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AN IMPROVED COLLOCATION TECHNIQUE FOR DISTRIBUTED-ORDER FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. This paper addresses spectral collocation techniques to treat with the distributed-order fractional partial differential equation (DOFPDE). We introduce a new shifted fractional order Jacobi orthogonal function (SFOJOF) outputted by Jacobi polynomials. Also, we state some corollaries and theorems related to new SFOJOF. The shifted Jacobi-Gauss-Lobatto collocation (SJ-GL-C) and shifted fractional order Jacobi-Gauss-Radau collocation (SFJ-GR-C) methods are developed for approximating the DOFPDEs. The basis of the shifted Jacobi polynomial is adapted for spatial discretization and another basis of SFOJOF is investigated for temporal discretization. Through the selected basis functions, the related conditions are automatically accomplished. The principal target in our technique is to transform the DOFPDE to a system of algebraic equations. Some numerical examples are given to test the accuracy and applicability of our technique.

Key words: Spectral collocation method; Gauss-Lobatto quadrature; Gauss-Radau quadrature; Caputo fractional derivative; Distributed-order fractional diffusion equation.

1. INTRODUCTION

Anomalous diffusion [1, 2] has gained a more concern in the scientific literature for being prescribed many physical phenomena [3–6], more applicable for crowded systems (for example, diffusion within porous media or protein diffusion through cells). Sub-diffusion [7] has been used as a measure of macromolecular crowding in the cytoplasm. Time and/or space fractional diffusion equations, where the first temporal derivative is exchanged by the fractional Caputo derivative of order $\delta_1 \in (0,1)$ whilst the spatial second order derivative is exchanged by the fractional Riemann-Liouville derivative of order $\delta_2 \in (1,2)$, are the more suitable mathematical models for diffusion processes [8]. There are many theoretical and numerical studies for time fractional sub-diffusion equations; see, for example [9–12]. The more general class of diffusion equation is called fractional diffusion equation of distributed order:

$$\begin{aligned} &D_t^{\mathcal{W}} u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + \mathcal{H}(x,t), 0 \le x \le \mathcal{L}, 0 \le t \le \mathcal{T}, \\ &u(0,t) = g_1(t), u(\mathcal{L},t) = g_2(t), u(x,0) = g_3(x), 0 \le x \le \mathcal{L}, 0 \le t \le \mathcal{T}. \end{aligned} \tag{1}$$

For physical significance and others analytical advantages, see [13–15] and the references therein.

Several numerical techniques are presented to solve the fractional differential equations [16–19]. On the top of this list, the spectral method [20–22] has been improved recently. Spectral methods [23–25] are exceedingly used to construct numerical algorithms for solving fractional differential equations. In the spectral methods, the numerical solution is approximated as a truncated sum of assured basis functions. Then we choose the coefficients such that the error is minimized. For spectral collocation method [26–28], the approximate solution is compelled to satisfy the discussed problem as possible. In other words, the residuals are letting to be zero at confirmed collocation points.

In the present paper, we extend the SJ-GL-C and SFJ-GR-C methods to solve the DOFPDEs. The shifted Jacobi-Gauss-Lobatto (SJ-GL) and shifted fractional-order Jacobi-Gauss-Radau (SFJ-GR) are chosen as interpolation nodes for spatial and temporal variables, respectively. Numerical solution of such equation is putted as a truncated series of basis functions of shifted Jacobi polynomials and SFOJOF, for spatial and temporal variables, respectively, thereafter we estimate the residuals of the aforesaid problem at the SJ-GL and SFJ-GR quadrature points. The shifted Legendre Gauss-Lobatto quadrature is used to numerically treat with the integral term (distributed-order fractional term). So, a system of algebraic equations is taken out. The precision of our technique is confirmed with various numerical problems.

Our work is ordered as follows. In Sec. 2, we list some mathematical fundamentals. In Sec. 3, a timespace approach for the one-dimensional DOFPDEs with homogeneous and nonhomogeneous conditions is derived. The competence of our numerical approach is exhibited by diverse examples in Sec. 4. Few remarks are mentioned in the last Section.

2. PROPERTIES OF SHIFTED FRACTIONAL JACOBI POLYNOMIALS

First, we state some basic definitions and theorems about Jacobi polynomials

$$\begin{aligned} &\mathcal{J}_{k+1}^{(\rho,\sigma)}(x) = (a_k^{(\rho,\sigma)} x - b_k^{(\rho,\sigma)}) \mathcal{J}_k^{(\rho,\sigma)}(x) - c_k^{(\rho,\sigma)} \mathcal{J}_{k-1}^{(\rho,\sigma)}(x), \quad k \ge 1, \\ &\mathcal{J}_0^{(\rho,\sigma)}(x) = 1, \quad \mathcal{J}_1^{(\rho,\sigma)}(x) = \frac{1}{2}(\rho+\sigma+2)x + \frac{1}{2}(\rho-\sigma), \end{aligned}$$

$$\mathcal{J}_k^{(\rho,\sigma)}(-x) = (-1)^k \mathcal{J}_k^{(\rho,\sigma)}(x), \quad \mathcal{J}_k^{(\rho,\sigma)}(-1) = \frac{(-1)^k \Gamma(k+\sigma+1)}{k!\Gamma(\sigma+1)}, \tag{2}$$

where $\rho,\ \sigma > -1$, $x \in [-1,1]$ and

$$a_k^{(\rho,\sigma)} = \frac{(2k+\rho+\sigma+1)(2k+\rho+\sigma+2)}{2(k+1)(k+\rho+\sigma+1)},$$
$$b_k^{(\rho,\sigma)} = \frac{(\sigma^2-\rho^2)(2k+\rho+\sigma+1)}{2(k+1)(k+\rho+\sigma+1)(2k+\rho+\sigma)},$$
$$c_k^{(\rho,\sigma)} = \frac{(k+\rho)(k+\sigma)(2k+\rho+\sigma+2)}{(k+1)(k+\rho+\sigma+1)(2k+\rho+\sigma)}.$$

Furthermore, the rth derivative of $\mathcal{J}_j^{(\rho,\sigma)}(x)$ is computed as

$$\mathcal{D}^r \mathcal{J}_j^{(\rho,\sigma)}(x) = \frac{\Gamma(j+\rho+\sigma+q+1)}{2^r \Gamma(j+\rho+\sigma+1)} \mathcal{J}_{j-r}^{(\rho+r,\sigma+r)}(x), \tag{3}$$

where r is an integer. Next, we list some basic definitions and theorems for shifted fractional Jacobi polynomial, see [29, 30] for more details.

