Romanian Reports in Physics **72**, 105 (2020)

TRAJECTORIES OF CHARGED PARTICLES UNDERGOING BROWNIAN MOTION IN A TIME VARIABLE MAGNETIC FIELD

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Received August 6, 2019

Abstract. The present paper aims to study the trajectory of a charged particle in Brownian Motion in a time variable magnetic field. The approach presented here builds on methods of statistical physics and produces an analytical general solution for the stochastic differential equation of motion. A numerical approach is also devised and example trajectories are presented for particular analytical shapes of the external magnetic field.

Key words: Brownian Motion, magnetic field.

1. INTRODUCTION

The motion of a Brownian charged particle is of paramount interest to many applications in physics, electrical engineering, astrophysics and even medical sciences [1]. The issue becomes even more interesting to applied fields of science if a time dependent magnetic field can be accommodated in the equation of motion of such a particle.

The paper begins by introducing some prerequisite information, regarding stochastic processes and motion of charged particles in magnetic fields (Section 2). Next, in Section 3, the equation of motion and differential equations for the means, variances and covariances of the positions and velocities of the particle are detailed for the case of a time-dependent magnetic field. A general numerical formula and some results for particular analytical shapes of the external magnetic field are discussed in Section 4. Conclusions follow in the last Section.

2. PREREQUISITES

There is a lot of literature to work with, both related to the analytical and numerical treatment of the phenomenon. We first present in short the theory pertaining to the Brownian motion of a neutral particle [2, Section 12.14], [3, Chapter 3], [1, Chapter 5], [4, Chapter 8], followed by a short description of classical motion of a charged particle in a constant and time dependent magnetic field [2, Chapter 26], [5, Chapter 5], [6], [7], [8]. Then in Section 3 we will merge the two problems into one.

2.1. CLASSICAL BROWNIAN MOTION OF A NEUTRAL PARTICLE

Mathematically, the equation

$$V(t+dt) - V(t) = -\gamma V(t)dt + \sigma dW(t) \tag{1}$$

is used to describe the velocity $V(t)$ of a particle in erratic motion in a fluid of a bulk viscosity γ [4, Section 7.1]. The erratic part is quantified by the second term in the right hand side, where $W(t)$ is the standard Wiener process and σ is an amplitude. The Wiener process is the solution to the equation

$$W(t+dt) - W(t) = \sqrt{dt} N_t^{t+dt}(0,1), \tag{2}$$

where $N_t^{t+dt}(0,1)$ is a number drawn from a unit normal distribution $\mathcal{N}(0,1)$ in the short time interval between the t and $t+dt$. The index t and exponent $t+dt$ notation is used to make clear the fact that for each timestep, a new (different) number is drawn. Consider a generic time t at which such a number is drawn and use the shorthand notation $N_t^{t+dt}(0,1) \equiv N_t$. The properties of the unit normal are such that

$$\langle N_t \rangle = 0 \quad \text{and} \quad \langle N_t N_{t'} \rangle = \delta_{tt'}, \tag{3}$$

where here and throughout the paper $\langle \cdot \rangle$ denotes the ensemble average of its argument.

The properties of the standard Wiener process are [1, Section 1.1]

$$\langle W(t) \rangle = 0 \quad \text{and} \quad \langle W(t)W(t') \rangle = min(t,t'), \tag{4}$$

$$\langle dW(t) \rangle = 0 \quad \text{and} \quad \langle dW(t)dW(t') \rangle = dt\delta_{tt'}. \tag{5}$$

The trajectory $X(t)$ of a particle with the velocity $V(t)$ is the solution of the equation

$$\frac{dX(t)}{dt} = V(t). \tag{6}$$

Both the trajectory and the velocity of a particle in Brownian motion are stochastic processes. Equation (1) was put forward in this form by Langevin to explain the erratic motion of a particle in a system of other particles and is called the Langevin equation. The motion is also known as Brownian motion due to historical reasons.

Numerical approaches for the solution of the stochastic differential equation describing classical Brownian motion span several decades *e.g.*, [9, 10].

2.2. CLASSICAL MOTION OF A CHARGED PARTICLE IN AN EXTERNAL MAGNETIC FIELD

The equation of motion for an electric charge in classical motion in a constant magnetic field is, in components, [5, Chapter 5]

$$dv_x = -\nu v_x dt + \Omega v_y dt, \tag{7}$$

$$dv_y = -\nu v_y dt - \Omega v_x dt, \tag{8}$$

$$dv_z = -\nu v_y dt \tag{9}$$

$$dx = v_x dt, \quad dy = v_y dt, \quad dz = v_z dt, \tag{10}$$

where ν is the friction coefficient and Ω is the (constant) Larmor frequency.

