Romanian Reports in Physics **73**, 111 (2021)

SCALING TRANSFORMATIONS, HETERO-BÄCKLUND TRANSFORMATIONS AND SIMILARITY REDUCTIONS ON A (2+1)-DIMENSIONAL GENERALIZED VARIABLE-COEFFICIENT BOITI-LEON-PEMPINELLI SYSTEM FOR WATER WAVES

XIN-YI GAO, YONG-JIANG GUO⋆, WEN-RUI SHAN⋆

State Key Laboratory of Information Photonics and Optical Communications, and School of Science, Beijing University of Posts and Telecommunications, Beijing 100876, China
Corresponding Authors' Emails: *yongerguo@bupt.edu.cn*, *shwr@bupt.edu.cn*

Received December 19, 2020

Abstract. To date, water waves are actively studied. On a (2+1)-dimensional generalized variable-coefficient Boiti-Leon-Pempinelli system for water waves, our scaling transformations and symbolic computation bring about a set of the hetero-Bäcklund transformations, while our symbolic computation results in a set of the similarity reductions, with respect to the horizontal velocity and elevation of the water wave, relying on the variable coefficients.

Key words: Water waves, (2+1)-dimensional generalized variable-coefficient Boiti-Leon-Pempinelli system, hetero-Bäcklund transformations, scaling transformations, similarity reductions, symbolic computation.

1. INTRODUCTION

To date, water waves have been actively studied [1–7]. Taking into account certain water waves propagating on the $x-y$ plane in an infinitely narrow channel of constant depth, Refs. [8–10] have investigated the (2+1)-dimensional generalized variable-coefficient Boiti-Leon-Pempinelli system:

$$u_{yt} = \alpha(t)\left(u^2 - u_x\right)_{xy} + \beta(t)v_{xxx} \ , \tag{1a}$$

$$v_t = \gamma(t)v_{xx} + \lambda(t)uv_x \ , \tag{1b}$$

with the real differentiable function $u(x,y,t)$ related to the horizontal velocity, the real differentiable function $v(x,y,t)$ related to the elevation of the water wave, the subscripts being the partial derivatives with respect to the variables x, y and t, while $\alpha(t)$, $\beta(t)$, $\gamma(t)$ and $\lambda(t)$ denoting the real non-zero differentiable functions of t [10].

Up to now, Ref. [8] has obtained a similarity reduction with some analytic solutions on System (1), Ref. [10] has constructed an auto-Bäcklund transformation on System (1) with some shock-wave-type solutions, while Ref. [9] has worked out some separation solutions for System (1) based on the extended tanh-function method.

More recent investigations have been seen on other nonlinear partial differential equations and their applications in diverse physical contexts, including, e.g, the studies of water waves and rogue (freak) waves in water systems and in other physical settings [11–22].

To our knowledge, scaling transformation and hetero-Bäcklund transformation on System (1) have not been obtained. It is noted that a hetero-Bäcklund transformation is also named the non-auto-Bäcklund transformation, as the relation between the solutions of different partial differential equations [23–25]. Making use of symbolic computation [26], we plan to find out such transformations. In addition, employing symbolic computation, we aim to design a set of the similarity reductions on System (1), which are different from that in Ref. [8].

2. SCALING TRANSFORMATIONS AND HETERO-BÄCKLUND TRANSFORMATIONS ON SYSTEM (1)

Brief review of the known knowledge:

- *Soon to be used are the three-dimensional Bell polynomials [27–29], i.e.,*

$$\begin{aligned} &Y_{mx,ry,nt}(w) \\ &\quad \equiv Y_{m,r,n}(w_{1,1,1},\cdots,w_{1,1,n},\cdots,w_{1,r,1},\cdots,w_{1,r,m},\cdots,w_{m,r,1},\cdots,w_{m,r,n}) \\ &\quad = e^{-w}\partial_x^m\partial_y^r\partial_t^n e^w \ , \end{aligned} \tag{2}$$

with $w(x,y,t)$ denoting a real non-zero C^∞ function of x, y and t, $w_{k,s,l} = \partial_x^k\partial_y^s\partial_t^l w$, $k = 0,\cdots,m$, $s = 0,\cdots,r$, $l = 0,\cdots,n$, while m, r and n meaning the nonnegative integers.