Definition 2.1 *Now, we define the new shifted fractional-order Jacobi orthogonal function outputted Jacobi polynomial. The shifted fractional order Jacobi orthogonal function is offered by*

$$\mathcal{J}_{\mathcal{L},j}^{(\varepsilon,\rho,\sigma)}(x) = \mathcal{J}_j^{(\rho,\sigma)}(2(\frac{x}{\mathcal{L}})^\varepsilon - 1), 0 < \varepsilon < 1, j = 0, 1, \cdots, 0 \le t \le \mathcal{L}. \tag{4}$$

Theorem 2.1 *For $\mathcal{W}_{\mathcal{L},f}^{(\varepsilon,\rho,\sigma)}(x) = \varepsilon(\mathcal{L}^\varepsilon - x^\varepsilon)^\rho x^{\varepsilon\sigma+\varepsilon-1}$, the set of shifted fractional Jacobi functions forms a complete $L^2_{\mathcal{W}_f^{(\varepsilon,\rho,\sigma)}}[0,\mathcal{L}]$-orthogonal system*

$$\int_0^{\mathcal{L}} \mathcal{J}_{\mathcal{L},i}^{(\varepsilon,\rho,\sigma)}(x)\, \mathcal{J}_{\mathcal{L},j}^{(\varepsilon,\rho,\sigma)}(x)\, \mathcal{W}_{\mathcal{L},f}^{(\varepsilon,\rho,\sigma)}(x)\, dx = \delta_{ij} h_{\mathcal{L},k}^{(\varepsilon,\rho,\sigma)}, \tag{5}$$

where $h_{\mathcal{L},k}^{(\varepsilon,\rho,\sigma)} = (\frac{\mathcal{L}^\varepsilon}{2})^{\rho+\sigma+1} h_i^{(\rho,\sigma)}$, $h_i^{(\rho,\sigma)}$ is the orthogonality constant for Jacobi polynomials.

Corollary 2.2 *Let $\mathcal{F}_N = span\{\mathcal{J}_{\mathcal{L},i}^{(\varepsilon,\rho,\sigma)} : 0 \le i \le N\}$, be the finite-dimensional fractional-polynomial space. Due to the orthogonal property, the function $\zeta(x) \in L^2_{\mathcal{W}_f^{(\varepsilon,\rho,\sigma)}}[0,\mathcal{L}]$ may be expressed as*

$$\zeta(x) = \sum_{i=0}^{\infty} \gamma_i \mathcal{J}_{\mathcal{L},i}^{(\varepsilon,\rho,\sigma)}(x), \quad \gamma_i = \frac{1}{h_{\mathcal{L},k}^{(\varepsilon,\rho,\sigma)}} \int_0^{\mathcal{L}} \mathcal{J}_{\mathcal{L},i}^{(\varepsilon,\rho,\sigma)}(x)\, \zeta(x)\, \mathcal{W}_{\mathcal{L},f}^{(\varepsilon,\rho,\sigma)}(x)\, dt.$$

Theorem 2.3 *The Caputo fractional derivative of order* δ*, of the shifted fractional-order Jacobi orthogonal function* $\mathcal{D}_c^{\delta}\mathcal{J}_{\mathcal{L},j}^{(\varepsilon,\rho,\sigma)}(x)$ *can obtained in terms of the shifted fractional-order Jacobi orthogonal functions as*

$$\mathcal{D}_c^{\delta}\mathcal{J}_{\mathcal{L},j}^{(\varepsilon,\rho,\sigma)}(x) = \sum_{n=0}^{\mathcal{K}} \varepsilon_{\delta}^{(n,j,\varepsilon,\rho,\sigma)} \mathcal{J}_{\mathcal{L},n}^{(\varepsilon,\rho,\sigma)}(x), \tag{6}$$

where

$$\varepsilon_{\delta}^{(n,j,\varepsilon,\rho,\sigma)} = \sum_{k=1}^{j}\sum_{s=0}^{n} \frac{E_k^{(\varepsilon,\rho,\sigma,j)} E_s^{(\varepsilon,\rho,\sigma,n)}}{h_{\mathcal{L},n}^{(\varepsilon,\rho,\sigma)}} \mathcal{R}_{k,s}(\varepsilon,\rho,\sigma,\delta),$$

and

$$\mathcal{R}_{k,s}(\varepsilon,\rho,\sigma,\delta) = \frac{\Gamma(k\varepsilon+1)\Gamma(\rho+1)\Gamma\left(k+s+\sigma-\frac{\delta}{\varepsilon}+1\right)\mathcal{L}^{\varepsilon(\rho+\sigma+k+s+1)-\delta}}{\Gamma(k\varepsilon-\delta+1)\Gamma\left(k+s+\rho+\sigma-\frac{\delta}{\varepsilon}+2\right)}.$$

Theorem 2.4 *The corresponding nodes and corresponding Christoffel numbers of the fractional Jacobi-Gauss, fractional Jacobi-Gauss-Radau, and fractional Jacobi-Gauss-Lobatto interpolation in the interval* $[0,\mathcal{L}]$ *are given by*

$$x_{\mathcal{L},\mathcal{K},s}^{(\rho,\sigma,\varepsilon)} = \mathcal{L}\left(\frac{x_{\mathcal{K},s}^{(\rho,\sigma)}+1}{2}\right)^{\frac{1}{\varepsilon}}, \quad \varpi_{\mathcal{L},\mathcal{K},s}^{(\varepsilon,\rho,\sigma)} = \left(\frac{\mathcal{L}^{\varepsilon}}{2}\right)^{\rho+\sigma+1} \varpi_{\mathcal{K},s}^{(\rho,\sigma)}, \; 0 \leqslant s \leqslant \mathcal{K},$$

where $t_{\mathcal{K},s}^{(\rho,\sigma)}$ *and* $\varpi_{\mathcal{K},s}^{(\rho,\sigma)}$, $0 \leqslant s \leqslant \mathcal{K}$, *are the nodes and Christoffel numbers of the standard Jacobi-Gauss, Jacobi-Gauss-Radau, and Jacobi-Gauss-Lobatto interpolation in the interval* $[-1,1]$.

