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NUMERICAL BEHAVIOR OF NONLINEAR DUFFING EQUATIONS WITH FRACTIONAL DAMPING

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Abstract. Duffing systems are a pattern for illustrating various physical processes and dynamical systems. These systems can be accurately modeled by fractional-order equations. In this paper, we present a numerical method to examine the behavior of fractionally damped Duffing equation *via* hybrid functions. We accomplish numerical simulations to demonstrate the efficiency and reliability of this technique. The obtained results show that the method provides a fast convergence and it has a good computational accuracy.

Key words: Duffing equations, fractional damping, hybrid functions, operational matrix, fractional derivatives.

1. INTRODUCTION

Fractional calculus can describe phenomena that cannot be modeled by integer-order calculus [1], such as robust synchronization problems [2], complex viscoelasticity [3], and other physical phenomena [4–6]. Many scientists are attempting to handle the fractional calculus to describe their models in physics and engineering [7–11]. Moreover, in the scientific literature the implementation and development of efficient methods for solving fractional differential models have been reported [9, 12–18].

The Duffing systems, as one of the important topics in science, arise in some branches of physics and engineering science and have been applied to describe various phenomena such as, signal processing [19], brain modelling [20], and ionization waves in plasmas [21]. The Duffing oscillator was first introduced by Georg Duffing in 1918; see, for example, Ref. [22]. Given the importance of these equations, the study of Duffing equations is receiving growing attention and there have been some attempts to obtain approximate solutions. For example, Yusufoglu [23] solved the Duffing equations by the Laplace decomposition algorithm. Cveticanin [24] assumed that the initial approximate solution is in the form of Jacobi elliptic function. Rad *et al.* [25] proposed a numerical method based on the radial basis functions to approximate the nonlinearly controlled Duffing oscillator. Merdan *et al.* [26] applied

the multi-step differential transform method to give approximate solutions of fractional nonlinear oscillatory and vibration equations. Alquran *et al.* [8] investigated the Duffing model involving fractional time derivative and multiple pantograph time delays.

We develop a new computational method for solving the Duffing oscillator of fractional order, based on an approximation that uses a hybrid function, which consists of block-pulse functions and Legendre polynomials. We notice that the Legendre polynomials defined on the interval $[-1, 1]$ are orthogonal functions and have beneficial properties that can be widely used in the approximation theory. Our approach is based upon representation of the second derivative term, in terms of a series expansion using hybrid functions with unknown coefficients. The operational matrix is used to convert the oscillator equation to the corresponding algebraic system, which can be solved to find the unknown function that not only simplifies but also speeds up the computation. Also, the existence of the error estimation bound confirms that the results obtained by using hybrid functions are reliable.

The Duffing equations with fractional damping do not have an analytical solution, then we will demonstrate the validity and the effectiveness of our approach for this purpose by comparing the numerical solution of the proposed method with the Runge-Kutta method for the classical Duffing equation.

The paper is organized as follows. In Sec. 2 the physical model is formulated. The description of the numerical method is given in Sec. 3. The obtained results are presented in Sec. 4 and the conclusions are given in Sec. 5.

2. PHYSICAL FORMULATION

Differential equations are used to model damped and driven oscillators [27, 28]. A physical system that involves a thin metal beam of mass m and two magnets that are strong enough, all housed in a stable frame, leads to the ordinary form of Duffing equation. The beam will bend to one of the magnets and stay there if no other force is acting on the system. A force that is applied periodically in a frame prevents the beam to stay at rest in any position. The greater the force that is acting on the frame, the more pronounced the oscillation will be between the two magnets. The beam will tend to gravitate towards one magnet, if the force of one of them is smaller. Figure 1 shows a Duffing oscillator system.

