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A SHORT PRESENTATION OF ŢIŢEICA'S PAPERS ON POSITRON THEORY

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Abstract. We present the computation of the elementary solutions of the Dirac equation, in the sense of Hadamard, and the subsequent evaluation of the vacuum polarization, worked out by Titeica, in four papers, published in the early forties of the previous century. Even if the quantum electrodynamics has chosen, about ten years later, a different evolution, Titeica's papers, which produced results confirmed by subsequent researches, remain valuable and interesting, from a historical perspective.

Key words: Dirac equation, positron theory, vacuum polarization, Hadamard elementary solutions, Hadamard descent method.

At the end of the thirties in the last century quantum electrodynamics was itself in a development stage that did not allow to foresee the later spectacular successes. Some problems that concerned researchers in those days come out also in the four articles by Ţiţeica [1–4] about positron theory and vacuum polarization. An English translation of these papers, originally published in French, can be found in [5]. Although in the period 1940-1942 when these papers were written the second quantization (named superquantization by the author) was known, significant progress in treating the interaction of the matter with the electromagnetic field will be done only in the next decade.

The main difficulties encountered by the present day reader stem from the ubiquitous use of the hole theory concepts as well as from the missing of a systematic perturbative approach offered *e.g.*, by the Feynman diagrams. To overcome this last difficulty, Dirac [6] proposed at his time an approximate treatment on the lines of Hartree's method of the self-consistent field [7]. Accordingly, each electron has its own wave function, and its motion in a definite electromagnetic field occurs in the same way for all electrons. As a rule, by reducing the motion of a system of particles to individual motions, the correlation effects of the various particles get lost. In the quantum mechanics of a system of electrons there is however a correlation type that cannot be neglected, namely the antisymmetry requierement for the wave function of the whole system. This is the general formulation of the Pauli exclusion principle for electrons. The simplest example, suitable for describing free electrons, is given by the two-point correlation functions. Regarded as density-matrix elements they provide a quantum mechanical version of the distribution functions from statistical mechanics.

The relativistic quantum mechanics of the electron allows for transitions between positive and negative energy states. Dirac proposal [8] permitting matter stability was that positive energy electrons cannot fall down into negative energy states because all the negative energy levels are occupied. If also all positive energy levels were vacant the system of electrons could not be as such observable, the system would be in the vacuum state. Any deviation from this distribution is however observable and signals the presence of particles. When a few negative energy levels are vacant, the resulting holes in the distribution of negative energies behave like particles of the same mass, positive energy, but also positive charge. The experimental discovery of the positron confirmed the conjecture that positrons are the holes of Dirac theory.

Despite conceptual advantages quantum theory did not immediately supplant hole theory. A reason is that the problem of the motion of the relativistic electron in external electromagnetic field could be successfully solved [6, 9] in this framework. Certain important steps towards the solution are described by Ţiţeica with his remarkable pedagogical skill in [1] and [2]. However the utilization of the hole-theory language makes it difficult for the present reader to follow several points in Ţiţeica's derivation. It is the purpose of this paper to clarify these points starting from quantum field theory, although the equivalence to Dirac's hole-theory seems not completely established [10].

In the following we shall adhere to the notations and conventions of [11], which are, in our opinion, closest to those used by Ţiţeica: spacetime coordinates with imaginary fourth coordinate $x = (\vec{x}, x_4 = ix^0 = ict)$, derivatives with respect to coordinates $\partial = (\nabla, \partial_4 = i\partial_0)$, Greek letters from the beginning of the alphabet for spinor indices, from the middle for the four-vectors, 4×4 hermitean γ-matrices obeying

$$\{\gamma_\rho, \gamma_\sigma\} = \gamma_\rho\gamma_\sigma + \gamma_\sigma\gamma_\rho = 2\delta_{\rho\sigma} \ \text{ with } \ \rho,\sigma = 1,\ldots,4. \tag{1}$$

The Dirac equation for the spinor field operator $\Psi(x) = \{\Psi_\alpha(x); \alpha = 1,\ldots,4\}$ in an external field given by the electrodynamic four-potential

$$A(x) = \{A_\rho(x); \rho = 1,\ldots,4\} = \left(\vec{A}(x), A_4(x) = iV(x)\right)$$

reads:

$$(\gamma\cdot\mathcal{D} + \mu)\,\Psi(x) = \left[\gamma\cdot\left(\partial + \frac{i\mathrm{e}}{\hbar c}A\right) + \mu\right]\Psi(x) = 0, \tag{2}$$

where $\mu = \dfrac{mc}{\hbar}$ is the Compton wavenumber of the electron with mass m and charge $-\mathrm{e} < 0$, and the dot $(\cdot)$ denotes the scalar product in Minkowski space.