Similar to those in Refs. [30, 31], on System (1), we construct the following scaling transformations:

$$\begin{aligned} &x \to \rho^1 x \ , \qquad y \to \rho^\sigma y \ , \qquad t \to \rho^2 t \ , \\ &\alpha(t) \to \rho^2\alpha(t) \ , \qquad \beta(t) \to \rho^2\beta(t) \ , \qquad \gamma(t) \to \rho^2\gamma(t) \ , \\ &\lambda(t) \to \rho^2\lambda(t) \ , \qquad u \to \rho^{-1}u \ , \qquad v \to \rho^{-\sigma}v \ , \end{aligned} \tag{3}$$

and assume that

$$\begin{aligned} u(x,y,t) &= \xi_1(t)w_x(x,y,t) + \xi_2(t) \ , \\ v(x,y,t) &= \xi_3(t)w_y(x,y,t) + \xi_4(t) \ , \end{aligned} \tag{4}$$

in which $\xi_1(t) \neq 0$, $\xi_2(t)$, $\xi_3(t) \neq 0$ and $\xi_4(t)$ mean the real differentiable functions with respect to t, ρ represents a positive constant and σ denotes an integer.

With the three-dimensional Bell polynomials and symbolic computation, in the case of

$$\xi_4(t) = \xi_4 \quad \text{only}, \qquad \xi_3(t) = \xi_3 \quad \text{only}, \qquad \gamma(t) = \frac{1}{2}\lambda(t)\xi_1(t) \ , \tag{5}$$

integration of Eqn. (1b) once with respect to y with the integration function vanishing brings about a three-dimensional-Bell-polynomial expression, i.e.,

$$Y_t(w) - \frac{1}{2}\lambda(t)\xi_1(t)Y_{2x}(w) - \lambda(t)\xi_2(t)Y_x(w) = 0 \ . \tag{6}$$

Further, assuming that

$$w(x,y,t) = \ln h(x,y,t) \ , \tag{7}$$

we are able to convert Eqn. (6) into a linear equation, i.e.,

$$h_t - \frac{1}{2}\lambda(t)\xi_1(t)h_{xx} - \lambda(t)\xi_2(t)h_x = 0 \ , \tag{8}$$

with $h(x,y,t)$ denoting a real non-zero differentiable function.

Similarly, if

$$\xi_1(t) = \xi_1 \quad \text{only} \ , \qquad \alpha(t)\xi_1^2 = \beta(t)\xi_3 - \alpha(t)\xi_1 \ , \tag{9}$$

integrating Eqn. (1a) once with respect to both x and y, respectively, with the integration functions vanishing, results in

$$Y_t(w) - \alpha(t)\xi_1 Y_{2x}(w) - 2\alpha(t)\xi_2(t)Y_x(w) = 0 \ , \tag{10}$$

which comes to be the same as Eqn. (6), under the constraint

$$\lambda(t) = 2\alpha(t) \ . \tag{11}$$

Thinking over all the aforementioned, employing the variable-coefficient constraints

$$\beta(t) = \mu_1\alpha(t) \ , \qquad \gamma(t) = \mu_2\alpha(t) \qquad \text{and} \qquad \lambda(t) = 2\alpha(t) \ , \tag{12}$$

as well as supposing that μ_1 and μ_2 represent the real non-zero constants, with the choices of

$$\xi_1(t) = \xi_1 = \mu_2 \ , \qquad \xi_3(t) = \xi_3 = \frac{\mu_2^2 + \mu_2}{\mu_1} \qquad \text{and} \qquad \xi_4(t) = \xi_4 \ , \tag{13}$$