Now, consider the case in which a charged particle is in deterministic motion in a time-dependent magnetic field. We will not discuss here the cause of this field but just study its effect on the trajectory of the particle.

When the external magnetic field is time-dependent, Faraday's law tells us that an induced electric field will appear; this electric field will obey the equation [2, Section 27.1.2]

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}. \tag{11}$$

We choose a coordinate system such that the external magnetic field is parallel to the $\hat{z}$ axis. Since the magnetic field is parallel to $\hat{z}$ and the curl of a vector is perpendicular to that vector, this leads to $\vec{E} \perp \hat{z}$. We may choose the axes in the perpendicular plane such that the electric field is parallel to one of the axes, *i.e.*, $\vec{E} \parallel \hat{x}$. With $\vec{B}(t) = (0,0,B(t))$ and $\vec{E}(x,y,z,t) = (E(x,y,z,t),0,0)$, let us write the Faraday Equation (11) in components:

$$\hat{x} : 0 = 0, \quad \hat{y} : \frac{\partial E}{\partial z} = 0, \quad \hat{z} : \frac{\partial E}{\partial y} = \frac{\partial B}{\partial t}. \tag{12}$$

From the second equation it results that E does not depend on z. From the third it results that E does not depend on x and that it depends on y linearly:

$$E(y,t) = y\frac{\partial B}{\partial t}. \tag{13}$$

In this configuration, the equation of motion is, in components:

$$dv_x = -\nu v_x dt + \Omega v_y dt + y d\Omega, \tag{14}$$

$$dv_y = -\nu v_y dt - \Omega v_x dt, \tag{15}$$

$$dv_z = -\nu v_y dt, \tag{16}$$

$$dx = v_x dt, \quad dy = v_y dt, \quad dz = v_z dt. \tag{17}$$

3. BROWNIAN MOTION OF A CHARGED PARTICLE IN A TIME DEPENDENT EXTERNAL MAGNETIC FIELD

The issue of Brownian Motion in a constant magnetic field has been discussed both analytically and numerically [4, 11–16] in an effort again spanning a couple of decades.

To analyze the effect of a time variable magnetic field, consider the framework in which Equations (11)-(13) hold. Then the Brownian Motion of a charged particle in a time dependent (deterministic) magnetic field is described by*

$$dV_x = -\nu V_x dt + \Omega V_y dt + Y d\Omega + \sqrt{\beta^2 dt} N_x, \tag{18}$$

$$dV_y = -\nu V_y dt - \Omega V_x dt + \sqrt{\beta^2 dt} N_y, \tag{19}$$

$$dV_z = -\nu V_y dt + \sqrt{\beta^2 dt} N_z, \tag{20}$$

$$dX = V_x dt, \quad dY = V_y dt, \quad dZ = V_z dt, \tag{21}$$

where ν is the friction coefficient and β is the amplitude of the noise term, such that the fluctuation dissipation theorem

$$\frac{\beta^2}{2\nu} = \frac{k_B T}{2} \tag{22}$$

is obeyed in a system with temperature T.

The term $Y d\Omega$ was obtained by the replacement of the term $-\frac{e}{m} E_x dt$ in the equation of motion with $-\frac{e}{m}\frac{\partial B}{\partial t} dt$. Then the notation $\Omega = -e/mB$ was used, and further

$$\frac{\partial B}{\partial t} = -\frac{m}{e}\frac{\partial \Omega}{\partial t},$$

such that the term induced by the time dependent magnetic field becomes

$$-\frac{e}{m} E_x dt = y d\Omega,$$

*Please note that throughout the paper we use the convention that sure processes are denoted by lower case letters and stochastic processes are denoted by upper case letters.

where we see that $d\Omega \sim dt$, *i.e.*, in future calculations, $d\Omega^2 \to 0$ and $d\Omega dt \to 0$.

With respect to Brownian Motion in a constant magnetic field (situation detailed in *e.g.*, [16]), the terms $\Omega V_y dt$ and $\Omega V_x dt$ have an additional time dependency through Ω and the term $Y d\Omega$ is completely new.