Theorem 2.5 *Let* $u(x) = v((\frac{1+x}{2})^{\frac{1}{\varepsilon}}) \in H^m_{\chi^{(\rho,\sigma)},*}(-1,1)$ *for some* $m \geq 1$ *and* $\phi = \varphi(\frac{1+x}{2})^{\frac{1}{\varepsilon}} \in \mathcal{F}_N^{(\rho,\sigma,\varepsilon)}$*. Then, for the fractional Jacobi-Gauss and the fractional Jacobi-Gauss-Radau integration, we have*

$$|(v,\varphi_{\chi^{(\rho,\sigma,\varepsilon)}}) - (v,\varphi_{\mathcal{K},\chi^{(\rho,\sigma,\varepsilon)}})| \leq CN^{-m} \parallel \partial^m v \parallel_{\chi^{(\rho+m,\sigma+m)}} \parallel \phi \parallel_{\chi^{(\rho,\sigma)}}, \tag{7}$$

where $H^m_{\chi^{(\rho,\sigma)},*}(-1,1)$ *is nonuniformly-weighted Sobolev space given by:*

$H^m_{\chi^{(\rho,\sigma)},*}(-1,1) := \{v : \partial_x^k v \in L^2_{\chi^{(\rho+k,\sigma+k)}}(-1,1) 0 \leq k \leq m\}$, $L^2_{\chi(-1,1)} := \{v : v$*is measurable on (-1,1) and* $\parallel v \parallel^2_{\chi} < \infty\}$ *and for the fractional Jacobi-Gauss-Lobatto integration, we have*

$$|(v,\varphi_{\chi^{(\rho,\sigma,\varepsilon)}}) - (v,\varphi_{\mathcal{K},\chi^{(\rho,\sigma,\varepsilon)}})| \leq CN^{-m} \parallel \partial^m v \parallel_{\chi^{(\rho+m-1,\sigma+m-1)}} \parallel \phi \parallel_{\chi^{(\rho,\sigma)}} . \tag{8}$$

3. SPECTRAL COLLOCATION METHOD

Based on SJ-GL-C and SFJ-GR-C methods, we introduce two numerical techniques to solve the DOFPDEs with homogeneous and nonhomogeneous conditions.

3.1. HOMOGENEOUS CONDITIONS

Here, we are occupied with utilizing the following DOFPDE

$$D_t^{\mathcal{W}} u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + \mathcal{H}(x,t),\, 0 \le x \le \mathcal{L},\, 0 \le t \le \mathcal{T}, \tag{9}$$

with the homogeneous conditions

$$u(0,t) = u(\mathcal{L},t) = u(x,0) = 0, \tag{10}$$

where $\mathcal{H}(x,t)$ is a given function.

Definition 3.1 *The distributed-order fractional $D_t^{\mathcal{W}}\chi(t)$ is defined as*

$$D_t^{\mathcal{W}}\chi(t) = \int_0^1 \mathcal{W}(\mu) D_t^{\mu}\chi(t) d\mu, \tag{11}$$

where $\mathcal{W}(\mu)$ is a given function and $D_t^{\mu}\chi(t)$ is the Caputo fractional derivatives of order μ.

Definition 3.2 *The Caputo fractional derivative of order μ is defined as*

$$D^{\mu}\chi(t) = \frac{1}{\Gamma(m-\mu)} \int_0^t (t-\eta)^{m-\mu-1} \frac{d^m\chi(\eta)}{d\eta^m} d\eta, \quad m-1 < \mu \le m,\; t > 0, \tag{12}$$

where m is the ceiling function of μ.

The approximate solution of $u_{Approx}(x,t)$ can be extended, using combination of basis functions of shifted Jacobi polynomials and SFOJOF as

$$u_{Approx}(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{M} \varsigma_{i,j} \chi_{\mathcal{L},i}^{(\rho_1,\sigma_1)}(x) \Omega_{\mathcal{T},j}^{(\rho_2,\sigma_2,\varepsilon)}(t), \tag{13}$$

where

$$\begin{aligned} \chi_{\mathcal{L},i}^{(\rho_1,\sigma_1)}(x) =& \mathcal{J}_{\mathcal{L},i}^{(\rho_1,\sigma_1)}(x) + \xi_i \mathcal{J}_{\mathcal{L},i+1}^{(\rho_1,\sigma_1)}(x) + \zeta_i \mathcal{J}_{\mathcal{L},i+2}^{(\rho_1,\sigma_1)}(x), \\ \Omega_{\mathcal{T},j}^{(\rho_2,\sigma_2,\varepsilon)}(t) =& \mathcal{J}_{\mathcal{T},j}^{(\rho_2,\sigma_2,\varepsilon)}(t) + \mathcal{V}_j \mathcal{J}_{\mathcal{T},j+1}^{(\rho_2,\sigma_2,\varepsilon)}(t). \end{aligned} \tag{14}$$

In contrast of the numerical approaches discussed in [31, 32], the parameters ξ_i, ζ_i, $\mathcal{V}_j$ ensure that both initial and boundary conditions are already satisfied. Below, we list the major procedure of the collocation method to solve the DOFPDEs with homogeneous conditions. Let

$$\begin{aligned} u_{Approx}(x,t) &= \sum_{i=0}^{\mathcal{N}}\sum_{j=0}^{\mathcal{M}} \varsigma_{i,j} \chi_{\mathcal{L},i}^{(\rho_1,\sigma_1)}(x) \Omega_{\mathcal{T},j}^{(\rho_2,\sigma_2,\varepsilon)}(t) \\ &= \sum_{i=0}^{\mathcal{N}}\sum_{j=0}^{\mathcal{M}} \varsigma_{i,j} \mathcal{S}_0^{i,j}(x,t), \end{aligned} \tag{15}$$

where $\mathcal{S}_0^{i,j}(x,t) = \chi_{\mathcal{L},i}^{(\rho_1,\sigma_1)}(x)\Omega_{\mathcal{T},j}^{(\rho_2,\sigma_2,\varepsilon)}(t)$.
Over and above that, the first and second partial derivatives are evaluated as

$$\begin{aligned}\frac{\partial u_{Approx}(x,t)}{\partial x} &= \sum_{i=0}^{\mathcal{N}}\sum_{j=0}^{\mathcal{M}} \varsigma_{i,j}\frac{\partial \chi_{\mathcal{L},i}^{(\rho_1,\sigma_1)}(x)}{\partial x}\Omega_{\mathcal{T},j}^{(\rho_2,\sigma_2,\varepsilon)}(t)\\ &= \sum_{i=0}^{\mathcal{N}}\sum_{j=0}^{\mathcal{M}} \varsigma_{i,j}\mathcal{S}_1^{i,j}(x,t),\end{aligned} \tag{16}$$