The ordinary Duffing oscillator can be described by

$$mu''(t) + \delta u'(t) + \gamma u(t) + \beta u^3(t) = F\cos\omega t, \tag{1}$$

where m and δ are the mass and damping coefficients, respectively, and they are always positive numbers. Also, γ is the linear stiffness of the spring, β represents the amount of nonlinearity in the restoring force, and F and ω are the excitation

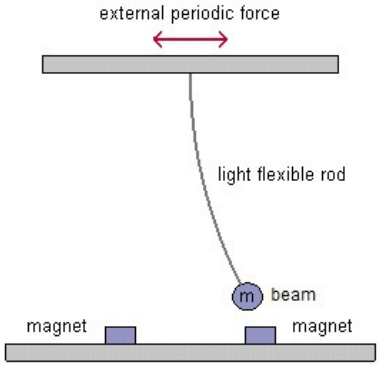

Fig. 1 – Duffing oscillator system.

amplitude and the excitation frequency, respectively. We suppose that the beam's position at the time $t = 0$ with velocity $u'(0) = d_1$ is $u(0) = d_0$. For $F = 0$, the equation is a system without driving force, for more details see [29]. The unknown function $u(t)$ is the position of the end of the beam and denotes the displacement of the nonlinear system. For simplicity, the Eq. (1) can be divided by m, then to obtain the fractional-order Duffing oscillator, the first derivative is replaced by the α fractional-order derivative of the Caputo's sense, as follows

$$u''(t) + \delta_0 D_*^{\alpha} u(t) + \gamma_0 u(t) + \beta_0 u^3(t) = F_0 \cos \omega t. \tag{2}$$

For the Riemann-Liouville fractional integral and Caputo's derivative operators, we have [1, 30]:

$$I^{\alpha} D_*^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0) \frac{t^k}{k!}, \quad n-1 < \alpha \leq n, \ \ n \in \mathbb{N}. \tag{3}$$

3. DESCRIPTION OF THE METHOD

First, we express the function approximation with the hybrid bases [31] and then we convert the Duffing equations into an algebraic system using the operational matrix.

3.1. FUNCTION APPROXIMATION BY HYBRID BASIS

The hybrid functions are defined on the interval $[0, 1)$ as [32],

$$h_{nm}(t) = \begin{cases} P_m(2Nt - 2n + 1), & t \in \left[\frac{n-1}{N}, \frac{n}{N}\right), \\ 0, & \text{o.w.}, \end{cases}$$

where n, m are the orders of block-pulse functions and Legendre polynomials, respectively, and $P_m(t)$ are the well-known Legendre polynomials. These functions are complete and orthogonal.

A function $u(t)$, which is a square integrable function defined over the interval $[0, 1)$, may be expanded as [33],

$$u(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} h_{nm}(t) \approx \sum_{n=1}^{N} \sum_{m=0}^{M-1} c_{nm} h_{nm}(t) = C^T H(t).$$

Here $C, H(t)$ are

$$C = \left[c_{10}, \ldots, c_{1(M-1)}, \ldots, c_{N0}, \ldots, c_{N(M-1)}\right]^T,$$

$$H(t) = \left[h_{10}(t), \ldots, h_{1(M-1)}(t), \ldots, h_{N0}(t), \ldots, h_{N(M-1)}(t)\right]^T. \tag{4}$$

In the following we obtain the error bound that verifies the convergence of the approximation. In other words, by applying the hybrid basis functions, we will have two parameters M and N, which help to improve the accuracy of our approximation.

We divide the interval $[0, 1]$ into subintervals $[\frac{n-1}{N}, \frac{n}{N}]$, so that $C^T H(t)$ is a polynomial of degree at most $(M-1)$ on each subinterval that approximates f. If $\tilde{f}(t)$ denotes the interpolating polynomial of $f(t)$ and we use the maximum error bound, then we have

$$\begin{aligned}
\|f(t) - C^T H(t)\|^2_{L^2([0,1])} &= \int_0^1 [f(t) - C^T H(t)]^2 dt \\
&\approx \sum_{n=1}^{N} \int_{\frac{n-1}{N}}^{\frac{n}{N}} [f(t) - C^T H(t)]^2 dt \\
&\leq \sum_{n=1}^{N} \int_{\frac{n-1}{N}}^{\frac{n}{N}} [f(t) - \tilde{f}(t)]^2 dt \\
&\leq \sum_{n=1}^{N} \int_{\frac{n-1}{N}}^{\frac{n}{N}} \left[\left(\frac{1}{N}\right)^M \frac{1}{M!} \sup_{\zeta \in [\frac{n-1}{N}, \frac{n}{N}]} f^{(M)}(\zeta) \right]^2 dt \\
&\leq \frac{L^2}{N^{2M} M!^2},
\end{aligned}$$

where $f(t) \in C^M[0, 1]$ and $f^{(M)}(t) \leq L$. In other words, by using the hybrid basis functions, we can obtain any required accurate approximation of $f(t)$ by choosing M, N sufficiently large and $C^T H(t) \to f(t)$.

