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# GEODESICS AND ELECTROMAGNETIC POTENTIALS ASSOCIATED TO A POINT LIKE PARTICLE REVISITED

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*Abstract.* We revisit the geodesic equations for both special and general relativity from the perspective of the field equations for a fermion field. Complete agreement with the known results is obtained. As an application of this method we compute the electromagnetic potentials created by a point like charge in the Minkowski space. The method can have applications in studying the electromagnetic potentials or other dynamical issues associated to a point like particle charged or not in both special or general relativity.

*Key words*: geodesic equations, field equations, fermionic field.

## 1. INTRODUCTION

In classical mechanics the path of a free particle massive or not is given by a straight line. Knowing the initial conditions like the initial position and velocity of the particle determine entirely the future path. With the advent of special relativity and the introduction of the Minkowski space the kinematics of a free particle becomes slightly more complicated [1]. The four momenta of the particle are defined as:

$$p^\mu = m\frac{dx^\mu}{d\tau}, \tag{1}$$

where $d\tau$ is the differential of the proper time for a massive particle:

$$d\tau = \frac{1}{\gamma}dt, \tag{2}$$

Here $\gamma = \frac{1}{1-\frac{v^2}{c^2}}$, with $\vec{v}$ the velocity of the particle, and $t$ is the regular time variable. One may write:

$$p^\mu = m\gamma\frac{dx^\mu}{dt}. \tag{3}$$

The four velocity of the particle is then defined as,

$$u^\mu = \frac{p^\mu}{m} = (\gamma c, \gamma\vec{v}). \tag{4}$$

One can introduce alternatively a four velocity as,

$$v^\mu = (c, \vec{v}), \tag{5}$$

which in natural units becomes:

$$v^\mu = (1, \frac{\vec{v}}{c}). \tag{6}$$

In general relativity where the space time is curved and the principle of equivalence works there are additional assumptions that one might need in order to determine the motion of the free particle [1, 7].

In general relativity one may extract the equation of the geodesic from the equivalence principle [2],

$$\frac{d^2x^\mu}{dt^2} = \\ -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} + \Gamma^0_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}\frac{dx^\mu}{dt}. \tag{7}$$

The geodesic equation may be derived also from the action [3, 4]:

$$S = \int d^4x ds, \tag{8}$$

by introducing the line element,

$$ds = \sqrt{-g_{\mu\nu}dx^\mu dx^\nu}. \tag{9}$$

with the result:

$$\frac{d^2x^\mu}{ds^2} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}. \tag{10}$$

In this short note we shall determine the geodesic equation for a point like particle from the field equation of the particle in the Minkowski or general relativity space-time. We thus aim to fill a gap in the conceptual framework that relates the dynamics and kinematics of a particle. Such an approach may be fruitful in determining not only the electromagnetic potentials created by a point like charge but also other problems connected with the kinematics or dynamics of a point like object.

In section II we derive the geodesic equation in the Minkowski space which is trivial. Section III contains a similar calculation in general relativity. In section IV we determine in our approach the electromagnetic potentials associated to a point like charge in the Minkowski space. Section V is dedicated to the conclusions.

## 2. GEODESIC IN SPECIAL RELATIVITY

We shall start with the action of a massive fermion in the Minkowski space (this can be easily generalized to a massless fermion):

$$S = \int d^4x \left[ i\bar{\Psi}\gamma^{\mu}\partial_{\mu}\Psi + m\bar{\Psi}\Psi \right]. \tag{11}$$

We constrain the fermion $\Psi$ to move on a path $y(y^0, x^0)$. This modifies the action as follows:

$$S = \int d^4x \Bigg[ [\frac{1}{2} i\bar{\Psi}\gamma^{\mu}\partial_{\mu}[\delta(x-y)\Psi] + h.c.] + \\ m\bar{\Psi}\delta(x-y)\Psi \Bigg]. \tag{12}$$

The fermions may be considered of mass dimension zero or a multiplicative constant may be applied in front of the action to restore the correct dimensionality.

