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LAMBERT W FUNCTION AND DIFFERENT FORMS OF WIEN'S DISPLACEMENT LAW

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Abstract. We shall use the standard physics problem of spectral distribution functions of blackbody radiation from standard physics textbooks that many students meet in their education, and we shall express solutions of transcendental equations in terms of Lambert W functions. Finding various forms of Wien's displacement law from spectral distribution functions requires solving transcendental equations. Using the Lambert W function, we derive closed-form analytical expressions for the wavelengths where maxima occur in the different spectral distributions of blackbody radiation. Transcendental equations can be solved by means of numerical methods but the possibility of obtaining the exact analytical solution of such equation *e.g.* through the Lambert W function is appealing. The reason is that a closed, analytical form of a function is very important as it provides the necessary asymptotic forms both at small and large values of the independent variable. In principle, exact analytical solutions to physics problems are always desirable and preferable to numerical ones. Our method is illustrated in a pedagogical manner for the benefit of students at the undergraduate level.

Key words: Lambert function, blackbody radiation, Wien peaks.

1. INTRODUCTION

The Wien's displacement law establishes the dependence of the wavelength λ_{max} that corresponds to the spectral maximum of the spectral radiation on the temperature T: $\lambda_{max}T = b$ where the b is Wien's constant. It means that as the temperature increases, the peak in the curve for spectral radiant density, when plotted as a function of wavelength, becomes displaced toward shorter wavelengths. The Wien's displacement law takes a various form when Planck's law is expressed in terms of wavelength than that in terms of frequency. Even today, Wien's displacement law is utilized spectroscopically to estimate the temperature of objects from a distance, for example in estimating of the temperature of the sun surface or other celestial object. The Wien's displacement law has been extensively studied and discussed in the last two decades, see for example [1–12].

In general, analytic solutions to many problems are available on introductory physics courses. However, such solutions are limited on upper-level and graduate courses and become very rare in actual research practice. Those problems without analytical solutions are usually solved using computerized numerical techniques. Therefore, the basic idea of this paper is to use the standard physics problem of spectral distribution functions of blackbody radiation and to express different forms of Wien's displacement law *via* the Lambert W function.

The Lambert W function has a rich variety of applications ranging from physics and computer science, to statistics and biology, see for example [13–22]. Besides their theoretical importance, the obtained results will be of interest to teachers involved in undergraduate physics.

2. DIFFERENT SPECTRAL FUNCTIONS FOR BLACKBODY RADIATION

The Planck's spectral distribution W_λ of blackbody radiation as a function of the wavelength λ and absolute temperature T is usually given in the form

$$W_\lambda d\lambda = \left[\frac{8\pi hc}{\lambda^5}\frac{1}{\exp(hc/\lambda kT)-1}\right]d\lambda. \tag{1}$$

The quantity $W_\lambda d\lambda$ is the power emitted in the wavelength interval $d\lambda$ per unit area from blackbody at absolute temperature T. In other words it is the power radiated per unit area per unit wavelength at a given temperature. The fundamental constants h, c, and k are Planck's constant, the speed of light on a vacuum, and Boltzmann's constant, respectively. Note that the power per unit area depends only on temperature, and not on other characteristic of the object.

The spectral distribution (1) can be expressed in terms of frequency $\nu = c/\lambda$. It should be noted if $d\lambda$ and $d\nu$ refer to the same interval of spectrum, we do not have $W_\lambda = W_\nu$ but $W_\lambda|d\lambda| = W_\nu|d\nu|$:

$$W_\nu d\nu = \left[\frac{8\pi h\nu^3}{c^3}\frac{1}{\exp(h\nu/kT)-1}\right]d\nu. \tag{2}$$

Equation (2) gives the energy density of the radiation in units of J/m^3 between the frequencies ν and $\nu + d\nu$.

Beside these (W_λ and W_ν) the most commonly used are another versions. The nature itself provides no physical (or biological) preference between these two, and indeed there are other useful dispersion rules. Namely, the spectral distribution (1) can be expressed in different equivalent forms such as:

$$W_\lambda d\lambda = \left[\frac{8\pi hc}{\lambda^5}\frac{1}{\exp(hc/\lambda kT)-1}\right]d\lambda, \tag{3}$$

$$= \left[\frac{4\pi hc}{\lambda^6}\frac{1}{\exp(hc/\lambda kT)-1}\right]d\lambda^2 \equiv W_{\lambda^2}d\lambda^2, \tag{4}$$

$$= \left[\frac{8\pi hc}{\lambda^4}\frac{1}{\exp(hc/\lambda kT)-1}\right]d\ln\lambda \equiv W_{\ln\lambda}d\ln\lambda, \tag{5}$$

$$= \left[\frac{16\pi hc}{\lambda^{9/2}}\frac{1}{\exp(hc/\lambda kT)-1}\right]d\sqrt{\lambda} \equiv W_{\sqrt{\lambda}}d\sqrt{\lambda}. \tag{6}$$