This is the system of equations that, together with initial conditions and a specified analytical shape for the time dependency of $B(t)$, describes the motion of a charged particle in a time dependent magnetic field, while subjected to friction and noise.

The general algorithm to solve this system has been put forward in concise form in [4] and contains the steps: 1) Check if the stochastic processes are normal, 2) If so, go through the steps required to find their means, variances and covariances, 3)Write the solution to Equations (18)-(21).

We will apply this algorithm here. Based on considerations explained in [4, Section 7], the stochastic processes involved are normal, i.e., they may be written as

$$X(t) = \mathcal{N}\left(\mu_X(t), \sigma_X^2(t)\right), \tag{23}$$

$$Y(t) = \mathcal{N}\left(\mu_Y(t), \sigma_Y^2(t)\right), \tag{24}$$

$$Z(t) = \mathcal{N}\left(\mu_Z(t), \sigma_Z^2(t)\right), \tag{25}$$

$$V_x(t) = \mathcal{N}\left(\mu_{vx}(t), \sigma_{vx}^2(t)\right), \tag{26}$$

$$V_y(t) = \mathcal{N}\left(\mu_{vy}(t), \sigma_{vy}^2(t)\right), \tag{27}$$

$$V_z(t) = \mathcal{N}\left(\mu_{vz}(t), \sigma_{vz}^2(t)\right). \tag{28}$$

The advantage of this approach is that the problem now reduces to finding differential equations for the sure variables means, variances and covariances of positions and velocities.

The next step is to find and solve the differential equations associated to the means, variances and covariances of these normal stochastic processes. Let us clearly denote our variables

$$varX = \sigma_X^2, \quad varY = \sigma_Y^2, \quad varV_x = \sigma_{vx}^2, \quad varV_y = \sigma_{vy}^2, \tag{29}$$

$$covXV_x = K_x, \quad covYV_y = K_y, \quad covXV_y = H_x, \quad covYV_x = H_y, \tag{30}$$

$$covV_xV_y = L_1, \quad covXY = L_2, \tag{31}$$

and recall that they are time dependent.

We follow the algorithm used in [16], but applied to our particular case of time variable magnetic field, to find that the means, variances and covariances obey the following differential equations

$$\frac{d\mu_X}{dt} = \mu_{vx}, \quad \frac{d\mu_Y}{dt} = \mu_{vy}, \tag{32}$$

$$\frac{d\mu_{vx}}{dt} = -\nu\mu_{vx} + \Omega\mu_{vy} + \mu_Y \frac{d\Omega}{dt}, \quad \frac{d\mu_{vy}}{dt} = -\nu\mu_{vy} - \Omega\mu_{vx}, \tag{33}$$

$$\frac{d\sigma_X^2}{dt} = 2K_x, \quad \frac{d\sigma_Y^2}{dt} = 2K_y, \tag{34}$$

$$\frac{d\sigma_{vx}^2}{dt} = -2\nu\sigma_{vx}^2 + 2\Omega L_1 + \beta^2 + 2H_y \frac{d\Omega}{dt}, \tag{35}$$

$$\frac{d\sigma_{vy}^2}{dt} = -2\nu\sigma_{vy}^2 - 2\Omega L_1 + \beta^2, \tag{36}$$

$$\frac{dL_2}{dt} = H_x + H_y, \tag{37}$$

$$\frac{dK_x}{dt} = -\nu K_x + \Omega H_x + \sigma_{vx}^2 + L_2 \frac{d\Omega}{dt}, \tag{38}$$

$$\frac{dH_x}{dt} = -\nu H_x - \Omega K_x + L_1, \tag{39}$$

$$\frac{dH_y}{dt} = -\nu H_y + \Omega K_y + L_1 + \sigma_Y^2 \frac{d\Omega}{dt}, \tag{40}$$

$$\frac{dK_y}{dt} = -\nu K_y - \Omega H_y + \sigma_{vy}^2, \tag{41}$$

$$\frac{dL_1}{dt} = -2\nu L_1 - \Omega\sigma_{vx}^2 + \Omega\sigma_{vy}^2 + K_y \frac{d\Omega}{dt}. \tag{42}$$