3.2. OPERATIONAL MATRIX OF THE FRACTIONAL INTEGRATION

The fractional integration of $H(t)$ (4), is given by

$$I^{\alpha}H(t) \approx P^{\alpha}_{\mu\times\mu}H(t),$$

where $\mu = N \times M$ and $P^{\alpha}_{\mu\times\mu}$ is the operational matrix for fractional integration, so that

$$P^{\alpha}_{\mu\times\mu} \approx \Phi_{\mu\times\mu}F^{\alpha}\Phi^{-1}_{\mu\times\mu}, \tag{5}$$

with

$$\Phi_{\mu\times\mu} = \left[H\left(\frac{1}{2\mu}\right)\ \ H\left(\frac{3}{2\mu}\right)\ \ldots\ H\left(\frac{2\mu-1}{2\mu}\right)\right],$$

and

$$F^{\alpha} = \frac{1}{\mu^{\alpha}}\frac{1}{\Gamma(\alpha+2)}\begin{bmatrix} 1 & \varepsilon_1 & \varepsilon_2 & \ldots & \varepsilon_{\mu-1} \\ 0 & 1 & \varepsilon_1 & \ldots & \varepsilon_{\mu-2} \\ 0 & 0 & 1 & \ldots & \varepsilon_{\mu-3} \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

with $\varepsilon_i = (i+1)^{\alpha+1} - 2i^{\alpha+1} + (i-1)^{\alpha+1}$, for $i = 1, 2, \cdots, \mu-1$ [34].

3.3. PERFORMANCE FOR FRACTIONAL DUFFING EQUATION

We consider the Duffing Eq. (2) and we firstly approximate $u''(t)$ by the hybrid functions as $u''(t) = C^T H(t)$. According to relation (3), we get

$$u(t) = C^T P^2_{\mu\times\mu}H(t) + u'(0)t + u(0),$$

and

$$D^{\alpha}_{*}u(t) = \begin{cases} C^T P^{2-\alpha}_{\mu\times\mu}H(t) + u'(0), & 0 < \alpha \leq 1, \\ C^T P^{2-\alpha}_{\mu\times\mu}H(t), & 1 < \alpha < 2, \end{cases}$$

where C is an unknown vector. Depending on the value of α, by substituting the above equations in Eq. (2) and using collocation points, we reduce the fractional Duffing equation to a nonlinear system of algebraic equations.

4. RESULTS AND DISCUSSION

We apply the method described in the previous Section to investigate the numerical behavior of Duffing equation solution with fractional order damping. As a test problem, we consider $\delta_0 = 0.4, \gamma_0 = 1.1, \beta_0 = 1, F_0 = 2.1$ and $\omega = 1.8$, with initial conditions $u(0) = 0.3$ and $u'(0) = -2.3$. We solve the corresponding fractional Duffing equation *via* the operational matrix method for different values of α.

We compare the approximate solutions with the Runge-Kutta method provided by the Mathematica symbolic package (as best approximation) and the Laplace transform decomposition algorithm (LTDA) [23], for the non fractional case, that is, for $\alpha = 1$. The results for classical Duffing equation are given in Table 1. It is clearly seen that the maximum absolute errors of the approximations, by using the presented numerical technique with $\mu = 16$, are less than 0.02%. Also, we have implemented the above technique for some different values of α between 0 and 2. The numerical behavior of $u(t)$ as solution of the fractional Duffing equation, with $\mu = 15$ and $\alpha = 0.25, 0.5, 1, 1.5, 1.75$, are shown in Fig. 2 and Table 2. It is clearly seen that

Table 1

The numerical solutions of the classical Duffing equation with different methods.