The field variables $\Psi(x)$ do not represent observables quantities by themselves, but one can construct bilinear expressions in these variables that have simple physical interpretation. An example is the four-vector current, which we write it below such as to account for the complete symmetry between negative (electrons) and positive (holes) charges

$$j_\rho(x) = \frac{i\mathrm{e}}{2}(\gamma_\rho)_{\beta\alpha}\left[\Psi_\alpha(x)\,,\,\bar{\Psi}_\beta(x)\right] \quad \text{with } \bar{\Psi}(x) = \Psi^\dagger(x)\gamma_4 \text{ the Dirac adjoint spinor.} \tag{3}$$

After applying on Eq. (3) the four-derivation ∂ one can eliminate $\gamma\cdot\partial\Psi$ and $\partial\bar{\Psi}\cdot\gamma$ with help of Dirac equation and its adjoint. One is left with the local conservation law $\partial\cdot j(x) = 0$.

The above manipulations have only an orientative value, the important question being where are these operators acting. In the absence of an external field the answer is simply the Fock space [12]. Second quantization methods allow then the construction of states with an arbitrary number of particles (including the vacuum state $|0>$ with no particles present), and make possible a computation of the matrix elements of field operators. The field operators Ψ and $\bar{\Psi}$ can be decomposed into positive and negative frequency parts

$$\Psi(x) = \Psi^{(+)}(x) + \Psi^{(-)}(x) \text{ and } \bar{\Psi}(x) = \bar{\Psi}^{(+)}(x) + \bar{\Psi}^{(-)}(x). \tag{4}$$

The positive frequency part signals the "absorption" of an electron and corresponds to the electron annihilation operator. The negative frequency part means the annihilation of an electron of negative energy, *i.e.*, the "emission" of a hole being equivalent to the creation of a positron. Similarly, the positive frequency part of $\bar{\Psi}$, $\bar{\Psi}^{(+)}$, will describe the annihilation of a positron, while $\bar{\Psi}^{(-)}$ the creation of electrons. Since $\bar{\Psi}^{(\pm)} = \overline{\Psi^{(\mp)}}$ one can agree to speak about electrons whenever $\Psi^{(+)}$ is involved, and about positrons in case $\Psi^{(-)}$ is occurring. In quantum field theory the vacuum state $|0>$ contains neither electrons, nor positrons, the absence of the latter implying that all negative energy levels are occupied. One recovers thus the vacuum definition in Dirac's theory of the positron. We note here the following properties of the field-theoretic vacuum state:

$$\Psi^{(+)}(x)|0> = \overline{\Psi^{(-)}}(x)|0> = <0|\Psi^{(-)}(x) = <0|\overline{\Psi^{(+)}}(x) = 0. \tag{5}$$

For the actual purpose the construction of the Fock space is not enough. Strictly speaking, since $\Psi(x)$ is an operator-valued distribution, expressions nonlinear in the quantum field like that in Eq. (3), do not make sense. More flexibility is gained by splitting the point x into two different spacetime points x' and x'', replacing so the commutator at x by the combination [9]

$$\mathcal{R}_{\alpha\beta}(x',x'') = -\frac{1}{2}\left[\Psi_\alpha(x')\,,\,\bar{\Psi}_\beta(x'')\right]. \tag{6}$$

Correspondingly, the expression in Eq. (3) is modified leading to the following regularized form of the current operator:

$$J_\rho(x', x'') = -i\mathrm{e}(\gamma_\rho)_{\beta\alpha}\mathcal{R}_{\alpha\beta}(x', x'') \quad (7)$$

whose matrix elements show a singularity arising along the entire light cone $(x' - x'')^2 = 0$ and not just at $x' = x''$. Note the occurence of the difference $x' - x''$ as a consequence of the translational invariance of free theory. In the next step one recognizes that by subtracting the vacuum expectation value of the regularized current from any other expectation value, a singularity-free result comes out. Hence the operator defined by

$$j_\rho(x', x'') = J_\rho(x', x'') + i\mathrm{e}(\gamma_\rho)_{\beta\alpha} < 0|\mathcal{R}_{\alpha\beta}(x', x'')|0 > \quad (8)$$

has finite matrix elements for all x' and x''. The current conservation law is recovered in the limit $x' = x''$, *i.e.*, $\lim_{x''\to x'}(\partial' + \partial'') \cdot j(x', x'') = 0$ holds, justifying the name (conserved) renormalized current operator for $j_\rho(x', x')$. The *c*-number containing the vacuum expectation value is referred to as the subtractive term. $< 0|\mathcal{R}_{\alpha\beta}(x', x'')|0 >$ obeys the free Dirac equation and its adjoint in x' and x'', respectively, has a characteristic singularity on the light cone (compensating the singularity of the vacuum expectation of the regularized current), and vanishes inside the light cone. Solutions of the Dirac equation with these properties were named following Hadamard [13] elementary solutions.