$$\begin{aligned}\frac{\partial^2 u_{Approx}(x,t)}{\partial x^2} &= \sum_{i=0}^{\mathcal{N}}\sum_{j=0}^{\mathcal{M}} \varsigma_{i,j}\frac{\partial^2 \chi_{\mathcal{L},i}^{(\rho_1,\sigma_1)}(x)}{\partial x^2}\Omega_{\mathcal{T},j}^{(\rho_2,\sigma_2,\varepsilon)}(t)\\ &= \sum_{i=0}^{\mathcal{N}}\sum_{j=0}^{\mathcal{M}} \varsigma_{i,j}\mathcal{S}_2^{i,j}(x,t),\end{aligned} \tag{17}$$

where

$$\mathcal{S}_1^{i,j}(x,t) = \frac{\partial \chi_{\mathcal{L},i}^{(\rho_1,\sigma_1)}(x)}{\partial x}\Omega_{\mathcal{T},j}^{(\rho_2,\sigma_2,\varepsilon)}(t), \quad \mathcal{S}_2^{i,j}(x,t) = \frac{\partial^2 \chi_{\mathcal{L},i}^{(\rho_1,\sigma_1)}(x)}{\partial x^2}\Omega_{\mathcal{T},j}^{(\rho_2,\sigma_2,\varepsilon)}(t),$$

$$\begin{aligned}\frac{\partial \chi_{\mathcal{L},i}^{(\rho_1,\sigma_1)}(x)}{\partial x} &= \frac{\partial \mathcal{J}_{\mathcal{L},i}^{(\rho_1,\sigma_1)}(x)}{\partial x} + \xi_i\frac{\partial \mathcal{J}_{\mathcal{L},i+1}^{(\rho_1,\sigma_1)}(x)}{\partial x} + \zeta_i\frac{\partial \mathcal{J}_{\mathcal{L},i+2}^{(\rho_1,\sigma_1)}(x)}{\partial x},\\ \frac{\partial^2 \chi_{\mathcal{L},i}^{(\rho_1,\sigma_1)}(x)}{\partial x^2} &= \frac{\partial^2 \mathcal{J}_{\mathcal{L},i}^{(\rho_1,\sigma_1)}(x)}{\partial x^2} + \xi_i\frac{\partial^2 \mathcal{J}_{\mathcal{L},i+1}^{(\rho_1,\sigma_1)}(x)}{\partial x^2} + \zeta_i\frac{\partial^2 \mathcal{J}_{\mathcal{L},i+2}^{(\rho_1,\sigma_1)}(x)}{\partial x^2}.\end{aligned} \tag{18}$$

A comparable procedure can be performed to the Caputo fractional derivative to get

$$D_t^{\mu}(\Omega_{\mathcal{T},j}^{(\rho_2,\sigma_2,\varepsilon)}(t)) = \Omega_{\mathcal{T},j}^{(\mu,\rho_2,\sigma_2,\varepsilon)}(t) = \mathcal{J}_{\mathcal{T},j}^{(\mu,\rho_2,\sigma_2,\varepsilon)}(t) + \mathcal{V}_j\mathcal{J}_{\mathcal{T},j+1}^{(\mu,\rho_2,\sigma_2,\varepsilon)}(t). \tag{19}$$

So

$$\begin{aligned}D_t^{\mu}u_{Approx}(x,t) &= \sum_{i=0}^{\mathcal{N}}\sum_{j=0}^{\mathcal{M}} \varsigma_{i,j}\chi_{\mathcal{L},i}^{(\rho_1,\sigma_1)}(x)D_t^{\mu}(\Omega_{\mathcal{T},j}^{(\rho_2,\sigma_2,\varepsilon)}(t))\\ &= \sum_{i=0}^{\mathcal{N}}\sum_{j=0}^{\mathcal{M}} \varsigma_{i,j}\chi_{\mathcal{L},i}^{(\rho_1,\sigma_1)}(x)\Omega_{\mathcal{T},j}^{(\mu,\lambda,\rho_2,\sigma_2)}(t)\\ &= \sum_{i=0}^{\mathcal{N}}\sum_{j=0}^{\mathcal{M}} \varsigma_{i,j}\mathcal{S}_3^{i,j}(x,t),\end{aligned} \tag{20}$$

where $\mathcal{S}_3^{i,j}(\mu,x,t) = \chi_{\mathcal{L},i}^{(\rho_1,\sigma_1)}(x)\Omega_{\mathcal{T},j}^{(\mu,\lambda,\rho_2,\sigma_2)}(t)$.
For any $\phi \in S_{2\mathcal{N}+1}[0,1]$ and using the shifted Legendre Gauss-Lobatto quadrature,

we have

$$\int_0^1 \phi(\mu)d\mu = \sum_{\kappa=0}^{\mathcal{K}} \varpi_{\mathcal{K},\kappa}\phi\left(\mu_{\mathcal{K},\kappa}\right), \tag{21}$$

thus, we can approximate the integral term $\int_0^1 \mathcal{W}(\mu)D_t^{\mu}u(x,t)d\mu$ as

$$\begin{aligned}\int_0^1 \mathcal{W}(\mu)D_t^{\mu}u_{Approx}(x,t)d\mu =& \int_0^1 \mathcal{W}(\mu)\sum_{i=0}^{\mathcal{N}}\sum_{j=0}^{\mathcal{M}}\varsigma_{i,j}\mathcal{S}_3^{i,j}(x,t)d\mu \\ =&\sum_{i=0}^{\mathcal{N}}\sum_{j=0}^{\mathcal{M}}\varsigma_{i,j}\Big(\int_0^1 \mathcal{W}(\mu)\mathcal{S}_3^{i,j}(\mu,x,t)d\mu\Big) \\ =&\sum_{i=0}^{\mathcal{N}}\sum_{j=0}^{\mathcal{M}}\sum_{\kappa=0}^{\mathcal{K}}\varsigma_{i,j}\varpi_{\mathcal{K},\kappa}\mathcal{W}(\mu_{\mathcal{K},\kappa})\mathcal{S}_3^{i,j}(\mu_{\mathcal{K},\kappa},x,t) \\ =&\sum_{i=0}^{\mathcal{N}}\sum_{j=0}^{\mathcal{M}}\varsigma_{i,j}\mathcal{S}_4^{i,j}(x,t),\end{aligned} \tag{22}$$

where $\mu_{\mathcal{K},\kappa}$, $\kappa=0,1,\cdots,\mathcal{K}$ are the shifted Legendre Gauss-Lobatto collocation points in the interval $[0,1]$ and $\mathcal{S}_4^{i,j}(x,t) = \sum_{\kappa=0}^{\mathcal{K}} \varpi_{\mathcal{K},\kappa}\mathcal{W}(\mu_{\mathcal{K},\kappa})\mathcal{S}_3^{i,j}(\mu_{\mathcal{K},\kappa},x,t)$.
Now, taking into account (22), we write (9) in the form:

$$\sum_{i=0}^{\mathcal{N}}\sum_{j=0}^{\mathcal{M}}\varsigma_{i,j}\mathcal{S}_4^{i,j}(x,t) = \sum_{i=0}^{\mathcal{N}}\sum_{j=0}^{\mathcal{M}}\varsigma_{i,j}\mathcal{S}_2^{i,j}(x,t)+\mathcal{H}(x,t), \quad (x,t)\in[0,\mathcal{L}]\times[0,\mathcal{T}]. \tag{23}$$