We rewrite the action in the Fourier space:

$$\frac{1}{2}\int d^4x \int \frac{d^4q}{(2\pi)^4} \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4r}{(2\pi)^4} \times \\ \exp[i(q-p)x]\exp[-i(x-y)r] \times \\ \bar{\Psi}(q) i\gamma^{\mu} \left[ -ir_{\mu} + i\frac{\partial y^{\rho}}{\partial x^{\mu}} r_{\rho} - 2ip_{\mu} \right] \Psi(p). \tag{13}$$

We consider the equation of motion for $\Psi(p)$ in the Fourier space and reinforce $\gamma^{\mu}p_{\mu} - m = 0$ to get:

$$\int d^4x \int \frac{d^4p}{(2\pi)^4} \frac{d^4r}{(2\pi)^4} \times \\ \exp[i(q-p)x]\exp[-i(x-y)r] \times \\ \gamma^{\mu} \left[ (r_{\mu}) - \frac{\partial y^{\rho}}{\partial x^{\mu}} (r_{\rho}) \right] \Psi(p) = 0. \tag{14}$$

Since the last equation should be independent on $q_{\mu}$ and this term appears in $r_{\mu}$ (this can be seen by applying retrospectively the delta function with a Jacobian factor) one can set the corresponding coefficient to zero which further yields:

$$\delta(x-y)\gamma^{\mu} \left[ -q_{\mu} + \frac{\partial y^{\rho}}{\partial x^{\mu}} q_{\rho} \right] = 0. \tag{15}$$

We multiply by $\gamma_\sigma p^\sigma$ and take the trace taking to get:

$$\delta(x-y)\Bigg[-p^\sigma q_\sigma + \frac{\partial y^\rho}{\partial x^\sigma} q_\rho p^\sigma\Bigg] = 0. \tag{16}$$

We take into account the fact that $y^\rho$ depends only on $x^0$ thus setting $\sigma = 0$:

$$\delta(x-y)\Bigg[-p^\rho q_\rho + \frac{\partial y^\rho}{\partial x^0} p^0 q_\rho\Bigg] = 0. \tag{17}$$

By applying the delta function to Eq. (17) and using the fact that the equation should be independent of $q^\rho$ one obtains:

$$\frac{\partial y^\rho}{\partial y^0} = \frac{p^\rho}{p^0}. \tag{18}$$

Note the complete agreement between Eq. (18) and Eq. (6).

From Eq. (18) one can deduce the path of the free particle in terms of the parameter $y^0$:

$$y^\rho = \frac{p^\rho}{p^0} y^0 + d, \tag{19}$$

where $d$ is a constant depending on the initial position. Note that usually one considers $\frac{dy^\rho}{ds}$ where $ds$ is the proper time, which shows that our formulation differs form the usual four velocity definition through a factor $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$.

## 3. GEODESIC IN GENERAL RELATIVITY

The next step is to consider the fermion action in general relativity:

$$S = \int d^4x \sqrt{-g}\Bigg[\bar{\Psi} i\gamma^\mu(\partial_\mu - \frac{i}{4}\omega_\mu^{ab}\sigma_{ab})\Psi\Bigg]. \tag{20}$$

Here $\omega_\mu^{ab}$ is the spin connection. We rewrite the action for a fermion constrained to move on path $y^\rho(y^0, x^0)$:

$$S = \int d^4x \sqrt{-g}\frac{1}{2}\Bigg[\bar{\Psi}(x) i\gamma^\mu \partial_\mu(\frac{1}{\sqrt{-g}}\delta(x-y)\Psi(x)) - \bar{\Psi}(x)\frac{i}{4}\omega_\mu^{ab}\sigma_{ab}\Psi(x)\frac{1}{\sqrt{-g}}\delta(x-y) + h.c.\Bigg]. \tag{21}$$