For later comparison we note here the vanishing of vacuum expectation of the operator in Eq. (8)

$$< 0|j_\rho(x', x'')|0 > = < 0|J_\rho(x', x'')|0 > + i\mathrm{e}(\gamma_\rho)_{\beta\alpha} < 0|\mathcal{R}_{\alpha\beta}(x', x'')|0 > = 0. \quad (9)$$

In order to go beyond the free field theory one needs a device for introducing the interaction. The usual assumption is that the interaction takes a long, but finite time such that the system can be supposed free both in the remote past and in the far future. When Ţiţeica wrote his papers one was aware that the vacuum in the far future, after having undergone the action of the electromagnetic field has to differ from the vacuum in the remote past, but the relationship between these two states provided by a unitary evolution operator was not known. The device will become available a few years later with the S-matrix proposal by Heisenberg [14].

In an arbitrary, time-dependent external field one cannot perform the decomposition Eq. (4). While this is possible for a stationary field, one is occasionally confronted, *e.g.*, for too strong fields, with interpretation difficulties. The usual procedure consists in computing vacuum expectation values by neglecting mutual interactions of the electrons in vacuum state, each electron being considered alone under the action of the existing electromagnetic field. In fact, since the external field cannot be separated from that produced by electrons, the field entering the calculation implicitely takes into account effects of the other electrons in vacuum, such that mutual interactions are not completely left out. The situation reminds the Hartree's calculation [7] of the electron orbits of an atom in the field modified by the electron itself.

Through the interaction with the electromagnetic field a new dynamical principle is introduced, the gauge invariance. The Dirac equation (2) is left invariant when the spinor quantum field $\Psi(x)$ and the external potential $A(x)$ are transformed with a gauge function $\Lambda(x)$, as follows:

$$A_\rho(x) \longrightarrow A_\rho(x) + \partial_\rho \Lambda(x) \text{ and } \Psi_\alpha(x) \longrightarrow \exp\left[-\frac{i\mathrm{e}}{\hbar c}\Lambda(x)\right]\Psi_\alpha(x). \tag{10}$$

A gauge invariant current operator regularized by point-splitting can be introduced by

$$\mathcal{J}_\rho(x',x'') = -i\mathrm{e}(\gamma_\rho)_{\beta\alpha} \exp\left[\frac{i\mathrm{e}}{\hbar c}\int_{x''}^{x'} A(x)\cdot \mathrm{d}x\right]\mathcal{R}_{\alpha\beta}(x',x'') \tag{11}$$

where the integral is taken along the straight line joining the points x'' and x'. On the way of obtaining a renormalized current, see Eq. (8), one has to define the vacuum expectation of $\mathcal{R}_{\alpha\beta}(x',x'')$ in the presence of an external field, $< \mathcal{R}_{\alpha\beta}(x',x'') >_A$, such that it compensates the singularities on the right hand side of (11). Then the operator

$$\mathcal{J}_\rho(x',x'') + i\mathrm{e}(\gamma_\rho)_{\beta\alpha} \exp\left[\frac{i\mathrm{e}}{\hbar c}\int_{x''}^{x'} A(x)\cdot \mathrm{d}x\right] < \mathcal{R}_{\alpha\beta}(x',x'') >_A \tag{12}$$

would be singularity free. However, the regular (finite) part is not uniquely specified by imposing the following two conditions [6] on the A-dependent expectation value: (i) $< \mathcal{R}_{\alpha\beta}(x',x'') >_A$ must be a solution of the Dirac equation (2) in x' and its adjoint in x'', with singularities of the same type as the elementary solution $< 0|\mathcal{R}_{\alpha\beta}(x',x'')|0 >$, and (ii) $< \mathcal{R}_{\alpha\beta}(x',x'') >_A$ must reduce to the elementary solution $< 0|\mathcal{R}_{\alpha\beta}(x',x'')|0 >$ for a vanishing potential $A = 0$. According to a theorem by Hadamard [13] the elementary solution of hyperbolic equations with second order partial derivatives behaves differently in odd and even spacetime dimensions. While in odd dimensions the elementary solution has only algebraic singularities (*e.g.*, half-integer powers of $(x' - x'')^2$), being thus perfectly determined, in even dimensions, due to an additional logarithmic singularity (of the form, *e.g.*, $\log\left[(x'-x'')^2/\mu^2\right]$) there are infinitely many elementary solutions. Certainly, the Dirac equation (2) contains only first order derivatives, can be however brought to a second order equation through the differential operator $\mathcal{D} - \mu$.

To find a unique four-dimensional answer, Ţiţeica formulated and completely solved the problem defined by the conditions (i) and (ii) in five dimensions. On choosing the additional coordinate (the fifth) and the Compton wavenumber μ as Fourier conjugates, he was able to render the "electron in five dimensions" massless, considerably simplifying the search for elementary solutions [15]. He then used the descent trick of Hadamard [13], which consists here in performing the Fourier transform with respect to the fifth coordinate, restoring thus the mass. The result is a unique elementary solution in four dimensions in the presence of the electromagnetic field. The calculation of the vacuum expectation value $< \mathcal{R}_{\alpha\beta}(x', x'') >_A$ for an even-dimensional spacetime by descent method is thoroughly presented in [1].