Combining the above-mentioned equations and equalizing the residual of (23) by zero give us

$$\sum_{i=0}^{\mathcal{N}}\sum_{j=0}^{\mathcal{M}}\mathcal{F}_{r,s}^{i,j}\varsigma_{i,j} =\mathcal{H}\Big(x_{\mathcal{L},N,r}^{(\rho_1,\sigma_1)},t_{\mathcal{T},\mathcal{M},s}^{(\rho_2,\sigma_2,\lambda)}\Big), \begin{cases} r=0,\cdots,\mathcal{N}, \\ s=0,\cdots,\mathcal{M}, \end{cases} \tag{24}$$

where

$$\mathcal{F}_{r,s}^{i,j} =\mathcal{S}_4^{i,j}(x_{\mathcal{L},N,r}^{(\rho_1,\sigma_1)},t_{M,s}^{(\rho_2,\sigma_2,\lambda)})-\mathcal{S}_2^{i,j}(x_{\mathcal{L},N,r}^{(\rho_1,\sigma_1)},t_{\mathcal{T},\mathcal{M},s}^{(\rho_2,\sigma_2,\lambda)}).$$

The previous system of algebraic equations can be solved for the unknown coefficients $\varsigma_{i,j}$, $i=0,\ldots,\mathcal{N}$, $j=0,\ldots,\mathcal{M}$.

3.2. NONHOMOGENEOUS CONDITIONS

In this Section we develop the aforesaid algorithm to treat the DOFPDEs,

$$D_t^{\mathcal{W}} u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + \mathcal{H}(x,t),\, 0 \le x \le \mathcal{L},\, 0 \le t \le \mathcal{T}, \tag{25}$$

with the nonhomogeneous conditions

$$u(0,t) = g_1(t),\, u(\mathcal{L},t) = g_2(t),\, u(x,0) = g_3(x),\, 0 \le x \le \mathcal{L},\, 0 \le t \le \mathcal{T}. \tag{26}$$

Here $\mathcal{H}(x,t)$, $g_1(t)$, $g_2(t)$, and $g_3(x)$ are given functions.
Now, we assume the following transformation

$$v(x,t) = u(x,t) - g_1(t) - \frac{x}{\mathcal{L}}(g_2(t) - g_1(t)) + \frac{g_1(0)(\mathcal{L}-x) - \mathcal{L} g_3(x) + x g_2(0)}{\mathcal{L}}.$$

Using the previous mapping, the problem (25)-(26) will transform into another one:

$$D_t^{\mathcal{W}} v(x,t) = \frac{\partial^2 v(x,t)}{\partial x^2} + \mathcal{G}(x,t),\, 0 \le x \le \mathcal{L},\, 0 \le t \le \mathcal{T}, \tag{27}$$

with the homogeneous initial-boundary conditions

$$v(0,t) = v(\mathcal{L},t) = v(x,0) = 0, \quad (x,t) \in [0,\mathcal{L}] \times [0,\mathcal{T}]. \tag{28}$$

The approximate solution can be presented as a truncated series:

$$v_{Approx}(x,t) = \sum_{i=0}^{\mathcal{N}} \sum_{j=0}^{\mathcal{M}} \varsigma_{i,j} \chi_{\mathcal{L},i}^{(\rho_1,\sigma_1)}(x) \Omega_{\mathcal{T},j}^{(\rho_2,\sigma_2,\varepsilon)}(t). \tag{29}$$

Based on the information included in this Subsection and the previous one, we obtain the following compact form

$$u_{Approx}(x,t) = \sum_{i=0}^{\mathcal{N}} \sum_{j=0}^{\mathcal{M}} \varsigma_{i,j} \chi_{\mathcal{L},i}^{(\rho_1,\sigma_1)}(x) \Omega_{\mathcal{T},j}^{(\rho_2,\sigma_2,\varepsilon)}(t) + x a_1(t) + a_0(t) + x(x-\mathcal{L}) b(x).$$

4. NUMERICAL RESULTS AND COMPARISONS

Based on the algorithms discussed above, we list some numerical results. The effectiveness and accuracy of this method appeared when we compared it with other methods.

Table 1

M_E of Example 4.1.

High-order difference schemes [33]					
	Scheme I	Scheme II		Scheme I	Scheme II
h	$N=300, J=50$	$N=10^5, J=20$	$(\Delta\mu)$	$N=2\times10^3, M=200$	$N=2\times10^4, M=200$
			$\frac{1}{2}$	1.485739×10^{-2}	2.960065×10^{-4}
$\frac{\pi}{4}$	4.840656×10^{-3}	1.552013×10^{-4}	$\frac{1}{4}$	3.691933×10^{-3}	1.949080×10^{-5}
$\frac{\pi}{8}$	1.226642×10^{-3}	59.533232×10^{-6}	$\frac{1}{8}$	9.200055×10^{-4}	1.234219×10^{-6}
$\frac{\pi}{16}$	3.093814×10^{-4}	5.915861×10^{-7}	$\frac{1}{16}$	2.283002×10^{-4}	7.608877×10^{-8}
$\frac{\pi}{32}$	7.920334×10^{-5}	3.537711×10^{-8}	$\frac{1}{32}$	5.545448×10^{-5}	3.485687×10^{-9}
	Our method with several choices of $(\rho_1, \sigma_1, \rho_2, \sigma_2)$ and ε				
	$(-\frac{1}{2},-\frac{1}{2},0,0)$			$(-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2})$	
$(\mathcal{N},\mathcal{M},\mathcal{K})$	$\varepsilon=\frac{1}{2}$	$\varepsilon=1$		$\varepsilon=\frac{1}{2}$	$\varepsilon=1$
(4,4,4)	7.87355×10^{-4}	3.27492×10^{-5}		1.49002×10^{-3}	327483×10^{-5}
(6,6,6)	2.75405×10^{-7}	2.75404×10^{-7}		2.75405×10^{-7}	2.75403×10^{-7}
(8,8,8)	1.08344×10^{-9}	1.08344×10^{-9}		1.08344×10^{-9}	1.08344×10^{-9}
(10,10,10)	3.52685×10^{-12}	3.52651×10^{-12}		3.52507×10^{-12}	3.52629×10^{-12}
(12,12,12)	9.76996×10^{-15}	8.65974×10^{-15}		8.65974×10^{-15}	8.65974×10^{-15}

