of positive quadratic function. The bilinear equations are derived form the corresponding Hirota's transformation and Bell polynomials for considered equations and then these are employed to procure new exact lump solutions along with some restriction conditions to guarantee positivity, analytic behavior, and localization of solutions. Dynamical features of developed lump solutions have been visualized in different planes by presenting density plots, three-dimensional and two-dimensional plots with some specific choices of the emerged free parameters to show the localizations of the solutions, which might be utilized to describe the nonlinear wave phenomena such as rogue waves and solitons. Furthermore, the peak and valley of the lump waves have been analyzed systematically as shown in the figures. To the best of our knowledge, the acquired solutions given in various cases have not been reported for a new form of the $(3+1)$-dimensional BKP-Boussinesq equation in the literature. It is worthy to implement this method as a promising and robust mathematical device for managing any NLPDEs, arising in mathematical physics, nonlinear mechanics, and other applied fields, by using the Hirota's bilinear equation. The findings reported in this work will be useful to design more lump solutions for other nonlinear systems.

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