As in the absence of the external field the conservation law for the operator in Eq. (12) is recovered in the limit of coincidenting points $x' = x''$, see text after Eq. (8).

The anticommutation relations between creation and annihilation operators [16], guarantee that the number of particles of given charge, momentum, and spin is either 0 or 1, and are consistent with both Pauli principle and vacuum definition Eq. (5). Conversely, since for free spinor quantum fields the right hand side of the expression

$$\mathcal{P}_{\alpha\beta}(x', x'') = \left\{ \Psi_\alpha(x')\, ,\, \bar{\Psi}_\beta(x'') \right\} \tag{13}$$

consists of anticommutators of annihilation and creation operators, the result is a c-number equal to the vacuum expectation value. One can easily prove that $< 0|\mathcal{P}_{\alpha\beta}(x', x'')|0 >$ satisfies the free Dirac equation, exhibits the same type of singularities as $< 0|\mathcal{R}_{\alpha\beta}(x', x'')|0 >$, but is vanishing outside the light cone. It represents therefore another, independent elementary solution of the free Dirac equation.

For $t' = t''$ Eq. (13) becomes proportional to $\delta(\vec{x}' - \vec{x}'')$. More precisely,

$$\left\{ \Psi_\alpha(x'), \Psi^\dagger_\beta(x'') \right\} = \delta_{\alpha\beta}\delta(\vec{x}' - \vec{x}'') \ \text{ for } \ t' = t'' \tag{14}$$

is one of the basic canonical anticommutation relations. Equation (14) holds also in the presence of an external field. As discussed above, it is difficult to give a proper definition of the vacuum state for an interacting field theory without using an S-matrix concept. Instead one can resort to the conditions (i) and (ii), and define the expectation value $< \mathcal{P}_{\alpha\beta}(x', x'') >_A$ as the elementary solution of Dirac equation Eq. (2) (and its adjoint) that reduces to the vacuum expectation value $< 0|\mathcal{P}_{\alpha\beta}(x', x'')|0 >$ for $A = 0$. The computation of the anticommutation function performed again by the descent method of Hadamard is the content of [2].

We would like to give now a couple of technical details, which present some interest because they do not use an explicit representation for γ-matrices or the structure of spinor amplitudes. The results can be thus taken over for arbitrary spacetime dimensionality, as has been done by Ţiţeica in [1] and [2]. For calculating the vacuum expectation values of (6) and (13) one introduces the following two-point correlation functions, named also vacuum fluctuations of the free spinor field:

$$\psi^{(+)}_{\alpha\beta}(x', x'') = < 0|\Psi_\alpha(x')\bar{\Psi}_\beta(x'')|0 > \ \text{ and } \ \psi^{(-)}_{\alpha\beta}(x', x'') = < 0|\bar{\Psi}_\beta(x'')\Psi_\alpha(x')|0 > . \tag{15}$$

They transform covariantly under the Lorentz group. Following the original paper by Dirac, Ţiţeica works with the density-matrix elements $\left(\Sigma_4^+\right)_{\alpha\beta}$ and $\left(\Sigma_4^-\right)_{\alpha\beta}$ for positive and negative frequencies, which are obtained from the correlation functions Eq. (15) by replacing $\bar{\Psi}$ with $\Psi^\dagger$. On taking into account the vacuum properties Eq. (5) one gets

$$\left(\Sigma_4^+\right)_{\alpha\beta}(x',x'') \quad =<0|\Psi_\alpha(x')\Psi_\beta^\dagger(x'')|0>=<0|\Psi_\alpha^{(+)}(x')\Psi_\beta^{(+)^\dagger}(x'')|0> \quad (16)$$

$$\left(\Sigma_4^-\right)_{\alpha\beta}(x',x'') \quad =<0|\Psi_\beta^\dagger(x'')\Psi_\alpha(x')|0>=<0|\Psi_\beta^{(-)^\dagger}(x'')\Psi_\alpha(x')|0>. \quad (17)$$

On inserting a complete orthonormal system of one-particle states $\{|n>\}$ in between the two Dirac field operators from Eq. (16), one can write

$$\left(\Sigma_4^+\right)_{\alpha\beta}(x',x'') = \sum_n <0|\Psi_\alpha^{(+)}(x')|n><0|\Psi_\beta^{(+)}(x'')|n>^\star, \quad (18)$$

where the sum entails integration over continuous (wave vector $\vec{q}$) and genuine sum over discrete (spin variable r and sign of the charge) quantum numbers. Since the sign of the charge is opposite to that of the frequency, in the above sum only intermediate states labelled by $|n+>=|\vec{q}\,r+>$ with $r=1,2$ are contributing. Similarly, in