4.1. EXAMPLE 1

We start with the DOFPDEs [33]:

$$
\begin{aligned}
&D_t^{\mathcal{W}}u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{8t^2\sin(x)(t(\ln(t)+6)-6)}{\ln(t)}, 0 \le x \le \pi, 0 \le t \le 0.5,\\
&u(0,t) = u(\pi,t) = u(x,0) = 0, 0 \le x \le \pi, 0 \le t \le 0.5,
\end{aligned}
\tag{30}
$$

where $\mathcal{W}(\mu) = \Gamma(4-\mu)$ and whose exact solution is known and is given by $u(x,t) = 8t^3\sin(x)$. To test the accuracy of our method, a comparison based on the maximum absolute errors (M_E) for Example 4.1 obtained by using our method and high-order difference schemes [33] (second-order method in space and distributed order (Scheme I); fourth-order method in space and distributed order (Scheme II)) is listed in Table 1. From these results, we obtain more accurate approximate solutions than those obtained by the two high-order difference schemes [33]. We have sketched in Figs. 1 and 2 t-direction and x-direction curves of the absolute errors of Example (4.1), respectively. Moreover, we have sketched in Fig. 3 the logarithmic graphs of M_E (*i.e.*, $log_{10}M_E$) obtained by the present method with different values of ($\mathcal{N}=\mathcal{M}=\mathcal{K}=2,4,6,\cdots,12$) at the following four cases

1. Case I: $(\rho_1, \sigma_1, \rho_2, \sigma_2, \varepsilon) = (-\frac{1}{2},-\frac{1}{2},0,0,\frac{1}{2})$,
2. Case II: $(\rho_1, \sigma_1, \rho_2, \sigma_2, \varepsilon) = (-\frac{1}{2},-\frac{1}{2},0,0,1)$,
3. Case III: $(\rho_1, \sigma_1, \rho_2, \sigma_2, \varepsilon) = (-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$,
4. Case IV: $(\rho_1, \sigma_1, \rho_2, \sigma_2, \varepsilon) = (-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},1)$.

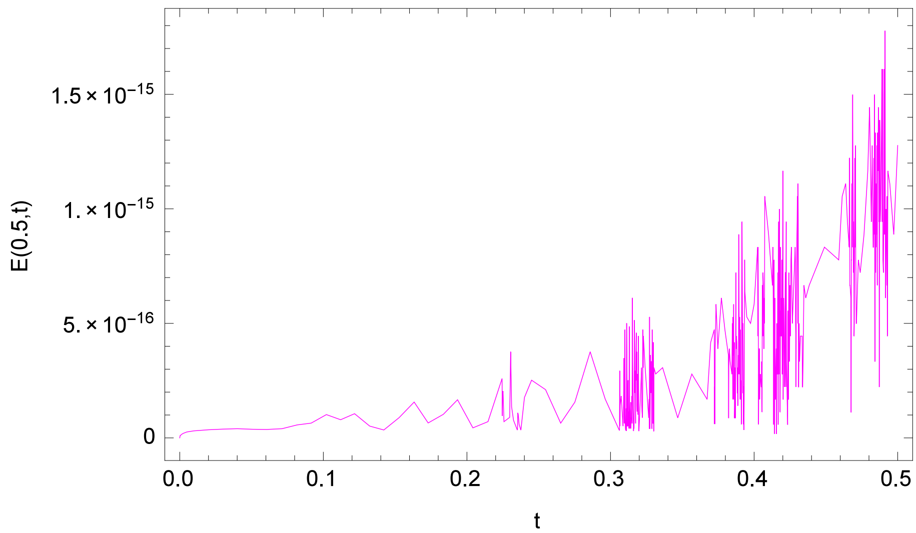

Fig. 1 – t-direction curve absolute errors of Example (4.1) where $\rho_1 = \sigma_1 = -\frac{1}{2}$, $\rho_2 = \sigma_2 = 0,\ \varepsilon = \frac{1}{2}$ and $\mathcal{N} = \mathcal{M} = 12, = \mathcal{K} = 8$.

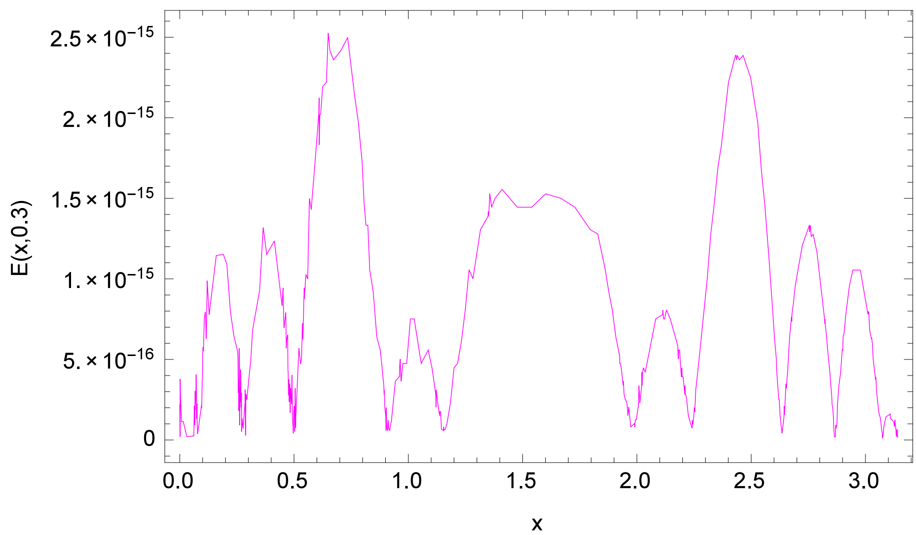

Fig. 2 – x-direction curve absolute errors of Example (4.1) where $\rho_1 = \sigma_1 = -\frac{1}{2}$, $\rho_2 = \sigma_2 = 0$, $\varepsilon = \frac{1}{2}$ and $\mathcal{N} = \mathcal{M} = 12, = \mathcal{K} = 8$.