We consider that the equation of motion for $\Psi(x)$ is fulfilled according to:

$$i\gamma^\mu(\partial_\mu - \frac{i}{4}\omega_\mu^{ab}\sigma_{ab})\Psi(x) = 0. \tag{22}$$

We apply the equation of motion for $\Psi$ to Eq. (21) and implement Eq. (22) to obtain:

$$\gamma^{\mu}\partial_{\mu}(\frac{1}{\sqrt{-g}}\delta(x-y))\Psi(x)=0. \tag{23}$$

This further leads to the operator equation:

$$\gamma^{\mu}\partial_{\mu}(\frac{1}{\sqrt{-g}}\delta(x-y))=0. \tag{24}$$

We multiply by $\gamma_{\beta}$ and take the trace which yields:

$$\partial_{\beta}(\frac{1}{\sqrt{-g}}\delta(x-y))=0. \tag{25}$$

We use the relation:

$$\partial_{\beta}\frac{1}{\sqrt{-g}}=-\Gamma^{\alpha}_{\beta\alpha}, \tag{26}$$

to further write Eq. (24) as,

$$\partial_{\beta}\delta(x-y))=\Gamma^{\alpha}_{\beta\alpha}(x)\delta(x-y). \tag{27}$$

We regard $y^{\rho}$ as a new set of coordinates related to $x^{\rho}$ through a singular transformation. Since $\Gamma^{\alpha}_{\beta\alpha}$ is a derivative of a scalar in the coordinate $x$ it should transform as a tensor according to:

$$\Gamma^{\alpha}_{\beta\alpha}(x)=\frac{\partial x^{\alpha}}{\partial y^{m}}\frac{\partial y^{n}}{\partial x^{\beta}}\frac{\partial y^{p}}{\partial x^{\alpha}}\Gamma^{m}_{np}(y). \tag{28}$$

We multiply Eq. (40) by $\frac{\partial y^{\mu}}{\partial y^{0}}$ and introduce the result in Eq. (28) to obtain:

$$\begin{aligned}\delta(x-y)\left[-\frac{\partial^{2}y^{\mu}}{\partial x^{\beta}\partial y^{0}}\right]=\\ \delta(x-y)\left[\frac{\partial y^{\mu}}{\partial y^{0}}\frac{\partial x^{\alpha}}{\partial y^{m}}\frac{\partial y^{n}}{\partial x^{\beta}}\frac{\partial y^{p}}{\partial x^{\alpha}}\Gamma^{m}_{np}(y)\right].\end{aligned} \tag{29}$$

Next we acknowledge that the coordinates $y^{\mu}$ depend only on $y^{0}$ and $x^{0}$. Then one may write:

$$dy^{\mu}=\frac{\partial y^{\mu}}{\partial y^{0}}dy^{0}+\frac{\partial y^{\mu}}{\partial x^{0}}dx^{0}, \tag{30}$$

Eq. (30) further leads to:

$$\begin{aligned}\frac{dy^{\mu}}{dy^{m}}=\delta_{\mu m}=\frac{\partial y^{\mu}}{\partial y^{0}}\frac{dy^{0}}{dy^{m}}+\frac{\partial y^{\mu}}{\partial x^{0}}\frac{\partial x^{0}}{\partial y^{m}}\\ \delta_{\mu m}=\frac{\partial y^{\mu}}{\partial y^{0}}\delta_{0m}+\frac{\partial y^{\mu}}{\partial x^{0}}\frac{\partial x^{0}}{\partial y^{m}},\end{aligned} \tag{31}$$

or furthermore to,

$$\frac{\partial y^\mu}{\partial x^0}\frac{\partial x^0}{\partial y^m} = \delta_{\mu m} - \frac{\partial y^\mu}{\partial y^0}\delta_{0m}. \tag{32}$$