$$\left(\Sigma_4^-\right)_{\alpha\beta}(x',x'') = \sum_n <n|\Psi_\alpha^{(-)}(x')|0><n|\Psi_\beta^{(-)}(x'')|0>^\star, \quad (19)$$

the sum extends over the intermediate states labelled by $|n->=|\vec{q}\,r->$ with $r=3,4$. One can easily show that $<0|\Psi_\alpha^{(+)}(x)|n+>$ and $<n-|\Psi_\alpha^{(-)}(x)|0>$ are plane wave solutions of the free Dirac equation

$$(\gamma\cdot\partial+\mu)\,\Psi(x)=0. \quad (20)$$

We assume that they form a complete orthonormal system of solutions. On introducing the four-vectors $q^{(\pm)} = \left(\vec{q}, \pm i\sqrt{\vec{q}^2+\mu^2}\right)$ and the complete orthonormal spinor amplitudes $u^{(r)}(\vec{q})$ with $r=1,\ldots,4$ one can set

$$<0|\Psi_\alpha^{(+)}(x)|n+>=u_\alpha^{(r)}(\vec{q})\frac{\exp\left(iq^{(+)}\cdot x\right)}{(2\pi)^{3/2}} \quad \text{and}$$

$$<n-|\Psi_\alpha^{(-)}(x)|0>=u_\alpha^{(r)}(\vec{q})\frac{\exp\left(iq^{(-)}\cdot x\right)}{(2\pi)^{3/2}}, \quad (21)$$

where in the first Eq. (21), $r=1,2$, and in the second one, $r=3,4$. On substituting (21) in Eqs. (18) and (19) one gets

$$\left(\Sigma_4^+\right)(x',x'') = (2\pi)^{-3}\int \mathrm{d}\vec{q}\exp\left[iq^{(+)}\cdot(x'-x'')\right]\sum_{r=1}^2 u^{(r)}(\vec{q})u^{(r)^\dagger}(\vec{q}); \quad (22)$$

$$\left(\Sigma_4^-\right)(x',x'') = (2\pi)^{-3}\int \mathrm{d}\vec{q}\exp\left[iq^{(-)}\cdot(x'-x'')\right]\sum_{r=3}^4 u^{(r)}(\vec{q})u^{(r)^\dagger}(\vec{q}). \quad (23)$$

From completeness and orthonormality of spinor amplitudes $u^{(r)}(\vec{q})$ one can show that the matrices $\sum_{r=1}^{2} u^{(r)}(\vec{q})u^{r\dagger}(\vec{q})$ and $\sum_{r=3}^{4} u^{(r)}(\vec{q})u^{r\dagger}(\vec{q})$ are mutually orthogonal projection operators on the two-dimensional subspaces of positive and negative frequencies. As noted by Dirac [6] it is possible to obtain the explicit form of these projection operators without resorting to a specific representation for spinor amplitudes and γ-matrices. Since spin variables play here no role, one can look for plane wave solutions $\psi(x;\vec{q})$ of wave vector $\vec{q}$ represented at time $t = 0$ by $(2\pi)^{-(3/2)}\exp(i\vec{q}\vec{x})$ times the unit matrix in spin space. By expressing the Dirac equation in its original form [17], *i.e.*, in terms of the matrices $\vec{\alpha} = i\gamma_4\vec{\gamma}$ and $\beta = \gamma_4$, one obtains a time-dependent Schrödinger equation for $\psi(t,\vec{x};\vec{q})$. On integrating it with the given initial condition one obtains the following result:

$$\psi(t,\vec{x};\vec{q}) = \frac{1}{2}\left(1+\frac{\vec{\alpha}\vec{q}+\beta\mu}{\sqrt{\vec{q}^2+\mu^2}}\right)\frac{\exp\left(iq^{(+)}x\right)}{(2\pi)^{3/2}} + \frac{1}{2}\left(1-\frac{\vec{\alpha}\vec{q}+\beta\mu}{\sqrt{\vec{q}^2+\mu^2}}\right)\frac{\exp\left(iq^{(-)}x\right)}{(2\pi)^{3/2}}. \tag{24}$$

Obviously, the occurrence of the four-vectors $q^{(+)}$ and $q^{(-)}$ is associated with positive and negative frequencies, respectively. Hence the projection operators have the form

$$\sum_{r=1}^{2} u^{(r)}(\vec{q})u^{r\dagger}(\vec{q}) = \frac{1}{2}\left(1+\frac{\vec{\alpha}\vec{q}+\beta\mu}{\sqrt{\vec{q}^2+\mu^2}}\right)\ ;\ \sum_{r=3}^{4} u^{(r)}(\vec{q})u^{r\dagger}(\vec{q}) = \frac{1}{2}\left(1-\frac{\vec{\alpha}\vec{q}+\beta\mu}{\sqrt{\vec{q}^2+\mu^2}}\right). \tag{25}$$