The motion on the z axis is the classical Brownian Motion, since the magnetic field does not affect motion parallel to itself. The differential equations and solutions are known, but we will write them here and throughout the paper for consistency

$$\frac{d\mu_Z}{dt} = \mu_{Vz}(t), \quad \frac{d\mu_{Vz}}{dt} = -\nu\mu_{Vz}(t), \tag{43}$$

$$\frac{d\sigma_Z^2}{dt} = 2covZV_z, \quad \frac{d\sigma_{Vz}^2}{dt} = -2\nu\sigma_{Vz}^2 + \beta^2, \tag{44}$$

$$\frac{dcovZV_z}{dt} = -\nu covZV_z + \sigma_{Vz}^2. \tag{45}$$

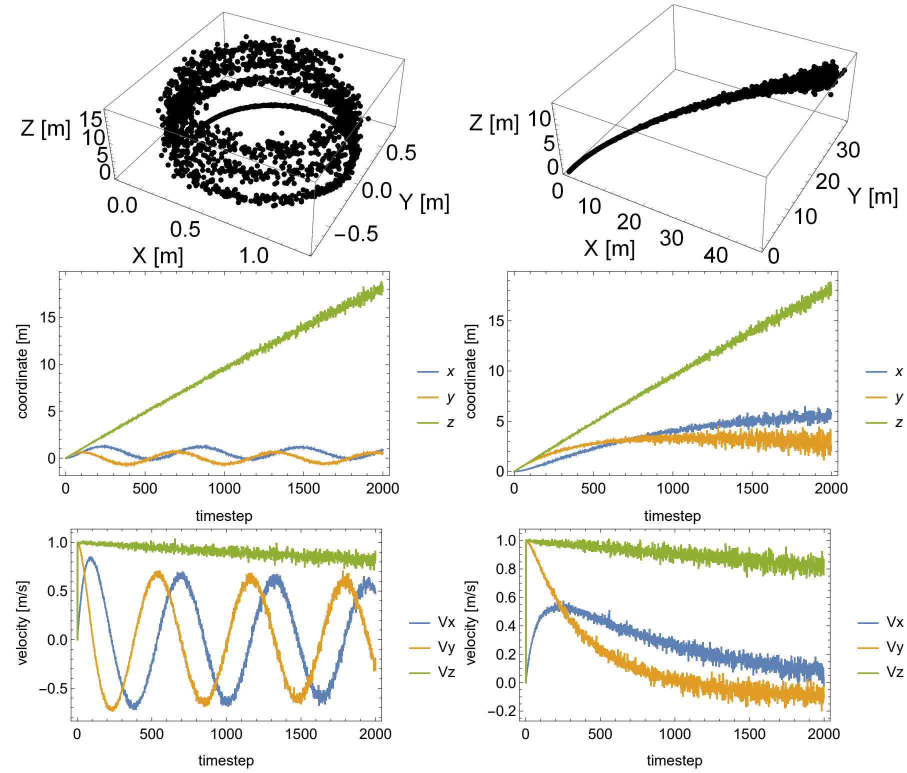

Fig. 1 – Trajectories and velocities for the parameter set
$X_0 = 0, Y_0 = 0, V_{x0} = 0, V_{y0} = 1, z_0 = 0; V_{z0} = 1, \nu = 0.01, \Omega_0 = 1, \beta = 0.01, h = 0.01$ and left: $\Omega(t) = \Omega_0(1 + e^{-t\Omega_0})$; right: $\Omega(t) = \Omega_0/(1 + \Omega_0 t)$.

In the Equations (32)-(45), only the parameters ν and β are constant; the other parameters are generally functions of time.

In the presence of the time dependent magnetic field, all the symmetries observed by [16] disappear. As a consequence, $\sigma_X \neq \sigma_Y$, $\sigma_{vx} \neq \sigma_{vy}$, $L_1 \neq 0$, $L_2 \neq 0$, $K_x \neq K_y$, $H_x \neq -H_y$.

As soon as the particular analytical shape of the external magnetic field is set, the Equations (32)-(45) may be numerically solved and so the solutions (23)-(28) are fully specified.

4. NUMERICAL APPROACH AND RESULTS

To produce trajectories and velocities for further analysis, one needs a numerical procedure. There is one available, in which the instantaneous value of a stochastic process is a linear combination between its mean value and a set of independent unit normals (as explained in, *e.g.*, [10, 16]). The coefficients in the linear combination depend on the variances and covariances of the stochastic processes dictating the behavior of the system.