we obtain the following set of the hetero-Bäcklund transformations:

$$u(x,y,t) = \mu_2\frac{h_x(x,y,t)}{h(x,y,t)} + \xi_2(t) \ , \tag{14a}$$

$$v(x,y,t) = \frac{\mu_2^2 + \mu_2}{\mu_1}\frac{h_y(x,y,t)}{h(x,y,t)} + \xi_4 \ , \tag{14b}$$

$$h_t - \mu_2\alpha(t)h_{xx} - 2\alpha(t)\xi_2(t)h_x = 0 \ . \tag{14c}$$

In other words, if $h(x,y,t)$ represents a solution of Eqn. (14c), Eqns. (14) comprise a set of the hetero-Bäcklund transformations linking $h(x,y,t)$ and a set of the solutions $u(x,y,t)$ and $v(x,y,t)$ on System (1).

We notice that Eqn. (14c) is a known linear partial differential equation, with the relevant information provided in Refs. [32, 33] and with the following special solutions hereby obtained:

$$h(x,y,t) = 1 + \exp\left[\sigma_1 x + \sigma_2(y) + \mu_2\sigma_1^2 \int \alpha(t)dt + 2\sigma_1 \int \alpha(t)\xi_2(t)dt\right] \ , \quad (15)$$

of which σ_1 denotes a real non-zero constant and $\sigma_2(y)$ means a real differentiable function of y.

As for the horizontal velocity and elevation of the water wave, Hetero-Bäcklund Transformations (14) are dependent on $\alpha(t)$, a variable coefficient in System (1), under Variable-Coefficient Constraints (12).

3. SIMILARITY REDUCTIONS ON SYSTEM (1)

For the purpose of forming a set of the similarity reductions on System (1), assuming* that

$$u(x,y,t) = \theta(x,y,t) + \omega(x,y,t)p[z(x,y,t)] \ , \quad (16a)$$

$$v(x,y,t) = \delta(x,y,t) + \kappa(x,y,t)q[z(x,y,t)] \ , \quad (16b)$$

which are similar to those in Refs. [34–37] but different from the beginning of Ref. [8], we take into consideration the case of $z_x = 0$, $z_y \neq 0$ and $z_t \neq 0$, with $\theta(x,y,t)$, $\omega(x,y,t) \neq 0$, $\delta(x,y,t)$, $\kappa(x,y,t) \neq 0$ and $z(x,y,t) \neq 0$ as some real differentiable functions to be determined.

Hence, a similarity reduction could result from each set of the aforementioned θ, ω, δ, κ and z we plan to find out.

As demanded to be a couple of the real ordinary differential equations (ODEs) for $p(z)$ and $q(z)$, with our symbolic computation, System (1) can be written as

$$\begin{aligned} &\omega z_y z_t p'' - 4\alpha(t)\omega\omega_x z_y p p' - 2\alpha(t)\left(\omega_x\omega_y + \omega\omega_{xy}\right)p^2 + \left[\left(\omega_t z_y + \omega_y z_t + \omega z_{yt}\right)\right. \\ &\quad \left. - \alpha(t)\left(2\theta_x\omega z_y + 2\theta\omega_x z_y - \omega_{xx} z_y\right)\right]p' - \beta(t)\kappa_{xxx} q + \left[\omega_{yt} - \alpha(t)\left(2\theta\omega_{xy}\right.\right. \\ &\quad \left.\left. + 2\theta_x\omega_y + 2\theta_y\omega_x + 2\theta_{xy}\omega - \omega_{xxy}\right)\right]p + \left[\theta_{yt} - \alpha(t)\left(2\theta\theta_{xy} + 2\theta_x\theta_y - \theta_{xxy}\right)\right. \\ &\quad \left. - \beta(t)\delta_{xxx}\right] = 0 \ , \end{aligned} \quad (17a)$$

$$\begin{aligned} &\kappa z_t q' - \lambda(t)\omega\kappa_x p q + \left[\kappa_t - \lambda(t)\theta\kappa_x - \gamma(t)\kappa_{xx}\right]q - \lambda(t)\omega\delta_x p + \left[\delta_t - \lambda(t)\theta\delta_x\right. \\ &\quad \left. - \gamma(t)\delta_{xx}\right] = 0 \ . \end{aligned} \quad (17b)$$