The substitution of (25) into Eqs. (22) and (23) leads to the matrix densities

$$\Sigma_4^{\pm}(x',x'') = \left[\frac{1}{i}\left(\partial_0' - \vec{\alpha}\nabla'\right) - \beta\mu\right]\int \frac{\mathrm{d}\vec{q}}{(2\pi)^3}\frac{\exp\left[iq^{(\pm)}\cdot(x'-x'')\right]}{\sqrt{\vec{q}^2+\mu^2}}. \tag{26}$$

Note the dependence on $x = x' - x''$, an effect of translation invariance of the free theory. On using again the γ-matrices, the corresponding Lorentz invariant correlation two-point functions (see Eq. (15)) are given by

$$\psi^{(\pm)}\left(x',x''\right) = \mp\left(\gamma\cdot\partial' - \mu\right)\int \frac{\mathrm{d}\vec{q}}{(2\pi)^3}\frac{\exp\left[iq^{(\pm)}\cdot(x'-x'')\right]}{\sqrt{\vec{q}^2+\mu^2}}. \tag{27}$$

With these expressions one can compute the vacuum field fluctuations

$$<0|\mathcal{R}_{\alpha\beta}(x',x'')|0> \quad \text{and} \quad <0|\mathcal{P}_{\alpha\beta}(x',x'')|0>$$

as

$$\begin{aligned} <0|\mathcal{R}_{\alpha\beta}(x',x'')|0> &= \frac{1}{2}\left[\psi_{\alpha\beta}^{(-)}(x',x'') - \psi_{\alpha\beta}^{(+)}(x',x'')\right] \\ <0|\mathcal{P}_{\alpha\beta}(x',x'')|0> &= \psi_{\alpha\beta}^{(-)}(x',x'') + \psi_{\alpha\beta}^{(+)}(x',x''). \end{aligned} \tag{28}$$

Both are solutions of the free Dirac equation Eq. (20). When written in the form

$$\begin{aligned} <0|\mathcal{R}_{\alpha\beta}(x',x'')|0> &= \frac{1}{2}\big(\gamma\cdot\partial'-\mu\big)_{\alpha\beta}\triangle^{(1)}\left(x'-x'',\mu^2\right) \\ <0|\mathcal{P}_{\alpha\beta}(x',x'')|0> &= i\big(\gamma\cdot\partial'-\mu\big)_{\alpha\beta}\triangle\left(x'-x'',\mu^2\right) \end{aligned} \tag{29}$$

with

$$\begin{aligned} \triangle^{(1)}\left(x,\mu^2\right) &= \int\frac{\mathrm{d}\vec{q}}{(2\pi)^3}\exp\left(i\vec{q}\vec{x}\right)\frac{\cos\left(x^0\sqrt{\vec{q}^2+\mu^2}\right)}{\sqrt{\vec{q}^2+\mu^2}} \\ \triangle\left(x,\mu^2\right) &= \int\frac{\mathrm{d}\vec{q}}{(2\pi)^3}\exp\left(i\vec{q}\vec{x}\right)\frac{\sin\left(x^0\sqrt{\vec{q}^2+\mu^2}\right)}{\sqrt{\vec{q}^2+\mu^2}}, \end{aligned} \tag{30}$$

they show characteristic singularities of the Klein-Gordon equation on the light cone $(x'-x'')^2=0$, *i.e.*, they are elementary solutions in the terminology of Hadamard. Furthermore, the four-dimensional density-matrix elements given for $\nu=2$ in Eq. (11) from [1], and in Eq. (2) from [2] are given by

$$\left(\Sigma_4^- - \Sigma_4^+\right)_{\alpha\beta}(x',x'') = <0|\mathcal{R}_{\alpha\gamma}(x',x'')|0>(\gamma_4)_{\gamma\beta} = <0|\left[\Psi_\alpha(x'),\Psi_\beta^\dagger(x'')\right]|0> \tag{31}$$

and

$$\left(\Sigma_4^- + \Sigma_4^+\right)_{\alpha\beta}(x',x'') = <0|\mathcal{P}_{\alpha\gamma}(x',x'')|0>(\gamma_4)_{\gamma\beta} = <0|\left\{\Psi_\alpha(x'),\Psi_\beta^\dagger(x'')\right\}|0>, \tag{32}$$

respectively.