Noticing that in Eq. (29) $\frac{\partial x^\alpha}{\partial y^m} = \frac{\partial x^0}{\partial y^m}\delta_{\alpha 0}$ (since only $x^0$ depends on $y^m$) and using the result in Eq. (32) we obtain:

$$\begin{aligned}
&\delta(x-y)\left[-\frac{\partial^2 y^\mu}{\partial x^0 \partial y^0}\right] = \\
&\delta(x-y)\left[\frac{\partial y^n}{\partial x^0}\frac{\partial y^p}{\partial x^0}\Gamma^\mu_{np}(y) - \frac{\partial y^\mu}{\partial y^0}\frac{\partial y^n}{\partial x^0}\frac{\partial y^p}{\partial x^0}\Gamma^0_{np}(y)\right].
\end{aligned} \tag{33}$$

We then apply the delta function to Eq. (44) to determine:

$$\begin{aligned}
&\frac{\partial^2 y^\mu}{\partial y^0 \partial y^0} = \\
&\frac{\partial y^\mu}{\partial y^0}\frac{\partial y^n}{\partial y^0}\frac{\partial y^p}{\partial y^0}\Gamma^0_{np} - \frac{\partial y^n}{\partial y^0}\frac{\partial y^p}{\partial y^0}\Gamma^\mu_{np}.
\end{aligned} \tag{34}$$

Equation (34) is identical to the geodesic equation in terms of the time variable written in Eq. (7).

## 4. ELECTROMAGNETIC POTENTIAL OF A POINT LIKE CHARGE

We write the electromagnetic Lagrangian with a charged fermion in our approach as:

$$\begin{aligned}
\mathcal{L} = &-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}i\bar{\Psi}\gamma^\nu\Psi\partial_\nu\frac{1}{M^4}\delta(a(x-y)) + \\
&i\bar{\Psi}\gamma^\nu D_\nu\Psi\frac{1}{M^4}\delta(a(x-y)) - m\bar{\Psi}\Psi\frac{1}{M^4}\delta(a(x-y)).
\end{aligned} \tag{35}$$

The constant $M$ is considered small and it is put to establish the correct dimensionality and has the following role in the representation of the delta function (Note that although we work in the Minkowski space the delta function is considered in the Euclidean variables for the hermicity of the action):

$$\delta(x-y) = \int \frac{d^4q}{(2\pi)^4}\exp[iq(x-y)], \tag{36}$$

with,

$$\int \frac{d^4q}{(2\pi)^4} = M^4. \tag{37}$$

The constant $a$ is introduced for later convenience.

We are interested in determining the electromagnetic potentials created by an on shell fermion. For that we imposed the equation of motion in first order for the gauge field and on shell in all orders for the fermion field:

$$\begin{aligned}&\partial_\mu\partial^\mu A^\nu(x) = e\bar{\Psi}(x)\gamma^\nu\Psi(x)\\&[i\gamma^\mu\partial_\mu - m]\Psi(x)\delta(x-y) = 0\end{aligned} \tag{38}$$

We rewrite the part of the Lagrangian left (the next order) after the equations of motion in Eq. (38) are applied:

$$\begin{aligned}\mathcal{L}_1 = &\frac{1}{2}i\bar{\Psi}\gamma^\nu\Psi\partial_\nu\frac{1}{M^4}\delta(a(x-y)) +\\&-e\bar{\Psi}\gamma^\nu A_\nu\Psi[\frac{1}{M^4}\delta(a(x-y)) - 1].\end{aligned} \tag{39}$$

We expand the delta function in Eq. (39) to obtain:

$$\begin{aligned}&\left[\int\frac{d^4q}{(2\pi)^4}\frac{1}{2}i\bar{\Psi}\gamma^\nu\Psi[iq_\nu[1+ia(x-y)q+...]\right] +\\&\left[\int\frac{d^4q}{(2\pi)^4} - e\bar{\Psi}\gamma^\nu A_\nu\Psi[iqa(x-y) - a^2(q(x-y))^2 + ....]\right] = 0.\end{aligned} \tag{40}$$