	Our method, M=4				
t values	N=4	N=8	N=16	LTDA	Runge-Kutta
0.1	0.083709	0.083616	0.083592	0.080942	0.083584
0.2	−0.104961	−0.105059	−0.105084	−0.106822	−0.105092
0.3	−0.265892	−0.265988	−0.266012	−0.266147	−0.266020
0.4	−0.399858	−0.399948	−0.399970	−0.399536	−0.399978
0.5	−0.508208	−0.508288	−0.508308	−0.508129	−0.508315
0.6	−0.592805	−0.592869	−0.592885	−0.593180	−0.592891
0.7	−0.656004	−0.656050	−0.656062	−0.655381	−0.656066
0.8	−0.700643	−0.700668	−0.700675	−0.700417	−0.700677
0.9	−0.729967	−0.729970	−0.729971	−0.732272	−0.729971

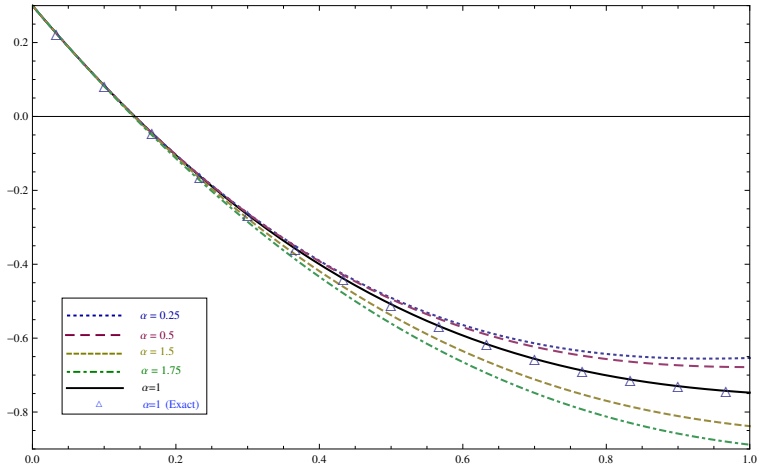

Fig. 2 – Numerical behavior of $u(t)$, with $\mu = 15$ and different values of α

Table 2

The numerical solutions of the fractional Duffing equation with $\mu = 15$ and different values of α.

t values	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 1.5$	$\alpha = 1.75$
0.1	0.0839201	0.0838979	0.0837282	0.0828998	0.0818005
0.2	−0.103624	−0.103812	−0.104952	−0.109001	−0.113287
0.3	−0.261669	−0.262359	−0.265889	−0.275955	−0.285148
0.4	−0.390382	−0.392107	−0.399863	−0.418623	−0.434003
0.5	−0.490728	−0.494191	−0.508221	−0.537945	−0.560311
0.6	−0.564440	−0.570469	−0.592825	−0.635141	−0.664769
0.7	−0.614026	−0.623513	−0.656032	−0.711736	−0.748345
0.8	−0.642750	−0.656573	−0.700679	−0.769596	−0.812329
0.9	−0.654533	−0.673485	−0.730009	−0.810919	−0.858357

the results are in good agreement with the Runge-Kutta method. Therefore, the operational matrix based on hybrid functions is verified to be accurate for solving the fractional Duffing equation. To test the efficiency of our numerical approach, we apply it for a system with the same parameters without driving force, $F_0 = 0$. According to Table 3, in this case, the above algorithm works well with high accuracy, too. Our results are digitally simulated using Mathematica, version 12.

Table 3

The numerical solutions of the Duffing equation without driving force with $\mu = 12$.

t Values	0.1	0.3	0.5	0.7	0.9
$\alpha = 0.75$	0.073386	−0.353034	−0.727417	−1.02347	−1.20811
$\alpha = 1$	0.073358	−0.353666	−0.730179	−1.03067	−1.22242
$\alpha = 1.25$	0.073293	−0.354762	−0.734358	−1.04065	−1.24093

5. CONCLUSION

The effect of fractional derivative damping on numerical solution of Duffing oscillators has been investigated. Our study shows that the properties of the hybrid functions and their operational matrices can be used to numerically solve of nonlinear Duffing equation with fractional order. By converting the Duffing equation to a nonlinear system, we obtain more accurate results, without computational complexity. These results confirm the usefulness of the operational matrix based on hybrid functions in approximating the Duffing oscillator equations with fractional order damping. The numerical algorithm developed in this study can be adopted to examine various dynamical behaviors of other physical systems.