In [3] the author develops a canonical formalism for the relativistic charged particle (charge $-$e, mass m) in an external electromagnetic field. The motion takes place on a world line $x=x(\tau)$, where τ denotes the proper time defined by (the dot means time derivative)

$$c\frac{\mathrm{d}\tau}{\mathrm{dt}} = \sqrt{c^2-\dot{\vec{x}}^2}. \tag{33}$$

While the usual canonical formulation based upon spatial coordinates $\vec{x}$ and canonic conjugate momenta $\vec{p}$ is necessary in order to pass to quantum mechanics, it does not include the constraint Eq. (33). This constraint can be derived by enlarging the system of canonical variables with two more conjugate variables s and P. On associating the s-independent Hamilton function

$$\mathcal{H}(\vec{x},s,\vec{p},P;t) = c\sqrt{P^2+\left(\vec{p}+\frac{\mathrm{e}}{c}\vec{A}\right)^2}-\mathrm{e}V \tag{34}$$

one can write down the eight canonical equations. A simple calculation gives

$$\dot{\vec{x}}^2+\dot{s}^2 = \left(\frac{\partial\mathcal{H}}{\partial\vec{p}}\right)^2+\left(\frac{\partial\mathcal{H}}{\partial P}\right)^2 = c^2, \tag{35}$$

which is equivalent to the constraint Eq. (33) provided that $s = c\tau$. Also from the canonical equation

$$\dot{P} = -\frac{\partial \mathcal{H}}{\partial s} = 0 \tag{36}$$

it follows $P = const = mc$.

We are now ready to construct the Dirac equation in a five-dimensional space-time comprising the usual four coordinates $(\vec{x}, x_4 = ict)$ and the additional coordinate s. Since the Clifford algebra in five-dimensions still uses 4×4 matrices, the five-dimensional wave function, denoted by $\kappa(x,s)$, has four components. The momentum P conjugate to s, which we identified with mc, has to be replaced through $-i\hbar\frac{\partial}{\partial s}$. This amounts to the substitution $\mu \longrightarrow -i\frac{\partial}{\partial s}$ in the Dirac equation in four dimensions. We further describe the electromagnetic field by a four-potential $A = (\vec{A}, A_4 = iV)$ depending on x, but not on s, and take the additional component A_s vanishing. Recalling the minimal prescription for introducing the electromagnetic field $\partial \longrightarrow \mathcal{D} = \partial + \frac{i\mathrm{e}}{\hbar c}A$, one obtains the five-dimensional Dirac equation

$$\left(\gamma \cdot \mathcal{D} - i\frac{\partial}{\partial s}\right)\kappa(x,s) = \left[\gamma \cdot \left(\partial + \frac{i\mathrm{e}}{\hbar c}A\right) - i\frac{\partial}{\partial s}\right]\kappa(x,s) = 0. \tag{37}$$

The construction of a conserved five-vector current density has been given in [18].

One can now make explicit the descent procedure by replacing the operator $-i\frac{\partial}{\partial s}$ with μ, multiplying then the whole equation by $\exp(-i\mu s)$, and integrating with respect to s from $-\infty$ to ∞. The four-component wave function

$$\psi(x) = \int_{-\infty}^{\infty} \exp(-i\mu s)\kappa(x,s)\mathrm{d}s. \tag{38}$$

satisfies the Dirac wave equation Eq. (2).

In [1] and [2] the elementary solutions have been obtained from the second-order Dirac equation in five dimensions. On setting

$$\kappa(x,s) = \left(\gamma \cdot \mathcal{D} + i\frac{\partial}{\partial s}\right)\omega(x,s) = \left[\gamma \cdot \left(\partial + \frac{i\mathrm{e}}{\hbar c}A\right) + i\frac{\partial}{\partial s}\right]\omega(x,s), \tag{39}$$

Equation (37) yields the following formula for the spinor $\omega(x,s)$:

$$\left[\partial \cdot \partial + \frac{\partial^2}{\partial s^2} + \frac{i\mathrm{e}}{\hbar c}(2A \cdot \partial + \partial \cdot A) - \frac{i\mathrm{e}}{\hbar c}\sigma_{\rho\sigma}F_{\rho\sigma}\right]\omega(x,s) = 0, \tag{40}$$

where $\sigma_{\rho\sigma} = -i/4\,[\gamma_\rho, \gamma_\sigma]$ and $F_{\rho\sigma} = \partial_\rho A_\sigma - \partial_\sigma A_\rho$ is the electromagnetic field strength. Note that $\partial \cdot \partial + \frac{\partial^2}{\partial s^2}$ represents the five-dimensional d'Alembertian. One concludes then, on simply dimensional grounds, that the elementary solution of (40) must be of the form $\left(x^2+s^2\right)^{-3/2}$ times a function regular in the five variables x and s. Note also the absence of the mass term in Eq. (40).