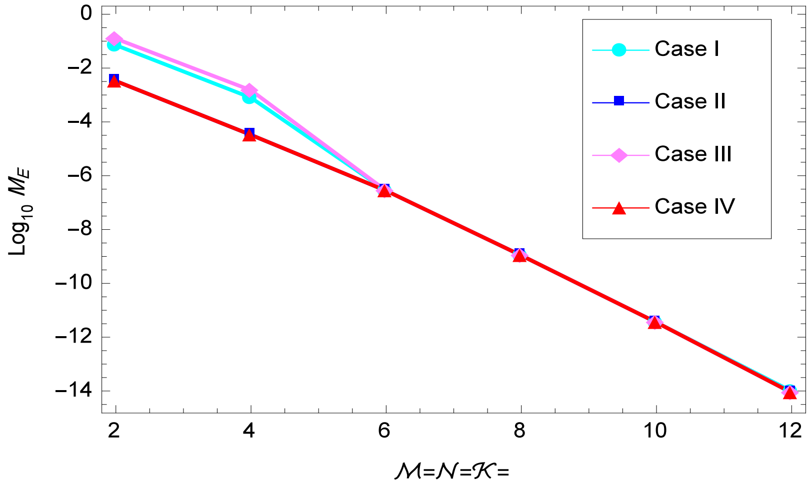

Fig. 3 – M_E convergence of Example (4.1).

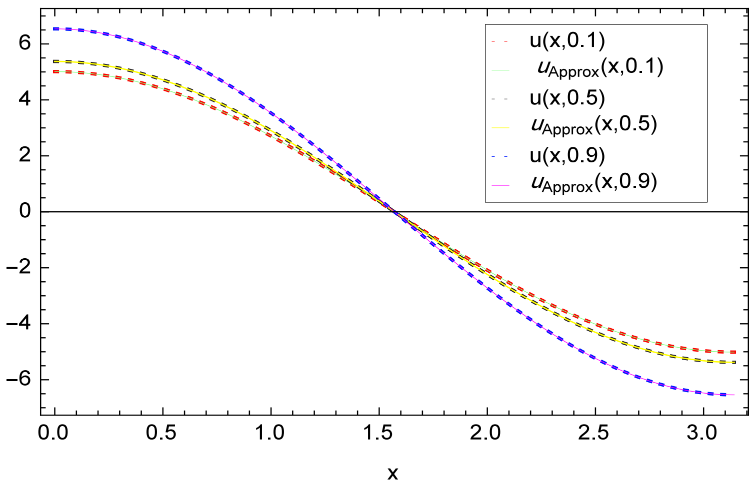

Fig. 4 – x-direction curves of the exact and approximate solutions of Example (4.3) where $\rho_1 = \sigma_1 = -\frac{1}{2}$, $\rho_2 = \sigma_2 = 0$, $\varepsilon = \frac{1}{2}$ and $\mathcal{N} = \mathcal{M} = 14$, $\mathcal{K} = 10$.

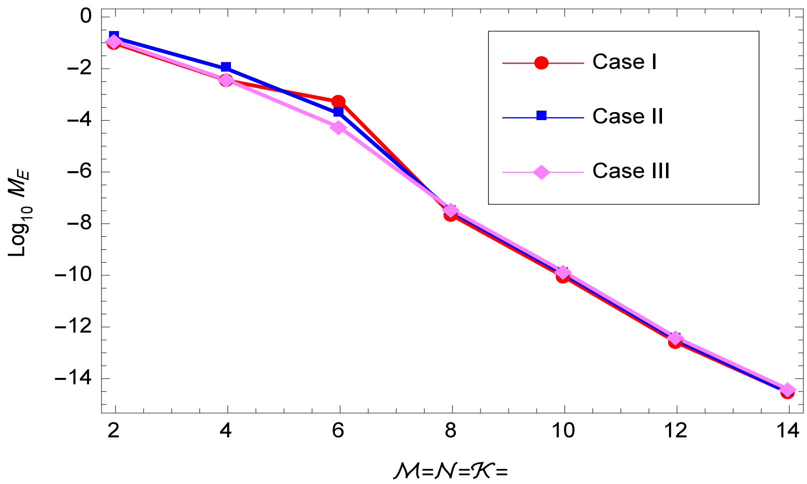

Fig. 5 – M_E convergence of Example (4.3).

4.2. EXAMPLE 2

Here, we test the following DOFPDE [33]:

$$D_t^{\mathcal{W}} u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + 2^{\kappa} t^{\kappa} \sin(x) \left(\frac{(t-1)\Gamma(\kappa+1)}{t \ln(t)} + 1 \right), 0 \leq x \leq \pi, 0 \leq t \leq 0.5,$$
$$u(0,t) = u(\pi,t) = u(x,0) = 0, 0 \leq x \leq \pi, 0 \leq t \leq 0.5, \tag{31}$$

where $\mathcal{W}(\mu) = \Gamma(\kappa + 1 - \mu)$. Its exact solution is

$$u(x,t) = (2t)^{\kappa} \sin(x).$$

Table 2 shows the better results for the M_E of our method, despite that the best result for M_E was acquired by using the high-order difference schemes [33] (second-order method in space and distributed order (Scheme I); fourth-order method in space and distributed order (Scheme II)).

Table 2

M_E of Example 4.2.