*rather than a more general set $u(x,y,t) = U[x,y,t,p(z)]$ and $v(x,y,t) = V[x,y,t,q(z)]$ (proof ignored)

Because of the ratios of the coefficients of different derivatives and powers of $p(z)$ and $q(z)$ as some functions of z only, Eqns. (17) could be rewritten as

$$p'' + \Omega_1(z)pp' + \Omega_2(z)p^2 + \Omega_3(z)p' + \Omega_4(z)q + \Omega_5(z)p + \Omega_6(z) = 0 \ , \quad (18a)$$
$$q' + \Gamma_1(z)pq + \Gamma_2(z)q + \Gamma_3(z)p + \Gamma_4(z) = 0 \ , \quad (18b)$$
$$\Omega_1(z)\omega z_y z_t = -4\alpha(t)\omega\omega_x z_y \ , \quad (18c)$$
$$\Omega_2(z)\omega z_y z_t = -2\alpha(t)\left(\omega_x\omega_y + \omega\omega_{xy}\right) \ , \quad (18d)$$
$$\Omega_3(z)\omega z_y z_t = (\omega_t z_y + \omega_y z_t + \omega z_{yt}) - 2\alpha(t)z_y\left(\theta_x\omega + \theta\omega_x\right) + \alpha(t)\omega_{xx}z_y \ , \quad (18e)$$
$$\Omega_4(z)\omega z_y z_t = -\beta(t)\kappa_{xxx} \ , \quad (18f)$$
$$\Omega_5(z)\omega z_y z_t = \omega_{yt} - \alpha(t)\left(2\theta\omega_{xy} + 2\theta_x\omega_y + 2\theta_y\omega_x + 2\theta_{xy}\omega - \omega_{xxy}\right) \ , \quad (18g)$$
$$\Omega_6(z)\omega z_y z_t = \theta_{yt} - \alpha(t)\left(2\theta\theta_{xy} + 2\theta_x\theta_y - \theta_{xxy}\right) - \beta(t)\delta_{xxx} \ , \quad (18h)$$
$$\Gamma_1(z)\kappa z_t = -\lambda(t)\omega\kappa_x \ , \quad (18i)$$
$$\Gamma_2(z)\kappa z_t = \kappa_t - \lambda(t)\theta\kappa_x - \gamma(t)\kappa_{xx} \ , \quad (18j)$$
$$\Gamma_3(z)\kappa z_t = -\lambda(t)\omega\delta_x \ , \quad (18k)$$
$$\Gamma_4(z)\kappa z_t = \delta_t - \lambda(t)\theta\delta_x - \gamma(t)\delta_{xx} \ , \quad (18l)$$

where $\Omega_i(z)$'s $(i = 1, \cdots, 6)$ and $\Gamma_j(z)$'s $(j = 1, \cdots, 4)$ represent some real to-be-determined functions of z only.

Based on the second freedom in Remark 2 in Ref. [34], Eqn. (18c) brings about

$$\omega(x,y,t) = -\frac{z_t}{4\alpha(t)}x \ , \qquad \Omega_1(z) = 1 \ , \quad (19)$$

and by reason of the first freedom in Remark 2 in Ref. [34], Eqn. (18d) gives rise to

$$z(x,y,t) = z(y,t) = \zeta_1(y) + \zeta_2(t) \ , \qquad \Omega_2(z) = 0 \ , \quad (20)$$

with $\zeta_1(y)$ denoting a real non-zero differentiable function of y, while $\zeta_2(t)$ meaning a real non-zero differentiable function of t.