It is then clear that in the first order:

$$A_\nu(x) = \left[-i\frac{1}{ea}\frac{q_\nu}{q(x-y)}\right]_e. \tag{41}$$

Here in the right hand side the variables are euclidean. Therefore in the Minkowski space one has:

$$A_\nu(x) = -\frac{1}{ea}\frac{q_\nu}{q(x-y)}. \tag{42}$$

The next step is to evaluate the constant $a$. For that we introduce the result in Eq. (42) in first line of Eq. (38):

$$\partial^\mu\partial_\mu\left[-\frac{1}{ea}\frac{q_\nu}{(x-y)^2}\right] = e\bar{\Psi}\gamma_\nu\Psi. \tag{43}$$

Next we will show that the left hand side of Eq. (43) really represents the current associated to a point like charge for an appropriate choice of $a$ and $q$. For that we need to show that:

$$\partial^\mu\partial_\mu\left[-\frac{1}{ea}\frac{q_\nu}{q(x-y)}\right] = e\int_{-T}^{T}d\tau\frac{dy_\nu}{d\tau}\delta(x-y). \tag{44}$$

This is equivalent to:

$$\frac{d}{d\tau}\partial^{\mu}\partial_{\mu}\left[-\frac{1}{ea}\frac{q_{\nu}}{q(x-y)}\right]=e\frac{dy_{\nu}}{d\tau}\delta(x-y), \tag{45}$$

and furthermore to:

$$\partial^{\mu}\partial_{\mu}\left[-\frac{1}{ea}\frac{q_{\nu}}{(q(x-y)^2)}\right]q^{\rho}\frac{dy_{\rho}}{d\tau}=e\frac{dy_{\nu}}{d\tau}\delta(x-y). \tag{46}$$

We immediately obtain that in order for Eq. (46) to make sense $q^{\nu}$ must be the momentum of the charge. Since $\frac{dy^{\nu}}{d\tau}=\frac{q^{\nu}}{m}$ one can further write:

$$\partial^{\mu}\partial_{\mu}\left[-\frac{1}{ea}\frac{1}{(q(x-y)^2)}q^2\right]=e\delta(x-y). \tag{47}$$

We can always pick $q$ only with the component $p_0$ and then apply the Lorentz transformation to generalize the result. Then we need to show:

$$\partial^{\mu}\partial_{\mu}\left[-\frac{1}{ea}\frac{1}{(z^2)}\right]=e\delta(z) \tag{48}$$

We multiply by $\exp[izr]$ and integrate over z to obtain:

$$\int d^4z(-r^2)(-\frac{1}{ea})\exp[irz]\frac{1}{(z^2)}=e, \tag{49}$$

and furthermore (since the Fourier transform of $\frac{1}{z^2}$ is just $-\frac{1}{r^2}$):

$$-\frac{1}{ea}=e, \tag{50}$$

from which we determine $a=-\frac{1}{e^2}$.

Finally one can update the equation (42) for the electromagnetic potentials associated to a point like charge with momentum $q_{\nu}$ and path $y_{\nu}$ as:

$$A_{\nu}(x)=e\frac{q_{\nu}}{q(x-y)}, \tag{51}$$

which is in perfect agreement with the results in the literature [8–10].

## 5. CONCLUSIONS

There are many avenues through which one may extract the geodesic equations in both special and general relativity. Here we used a method which leads to the correct results for both special and general relativity and that might be very useful in determining dynamical properties associated to a point like particle.

We also determined in our approach the electromagnetic potentials associated to a point like charge in the Minkowski space. The method can be used for calculating the electromagnetic potentials associated to a point like charge in general relativity, topic which is known to be plagued with issues. Such a derivation and other applications of the method will be left for further work.

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