High-order difference schemes [33]						
	$\kappa=2$		$\kappa=\frac{3}{2}$		$\kappa=1$	
	Scheme I	Scheme II	Scheme I	Scheme II	Scheme I	Scheme II
τ	$M=300, J=100$	$M=100, J=100$	$M=300, J=100$	$M=100, J=100$	$M=300, J=100$	$M=100, J=100$
$\frac{1}{20}$	2.604194×10^{-3}	2.604625×10^{-3}	1.084362×10^{-3}	1.084956×10^{-3}	1.864457×10^{-2}	1.864251×10^{-2}
$\frac{1}{40}$	7.424869×10^{-4}	7.426864×10^{-4}	3.821180×10^{-4}	3.824538×10^{-4}	1.087577×10^{-2}	1.087337×10^{-2}
$\frac{1}{80}$	2.042197×10^{-4}	2.042987×10^{-4}	1.725077×10^{-4}	1.716824×10^{-4}	6.198339×10^{-3}	6.196568×10^{-3}
$\frac{1}{160}$	5.557797×10^{-5}	5.563078×10^{-5}	7.644502×10^{-5}	7.589344×10^{-5}	3.405682×10^{-3}	3.404474×10^{-3}
$\frac{1}{320}$	1.483300×10^{-5}	1.485131×10^{-5}	3.258713×10^{-5}	3.225094×10^{-5}	1.824234×10^{-3}	1.823451×10^{-3}
Our method with two choices of $(\rho_1, \sigma_1, \rho_2, \sigma_2)$ at $\varepsilon=\frac{1}{2}$						
	$\kappa=2$		$\kappa=\frac{3}{2}$		$\kappa=1$	
$(\mathcal{N},\mathcal{M},\mathcal{K})$	$(0,0,0,0)$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$(0,0,0,0)$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$(0,0,0,0)$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$
(4,4,4)	3.51972×10^{-5}	2.65477×10^{-5}	3.66038×10^{-5}	2.92701×10^{-5}	3.76473×10^{-5}	2.92701×10^{-5}
(6,6,6)	1.98928×10^{-7}	1.47921×10^{-7}	2.04539×10^{-7}	1.52748×10^{-7}	2.07881×10^{-7}	1.52748×10^{-7}
(8,8,8)	7.77546×10^{-10}	5.42187×10^{-10}	7.83586×10^{-10}	5.5101×10^{-10}	7.86503×10^{-10}	5.5101×10^{-10}
(10,10,10)	2.63967×10^{-12}	1.99496×10^{-12}	2.65443×10^{-12}	5.5101×10^{-10}	2.66176×10^{-12}	2.05358×10^{-12}
(12,12,12)	6.55032×10^{-15}	5.27356×10^{-15}	6.27276×10^{-15}	5.27356×10^{-15}	6.27276×10^{-15}	5.27356×10^{-15}

Table 3

M_E of Example 4.3.

Our method with several choices of $(\rho_1, \sigma_1, \rho_2, \sigma_2)$ and ε						
	$(0,0,0,0)$		$(-\frac{1}{2},0,\frac{1}{2},0)$		$(-\frac{1}{2},-\frac{1}{2},0,0)$	
$(\mathcal{N},\mathcal{M},\mathcal{K})$	$\varepsilon=\frac{1}{2}$	$\varepsilon=\frac{1}{3}$	$\varepsilon=\frac{1}{2}$	$\varepsilon=\frac{1}{3}$	$\varepsilon=\frac{1}{2}$	$\varepsilon=\frac{1}{3}$
(4,4,4)	5.44253×10^{-4}	3.448×10^{-3}	9.49768×10^{-4}	9.54743×10^{-3}	7.86932×10^{-4}	3.58341×10^{-3}
(6,6,6)	3.69131×10^{-6}	5.16439×10^{-4}	5.08269×10^{-6}	1.76717×10^{-4}	5.84168×10^{-6}	5.21758×10^{-5}
(8,8,8)	2.08162×10^{-8}	2.08162×10^{-8}	2.81161×10^{-8}	2.81161×10^{-8}	3.31385×10^{-8}	3.31385×10^{-8}
(10,10,10)	8.48404×10^{-11}	8.48395×10^{-11}	1.09116×10^{-10}	1.09118×10^{-10}	1.31384×10^{-10}	1.31384×10^{-10}
(12,12,12)	2.48468×10^{-13}	2.49356×10^{-13}	3.11806×10^{-13}	3.12694×10^{-13}	3.78142×10^{-13}	3.78142×10^{-13}
(14,14,14)	3.33067×10^{-15}	2.83107×10^{-15}	2.83107×10^{-15}	2.83107×10^{-15}	3.71925×10^{-15}	3.71925×10^{-15}

4.3. EXAMPLE 3

Finally, we test the DOFPDE:

$$\begin{aligned}&D_t^{\mathcal{W}}u(x,t)=\frac{\partial^2 u(x,t)}{\partial x^2}+\mathcal{H}(x,t),\, 0\le x\le\pi,\, 0\le t\le 0.5,\\&u(0,t)=u(\pi,t)=u(x,0)=0,\, 0\le x\le\pi,\, 0\le t\le 1,\end{aligned}\tag{32}$$

where

$$\mathcal{W}(\mu)=\Gamma(5-\mu),$$

$\mathcal{H}(x,t)=$

$$\cos(x)\frac{6t^3\ln(t)(4\ln(t)-1)+t^2\left(6\ln^2(t)-8\ln(t)+4\right)-2t(\ln(t)(6\ln(t)-5)+2)}{\ln^3(t)}$$

$+\cos(x)(t^3+t^2+5)$.

Its exact solution is

$$u(x,t)=\left(t^3+t^2+5\right)\cos(x).$$

Based on the M_E acquired by our method, we have summarized some numerical results in Table 3. The results reveal the effectiveness, appropriateness, and high accuracy of our method. Also, we can observe that our numerical solutions coincide closely with the exact ones, see Fig. 4. Moreover, we sketched in Fig. 5 the logarithmic graphs of M_E (i.e., $log_{10}M_E$) obtained by the present method with different values of ($\mathcal{N}=\mathcal{M}=\mathcal{K}=2,4,6,\cdots,14$) at the following three cases:

1. Case I: $(\rho_1,\sigma_1,\rho_2,\sigma_2,\varepsilon)=(0,0,0,0,\frac{1}{3})$,
2. Case II: $(\rho_1,\sigma_1,\rho_2,\sigma_2,\varepsilon)=(-\frac{1}{2},0,\frac{1}{2},0,\frac{1}{3})$,
3. Case III: $(\rho_1,\sigma_1,\rho_2,\sigma_2,\varepsilon)=(-\frac{1}{2},-\frac{1}{2},0,0,\frac{1}{3})$.

5. CONCLUSION

We have presented a collocation technique for solving the aforesaid problem with fractional Jacobi polynomials. The algorithm is efficient, applicable to various operators, and can readily be extended to multi-dimensions. By employing fractional Jacobi polynomials for both temporal and spatial variables, the schemes are treating with the fractional differential operator of distributed order. All numerical computations were fulfilled in reasonable accuracy and with relatively small number of degrees of freedom. We can indicate that our numerical method can also accommodate other methods. For example, the spectral tau approach for non-smooth temporal solution may be fallen apart. Employing fractional-order Jacobi functions instead of the classical Jacobi functions stopped this deterioration.

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