Due to the second freedom in Remark 2 in Ref. [34], Eqn. (18f) makes for

$$\kappa(x,y,t) = \frac{z_y z_t^2}{96\alpha(t)\beta(t)}x^4 \ , \qquad \Omega_4(z) = 1 \ , \quad (21)$$

as a result that Eqn. (18i) comes to

$$\lambda(t) = \phi_1\alpha(t) \ , \qquad \Gamma_1(z) = \phi_1 \ , \quad (22)$$

with ϕ_1 as a real non-zero constant.

Because the first freedom in Remark 2 in Ref. [34] helps us simplify Eqns. (18j) and (18k) into

$$\beta(t) = \phi_2\alpha(t) \ , \qquad \delta(x,y,t) = 0 \ , \tag{23a}$$

$$\zeta_1(y) = \psi_1 y \ , \qquad \zeta_2(t) = \psi_2 \int \alpha(t)dt \ , \tag{23b}$$

$$\theta(x,y,t) = \theta(x,t) = -3\frac{\gamma(t)}{\lambda(t)}x^{-1} \ , \qquad \Gamma_2(z) = \Gamma_3(z) = 0 \ , \tag{23c}$$

we, for Eqns. (18e), (18g), (18h) and (18l), work out

$$\Omega_3(z) = \Omega_5(z) = \Omega_6(z) = \Gamma_4(z) = 0 \ , \tag{24}$$

with ψ_1 and ψ_2 representing two real non-zero constants.

In consequence, using symbolic computation, we figure out the following ODEs:

$$q + pp' + p'' = 0 \ , \tag{25a}$$

$$q' + \phi_1 pq = 0 \ , \tag{25b}$$

which could be simplified into an ODE as

$$p''' + (\phi_1 + 1)\,pp'' + p'^2 + \phi_1 p^2 p' = 0 \ , \tag{26}$$

with

$$q = -\left(pp' + p''\right) \ . \tag{27}$$

As a whole, under the variable-coefficient constraints

$$\lambda(t) = \phi_1\alpha(t) \qquad \text{and} \qquad \beta(t) = \phi_2\alpha(t) \ , \tag{28}$$

we finish up with a set of the similarity reductions on System (1), i.e.,

$$u(x,y,t) = -\frac{\psi_2}{4}xp[z(x,y,t)] - \frac{3\gamma(t)}{\phi_1\alpha(t)}x^{-1} \ , \tag{29a}$$

$$v(x,y,t) = -\frac{\psi_1\psi_2^2}{96\phi_2}x^4\left\{p[z(x,y,t)]p'[z(x,y,t)] + p''[z(x,y,t)]\right\} \ , \tag{29b}$$

$$z(x,y,t) = z(y,t) = \psi_1 y + \psi_2\int\alpha(t)dt \ , \tag{29c}$$

$$p''' + (\phi_1 + 1)\,pp'' + p'^2 + \phi_1 p^2 p' = 0 \ . \tag{29d}$$

Eqn. (29d) is a known ODE, classified into the ODE (137.1) category in Ref. [38] and hereby solved as

$$p(z) = \frac{2}{z + \eta_1} \ , \tag{30}$$

with η_1 denoting a real constant.

Similarity Reductions (29) are different from that in Ref. [8].

As for the horizontal velocity and elevation of the water wave, Similarity Reductions (29) rely on $\alpha(t)$ and $\gamma(t)$, two variable coefficients in System (1), under Variable-Coefficient Constraints (28).

Acknowledgements. This work has been supported by the National Nature Science Foundation of China under Grant No. 11871116 and Fundamental Research Funds for the Central Universities of China under Grant No. 2019XD-A11.