This can happen in five dimensions by taking an additional spatial coordinate $x_5 = s$ as Fourier conjugate to the Compton wave number μ. From a mathematical point of view the parameter μ can take any real value. Let us denote with $\Psi(x;\mu)$ the corresponding solution of Dirac equation (20). On multiplying Eq. (20) by $\exp(i\mu s)/(2\pi)$ and integrating over μ from $-\infty$ to ∞ under the assumption of the existence of Fourier transform, one finds

$$(\gamma\cdot\partial - i\partial_s)\,\Xi(x,s) = 0 \tag{41}$$

with

$$\Xi(x,s) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\mu}{2\pi}\exp(i\mu s)\Psi(x;\mu). \tag{42}$$

Here $\Xi(x,s) = \{\Xi_\alpha(x,s); \alpha = 1,\ldots,4\}$ is a four-component spinor in five dimensions. More generally, 2^ν-component spinors can be defined in spacetimes with 2ν or $2\nu+1$ dimensions. The massive Dirac equation Eq. (20) results from Eq. (41) by inverting the Fourier transform Eq. (42) as

$$\Psi(x) = \int_{-\infty}^{\infty} \mathrm{d}s \exp(-i\mu s)\,\Xi(x,s). \tag{43}$$

The above procedure allows to get the matrix densities $\Sigma_5^{\pm}(x',s';x'',s'')$ in five dimensions by Fourier transforming Eq. (26) with respect to $s'-s''$

$$\begin{aligned}
&\Sigma_5^{\pm}(x',s';x'',s'') = \\
&\int_{-\infty}^{\infty}\frac{\mathrm{d}\mu}{2\pi}\exp\left[i\mu\left(s'-s''\right)\right]\left[\frac{1}{i}\left(\partial_0' - \vec{\alpha}\nabla'\right) - \beta\mu\right]\int\frac{\mathrm{d}\vec{q}}{(2\pi)^3}\frac{\exp\left[iq^{(\pm)}\cdot(x'-x'')\right]}{\sqrt{\vec{q}^2+\mu^2}} \\
&= \frac{1}{i}\left(\partial_0' - \vec{\alpha}\nabla' - \beta\partial_s'\right)\int\frac{\mathrm{d}\vec{q}\,\mathrm{d}\mu}{(2\pi)^4}\frac{\exp\left\{i\left[q^{(\pm)}\cdot(x'-x'')+\mu\left(s'-s''\right)\right]\right\}}{\sqrt{\vec{q}^2+\mu^2}}.
\end{aligned} \tag{44}$$

(To avoid here any confusion concerning the meaning of μ in five dimensions Ţiţeica sets it equal to the last component l of the wavevector $\vec{q}l$). On taking into account translational invariance one finds for the five-dimensional vacuum fluctuations in the absence of an external field the expressions

$$\frac{1}{2}\left[\kappa_{\alpha\beta}^{(-)}(x,s) - \kappa_{\alpha\beta}^{(+)}(x,s)\right] = \left(\gamma\cdot\partial + i\frac{\partial}{\partial s}\right)_{\alpha\beta}\left[\frac{\theta(x^2+s^2)}{8\pi^2}\left(x^2+s^2\right)^{-3/2}\right]; \tag{45}$$

$$\kappa_{\alpha\beta}^{(-)}(x,s) - \kappa_{\alpha\beta}^{(+)}(x,s) = -i\left(\gamma\cdot\partial + i\frac{\partial}{\partial s}\right)_{\alpha\beta}\left[\epsilon(x^0)\frac{\theta(-x^2-s^2)}{4\pi^2}\left(-x^2-s^2\right)^{-3/2}\right]. \tag{46}$$

Here $\theta(\xi)$ is the Heaviside step function and $\epsilon(x^0)$ is the signum of x^0.

The last paper [4] deals with vacuum polarization. The physics behind is quite clear: an external electromagnetic field will act on the vacuum fluctuations of charged matter quantum fields in much the same way as on the charges of a material medium. It can lead to a change in the current distribution and thereby to a modification of the Maxwell equations. Put in a more formal way this amounts to the calculation of vacuum expectation value $< \ \dots \ >_A$ of the operator in Eq. (12). A perturbative approach for the computation of $< J_\rho(x', x'') >_A$ has been proposed by Heisenberg [9] and subsequently developed by other authors [19–21]; for a quantum field-theoretic approach, see [22]. The subtractive (second) term in (12) is calculated by Ţiţeica by applying the descent method directly on the first order Dirac equation (37). A comparison of the latter equation with Eq. (40) shows a singularity of the form $\left(x^2 + s^2\right)^{-5/2}$ for the elementary solution, since $\kappa(x, s)$ needs one more derivative than $\omega(x, s)$. On restricting the calculation to lowest order in the fine structure constant $\mathrm{e}^2/(\hbar c)$, and on using the perturbative results quoted by [9] and [21], Ţiţeica found no modification of the Maxwell equations induced by a plane wave electromagnetic field.

Today it seems difficult to make a realistic estimation of the above four papers, since they appeared eighty years ago in scientific journals of local spreading. For this reason, despite an increasing interest in the addressed topics after the Second World War, the scientific community could not notice the elegant computation of elementary solutions of the Dirac equation, nor the subsequent evaluation of the vacuum polarization. Of course, all the problems, first solved in [1–4], have been successfully approached later on, and became solutions in agreement with those found by Ţiţeica. Even so, we consider meritorious a republication of these papers, which will thus benefit from modern typographical performances. Personally, I enjoyed once again the clarity and elegance of Ţiţeica's phrase that made the irrefutable charm of his lectures.

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