REFERENCES

1. D. Baleanu, K. Diethelm, E. Scalas, J. J. Trujillo, *Fractional Calculus: Models and Numerical Methods*, World Scientific, Singapore (2012).
2. C. Li, W. Deng, Int. J. Mod. Phys. B **20**, 791 (2006).
3. Y. Charles Lu, Int. J. Mech. Mater. Des. **3**, 329 (2006).
4. Y. Zhang, Y. F. Pu, J. R. Hu, J. L. Zhou, Appl. Math. Inf. Sci. **6**, 299 (2012).
5. X. Lin, S. Zhou, H. Li, Appl. Math. Inf. Sci. **7**, 233 (2013).
6. D. Baleanu, S. S. Sajjadi, A. Jajarmi, O. Defterli, J. H. Asad, Rom. Rep. Phys. **73**, 105 (2021).
7. M. Ali, M. Alquran, I. Jaradat, N. A. Afouna, D. Baleanu, Rom. Rep. Phys. **72**, 103 (2020).
8. M. Alquran, I. Jaradat, D. Baleanu, R. Abdel-Muhsen, Rom. J. Phys. **64**, 103 (2019).
9. K. Nouri, S. Elahi-Mehr, L. Torkzadeh, Rom. Rep. Phys. **68**(2), 503 (2016).
10. D. Kumar, J. Singh, D. Baleanu, Rom. Rep. Phys. **69**, 103 (2017).
11. A. Izadkhah, K. Nouri, A. Nikoobin, Rom. J. Phys. **65**, 103 (2020).
12. A. H. Bhrawy, D. Baleanu, Rep. Math. Phys. **72**, 219 (2013).
13. E. H. Doha, A. H. Bhrawy, S. S. Ezz-Eldien, Cent. Eur. J. Phys. **11**, 1494 (2013).
14. A. H. Bhrawy, M. M. Tharwat, A. Yildirim, Appl. Math. Model. **37**, 4245 (2013).
15. K. A. Abro, I. A. Abro, A. Yildirim, Physica A **541**, 123306 (2020).
16. K. A. Abro, A. Yildirim, Iran. J. Sci. Technol. Trans. Sci. **43**, 2307 (2019).
17. M. I. Syam, M.-N. Y. Anwar, A. Yildirim, M. M. Syam, Int. J. Appl. Comput. Math. **5**, 38 (2019).
18. D. Baleanu, A. K. Golmankhaneh, A. K. Golmankhaneh, Rom. J. Phys. **54**(9–10), 823 (2009).
19. J. Wang, J. Zhou, B. Peng, Kybernetes **38**, 1662 (2009).
20. E. C. Zeeman, Bull. Inst. Math. Appl. **12**, 207 (1976).
21. H. G. Enjieu Kadji, B. R. Nana Nbendjo, J. B. Chabi Orou, P. K. Talla, Phys. Plasmas **15**, 032308 (2008).
22. J. Stokes, *Nonlinear Vibrations*, Intersciences, New York (1950).
23. E. Yusufoglu, Appl. Math. Comput. **177**, 572 (2006).
24. L. Cveticanin, Chaos Solitons Fractals **30**, 1221 (2006).
25. J. A. Rad, S. Kazem, K. Parand, Comput. Math. Appl. **64**, 2049 (2012).
26. M. Merdan, A. Gokdogan, A. Yildirim, Meccanica **48**, 1201 (2013).
27. D. Baleanu, A. Jajarmi, J. H. Asad, Rom. Rep. Phys. **71**, 103 (2019).
28. M. Alquran, I. Jaradat, D. Baleanu, M. Syam, Rom. J. Phys. **64**, 107 (2019).
29. I. Kovacic, M. J. Brennan, *The Duffing Equation: Nonlinear Oscillators and Their Behaviour*, Wiley, United Kingdom (2011).
30. K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer-Verlag, Berlin (2010).
31. K. Maleknejad, L. Torkzadeh, Rom. J. Phys. **60**(1–2), 87 (2015).
32. K. Maleknejad, B. Basirat, E. Hashemizadeh, Comput. Math. Appl. **61**, 2821 (2011).
33. K. Maleknejad, M. Tavassoli Kajani, Appl. Math. Comput. **145**, 623 (2003).
34. A. Kilicman, Z. A. A. Al Zhour, Appl. Math. Comput. **187**, 250 (2007).