REFERENCES

1. A. M. Wazwaz, Rom. Rep. Phys. 72, 110 (2020); M. Dolineanu and T. O. Cheche, Rom. Rep. Phys. 71, 903 (2019)
2. A. A. Abubakar, B. S. Yilbas and H. Al-Qahtani, J. Fluids Eng. 143, 021402 (2021); C. C. Ding, Y. T. Gao and G. F. Deng, Nonlinear Dyn. 97, 2023 (2019); Y. Shen and B. Tian, Appl. Math. Lett., in press, doi:10.1016/j.aml.2021.107301 (2021); Y. J. Feng, Y. T. Gao, L. Q. Li and T. T. Jia, Eur. Phys. J. Plus 135, 272 (2020); Y. Shen, B. Tian and T. Y. Zhou, Eur. Phys. J. Plus, in press, doi:10.1140/epjp/s13360-021-01323-0 (2021); J. J. Su, Y. T. Gao, G. F. Deng and T. T. Jia, Phys. Rev. E 100, 042210 (2019)
3. K. Sehic, H. Bredmose, J. D. Sorensen and M. Karamehmedovic, J. Eng. Math. 126, 1 (2021); T. T. Jia, Y. T. Gao, G. F. Deng and L. Hu, Nonlinear Dyn. 98, 269 (2019); X. Y. Gao, Y. J. Guo and W. R. Shan, Chaos Solitons Fract. 142, 110367 (2021); J. J. Su, Y. T. Gao and C. C. Ding, Appl. Math. Lett. 88, 201 (2019); L. Hu, Y. T. Gao, S. L. Jia, J. J. Su and G. F. Deng, Mod. Phys. Lett. B 33, 1950376 (2019)
4. F. Zouari, S. Nasraoui, M. Louati and M. S. Ghidaoui, J. Sound Vib. 491, 115706 (2021); Y. J. Feng, Y. T. Gao, T. T. Jia and L. Q. Li, Mod. Phys. Lett. B, 33, 1950354 (2019); Y. Q. Chen, B. Tian, Q. X. Qu, H. Li, X. H. Zhao, H. Y. Tian and M. Wang, Mod. Phys. Lett. B 34, 2050287 (2020); L. Hu, Y. T. Gao, T. T. Jia, G. F. Deng and L. Q. Li, Z. Angew. Math. Phys. 72, 75 (2021); X. Zhao, B. Tian, H. Y. Tian and D. Y. Yang, Nonlinear Dyn. 103, 1785 (2021)
5. C. I. Martin and B. Basu, Nonlinear Anal.-Real 57, 103176 (2021); Y. Shen, B. Tian, S. H. Liu and D. Y. Yang, Phys. Scr. 96, 075212 (2021); X. Y. Gao, Y. J. Guo and W. R. Shan, Phys. Lett. A 384, 126788 (2020); C. C. Hu, B. Tian, H. M. Yin, C. R. Zhang and Z. Zhang, Comput. Math. Appl. 78, 166 (2019); M. Wang, B. Tian, Y. Sun and Z. Zhang, Comput. Math. Appl. 79, 576 (2020)
6. J. Gemmrich, T. Mudge and J. Thomson, Ocean Dyn. 71, 343 (2021); S. H. Liu, B. Tian, Q. X. Qu, H. Li, X. H. Zhao, X. X. Du and S. S. Chen, Int. J. Comput. Math., in press, https://doi.org/10.1080/00207160.2020.1805107 (2021); Y. Shen, B. Tian, C. R. Zhang, H. Y. Tian and S. H. Liu, Mod. Phys. Lett. B, in press, doi:10.1142/S0217984921502614 (2021); M. Wang, B. Tian, Q. X. Qu, X. X. Du, C. R. Zhang and Z. Zhang, Eur. Phys. J. Plus, 134, 578 (2019)
7. D. Clamond, J. Phys. A 54, 045701 (2021); X. Y. Gao, Y. J. Guo and W. R. Shan, Appl. Math. Lett. 104, 106170 (2020); X. Y. Gao, Y. J. Guo and W. R. Shan, Acta Mech. 231, 4415 (2020); X. Y. Gao, Y. J. Guo and W. R. Shan, Commun. Theor. Phys. 72, 095002 (2020); X. Y. Gao, Y. J. Guo and W. R. Shan, Eur. Phys. J. Plus 135, 689 (2020); X. Y. Gao, Y. J. Guo and W. R. Shan, Chin. J. Phys. 70, 264 (2021)
8. R. M. El-Shiekh, Comput. Math. Appl. 73, 1414 (2017)
9. W. G. Ni and C. Q. Dai, Appl. Math. Comput. 270, 434 (2015)
10. X. Y. Gao, J. Math. Phys. 56, 014101 (2015)
11. W. Liu and A. M. Wazwaz, Rom. J. Phys. 64, 111 (2019); T. T. Jia, Y. T. Gao, X. Yu and L. Q. Li, Appl. Math. Lett. 114, 106702 (2021); H. Y. Tian, B. Tian, C. R. Zhang and S. S. Chen, Int. J.