It is apparent that for each time step t the normal random variables $X(t)$, $Y(t)$, $V_x(t)$ and $V_y(t)$ are dependent. Also, the normal random variables $Z(t)$ and $V_z(t)$ are dependent. Since a set of normal dependent random variables is uniquely determined by their mean, variances and covariances, one may write the random variables as linear combinations of standard independent unit normals.

So the next step is to find the coefficients $a_i, b_i, c_i, d_i, e_i, f_i$ in the expressions

$$X = \mu_X + a_1 N_1 + a_2 N_2 + a_3 N_3 + a_4 N_4, \tag{46}$$

$$Y = \mu_Y + b_1 N_1 + b_2 N_2 + b_3 N_3 + b_4 N_4, \tag{47}$$

$$V_x = \mu_{vx} + c_1 N_1 + c_2 N_2, \tag{48}$$

$$V_y = \mu_{vy} + d_1 N_1 + d_2 N_2, \tag{49}$$

$$Z = \mu_Z + e_1 N_{1z} + e_2 N_{2z}, \tag{50}$$

$$V_z = \mu_{vz} + f_1 N_{1z} + f_2 N_{2z}, \tag{51}$$

with the constraint that they will obey Equations (32)-(45) for the means, variances and covariances. By imposing these constraints, we find that

$$a_2 = \frac{H_x}{\sigma_{vy}} \quad b_2 = \frac{K_y}{\sigma_{vy}}, \quad c_2 = \frac{L_1}{\sigma_{vy}}, \quad d_2 = \sigma_{vy}, \tag{52}$$

$$c_1 = \sqrt{\sigma_{vx}^2 - c_2^2}, \quad a_1 = \frac{K_x - a_2 c_2}{c_1}, \quad b_1 = \frac{H_y - b_2 c_2}{c_1}, \quad d_1 = 0, \tag{53}$$

$$a_3 = \sqrt{\sigma_X^2 - a_1^2 - a_2^2}, \quad b_3 = \frac{L_2 - a_1 b_1 - a_2 b_2}{a_3}, \tag{54}$$

$$a_4 = 0, \quad b_4 = \sqrt{\sigma_Y^2 - b_1^2 - b_2^2 - b_3^2}, \tag{55}$$

$$e_1 = \frac{covZV_z}{\sigma_{vz}}, \quad e_2 = \sqrt{\sigma_Z^2 - \frac{cov^2 ZV_z}{\sigma_{vz}^2}}, \tag{56}$$

$$f_1 = \sigma_{vz}, \quad f_2 = 0. \tag{57}$$

Please note that generally the coefficients $a_i, b_i, c_i, d_i, e_i, f_i$ are time dependent.

Equations (46)-(51) represent an update procedure for the numerical solution of the system of SDEs (18)-(21).

Using these procedure, sample trajectories and velocities are obtained. Figure 1 left hand side shows a plot of the trajectory for the specified parameter set and $\Omega(t) = \Omega_0(1+e^{-t\Omega_0})$. A plot for the same numerical value of parameters but a different time dependency of the external magnetic field, $\Omega(t) = \Omega_0/(1+\Omega_0 t)$, is shown in the right hand side. Even by such a superficial analysis, the importance of the time dependency of the magnetic field is apparent. For the second case, although all parameters are otherwise identical, the noise seems to be taking over the behavior.

5. CONCLUSIONS

The equation of motion of a charged particle undergoing Brownian Motion in a time dependent magnetic field contains several competing terms. There is bulk friction and a stochastic term, both quantifying the interaction of the particle with the rest of the constituents of the medium. The presence of an external magnetic field (say parallel to the O_z axis) assures that this motion occurs around magnetic field lines such that the projection of the motion on the plane perpendicular to the magnetic field is (roughly) a circle. If this magnetic field is time dependent, as in the case considered in this paper, a series of novel effects appear, most notably the presence of a new term in the equation of motion, proportional to ydB. Aside from the rotation frequency itself being time dependent, symmetry is thus lost in the xOy plane. The stochastic processes in the xOy become more coupled and their statistical characteristics differ from the ones documented for the case of constant magnetic field. The z motion is still decoupled.

This paper offers an analytical and numerical solution to the issue of the random motion of a charged particle in a time variable magnetic field. The envisaged applications are manifold and will be developed in a future paper.

Acknowledgements. This work was supported by a grant of the Romanian Ministry of Research and Innovation, CNCS - UEFISCDI, project number PN-III-P1-1.1-PD-2016-0215, within PNCDI III.

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