Comput. Math. doi:10.1080/00207160.2021.1900568 (2021)

12. C. B. Ward and P. G. Kevrekidis, Rom. J. Phys. 64, 112 (2019); G. F. Deng, Y. T. Gao, C. C. Ding and J. J. Su, Chaos Solitons Fract. 140, 110085 (2020); L. Q. Li, Y. T. Gao, L. Hu, T. T. Jia, C. C. Ding and Y. J. Feng, Nonlinear Dyn. 100, 2729 (2020)
13. Z. Li, H. Wei and P. He, Rom. Rep. Phys. 71, 110 (2019); D. Wang, Y. T. Gao, J. J. Su and C. C. Ding, Mod. Phys. Lett. B 34, 2050336 (2020)
14. B. A. Malomed and D. Mihalache, Rom. J. Phys. 64, 106 (2019); C. C. Ding, Y. T. Gao, G. F. Deng and D. Wang, Chaos Solitons Fract. 133, 109580 (2020); G. F. Deng, Y. T. Gao, J. J. Su, C. C. Ding and T. T. Jia, Nonlinear Dyn. 99, 1039 (2020)
15. P. Li, J. Li, B. Han, H. Ma and D. Mihalache, Rom. Rep. Phys. 71, 106 (2019); F. Y. Liu, Y. T. Gao, X. Yu, C. C. Ding, G. F. Deng and T. T. Jia, Chaos Solitons Fract. 144, 110559 (2021); D. Wang, Y. T. Gao, C. C. Ding and C. Y. Zhang, Commun. Theor. Phys. 72, 115004 (2020)
16. C. Hou, L. Bu, F. Baronio, D. Mihalache and S. Chen, Rom. Rep. Phys. 72, 405 (2020); X. Zhao, B. Tian, Q. X. Qu, Y. Q. Yuan, X. X. Du and M. X. Chu, Mod. Phys. Lett. B 34, 2050282 (2020); Y. Q. Chen, B. Tian, Q. X. Qu, H. Li, X. H. Zhao, H. Y. Tian and M. Wang, Int. J. Mod. Phys. B 34, 2050226 (2020)
17. L. Kaur and A. M. Wazwaz, Rom. Rep. Phys. 71, 102 (2019); D. Y. Yang, B. Tian, Q. X. Qu, C. R. Zhang, S. S. Chen and C. C. Wei, Chaos Solitons Fract., in press, https://doi.org/10.1016/j.chaos.2020.110487 (2021); M. Wang, B. Tian, Y. Sun, H. M. Yin and Z. Zhang, Chin. J. Phys. 60, 440 (2019)
18. M. Crabb and N. Akhmediev, Rom. Rep. Phys. 72, 118 (2020); X. X. Du, B. Tian, Q. X. Qu, Y. Q. Yuan and X. H. Zhao, Chaos Solitons Fract. 134, 109709 (2020); S. S. Chen, B. Tian, J. Chai, X. Y. Wu and Z. Du, Wave. Random Complex 30, 389 (2020)
19. A. Ankiewicz, M. Bokaeeyan and W. Chang, Rom. Rep. Phys. 72, 119 (2020); C. R. Zhang, B. Tian, Q. X. Qu, L. Liu and H. Y. Tian, Z. Angew. Math. Phys. 71, 18 (2020); H. Y. Tian, B. Tian, Y. Q. Yuan and C. R. Zhang, Phys. Scr. 96, 045213 (2021)
20. A. M. Wazwaz, Rom. Rep. Phys. 69, 108 (2017); S. H. Liu, B. Tian, Q. X. Qu, C. R. Zhang, C. C. Hu and M. Wang, Mod. Phys. Lett. B 34, 2050243 (2020); M. Wang, B. Tian, C. C. Hu and S. H. Liu, Appl. Math. Lett. 119, 106936 (2021)
21. E. V. Krishnan, Q. Zhou and A. Biswas, Proc. Romanian Acad. A 17, 137 (2016); X. X. Du, B. Tian, Y. Q. Yuan and Z. Du, Ann. Phys. (Berlin) 531, 1900198 (2019); D. Y. Yang, B. Tian, Q. X. Qu, H. Li, X. H. Zhao, S. S. Chen and C. C. Wei, Phys. Scr. 96, 045210 (2021)
22. H. Leblond and D. Mihalache, Phys. Rep. 523, 61 (2013); S. S. Chen, B. Tian, Y. Sun and C. R. Zhang, Ann. Phys. (Berlin) 531, 1900011 (2019); C. R. Zhang, B. Tian, Y. Sun and H. M. Yin, EPL 127, 40003 (2019)
23. A. V. Tsiganov, Theor. Math. Phys. 192, 1350 (2017)
24. A. V. Tsiganov, Regul. Chaotic Dyn. 20, 74 (2015)
25. X. Y. Gao, Y. J. Guo and W. R. Shan, Chaos Solitons Fract. 138, 109950 (2020)
26. X. Y. Gao, Y. J. Guo, W. R. Shan, Y. Q. Yuan, C. R. Zhang and S. S. Chen, Appl. Math. Lett. 111, 106627 (2021); F. Y. Liu, Y. T. Gao, X. Yu, L. Q. Li, C. C. Ding and D. Wang, Eur. Phys. J. Plus, in press, https://doi.org/10.1140/epjp/s13360-021-01469-x (2021)
27. E. T. Bell, Ann. Math. 35, 258 (1934)
28. F. Lambert, I. Loris, J. Springael and R. Willox, J. Phys. A 27, 5325 (1994)
29. Y. F. Wang, B. Tian and Y. Jiang, Appl. Math. Comput. 292, 448 (2017)
30. F. Lambert and J. Springael, Chaos Solitons Fract. 12, 2821 (2001)
31. F. Lambert and J. Springael, Acta Appl. Math. 102, 147 (2008)
32. H. F. Pu, K. Wang, J. W. Qiu and X. L. Chen, Appl. Math. Model. 88, 916 (2020)

33. J. Rodrigo-Ilarri, M. E. Rodrigo-Clavero, E. Cassiraga and L. Ballesteros-Almonacid, Int. J. Environ. Res. Public Health 17, 3280 (2020)
34. Q. T. Chen and Y. D. Shang, AIMS Math. 4, 231 (2019)
35. B. Zhang and H. C. Hu, Z. Naturforsch. A 73, 357 (2018)
36. H. L. An, J. J. Yang and M. W. Yuen, Commun. Theor. Phys. 63, 613 (2015)
37. P. Clarkson and M. Kruskal, J. Math. Phys. 30, 2201 (1989)
38. D. Zwillinger, Handbook of Differential Equations, 3rd Ed., Acad., San Diego (1997).