Similarly, the spectral distribution (2) as a function of frequency can be also expressed in various forms:

$$W_\nu d\nu = \left[\tfrac{8\pi h\nu^3}{c^3}\tfrac{1}{\exp(h\nu/kT)-1}\right]d\nu, \tag{7}$$

$$= \left[\tfrac{4\pi h\nu^2}{c^3}\tfrac{1}{\exp(h\nu/kT)-1}\right]d\nu^2 \equiv W_{\nu^2}d\nu^2, \tag{8}$$

$$= \left[\tfrac{8\pi h\nu^4}{c^3}\tfrac{1}{\exp(h\nu/kT)-1}\right]d\ln\nu \equiv W_{\ln\nu}d\ln\nu, \tag{9}$$

$$= \left[\tfrac{16\pi h\nu^{7/2}}{c^3}\tfrac{1}{\exp(h\nu/kT)-1}\right]d\sqrt{\nu} \equiv W_{\sqrt{\nu}}d\sqrt{\nu}. \tag{10}$$

In addition to W_λ, the radiant emittance I_λ (spectral emissive power of blackbody) is a closely related quantity: $W_\lambda = (4/c)I_\lambda$ so that:

$$I_\lambda = \frac{2\pi hc^2}{\lambda^5}\frac{1}{\exp(hc/\lambda kT)-1}. \tag{11}$$

The radiant emittance can be expressed in terms of frequency I_ν by using $I_\lambda|d\lambda| = I_\nu|d\nu|$:

$$I_\nu = \frac{2\pi h\nu^3}{c^2}\frac{1}{\exp(h\nu/kT)-1}. \tag{12}$$

The photon distribution functions are of special interest to atmospheric physicist. In atmospheric physics the photon distribution functions are preferred over Planck energy distribution functions because in solar photolysis the basic construct are photons. The quantity J_λ is emittance of photons per cm^2 per s per unit λ so that it is a spectral photon emittance, and J_λ is related to the spectral radiant emittance by [23]:

$$\left(\frac{hc}{\lambda}\right)J_\lambda = I_\lambda. \tag{13}$$

Such functions can be expressed in terms of wavelength and frequency:

$$J_\lambda = \frac{2\pi c/\lambda^4}{\exp(hc/\lambda kT)-1}, \tag{14}$$

$$J_\nu = \frac{2\pi\nu^2/c^2}{\exp(h\nu/kT)-1}. \tag{15}$$

The spectral brightness function, B_ν, was measured by the FIRAS (far infrared astronomical spectrometer) instrument on board the COBE satellite [24]. The quantity B_ν is the incident energy flux in units of $W/(m^2 srHz)$ (sr is a steradian, a unit for solid angle), and is given by $B_\nu = (c/4\pi)W_\nu$. Thus the Planck brightness function is [24]:

$$B_\nu = \frac{2h\nu^3/c^2}{\exp(h\nu/kT)-1}. \tag{16}$$

The quantity B_ν can be also expressed in terms of wavelength *i.e.* B_λ *versus* λ, where B_λ is the emitted power per unit area per steradian per wavelength interval and is given by

$$B_\lambda = \frac{2hc^2/\lambda^5}{\exp(hc/\lambda kT)-1}. \tag{17}$$

The functions B_ν and B_λ describe the same physics, but they have different shapes, due to the nonlinear change of variable from wavelength to frequency. As a consequence, the two curves peaks appear at different locations in the spectrum.

3. LAMBERT W FUNCTION

The definition of the Lambert $W(z)$ is that it is the function that solves the transcendental equation:

$$z = W(z)e^{W(z)}, \tag{18}$$

where z is a complex number and e is the base of the natural logarithm. This equation always has an infinite number of solutions, most of them complex, and so $W(z)$ is a multi-valued function. The different possible solutions are labeled by an integer variable called the branch of W *i.e.* $W_k(z)$ for any $k = 0, \pm1, \pm2, ...$ When $z \equiv x$ is a real number $x = Re(z)$, equation (18) has two real solutions in certain restricted x-intervals in which case they are $W_0(x)$ and $W_{-1}(x)$. The branch $W_0(x)$ is defined for $x \in [-e^{-1}, \infty)$, whereas $W_{-1}(x)$ is defined for $x \in [-e^{-1}, 0)$. These two branches meet at the common point $W_0(-e^{-1}) = W_1(-e^{-1}) = -1$. Outside the said intervals, these two branches of $W(x)$ are complex-valued even for a real argument x. It should be noted that even if z is real, the branches other than $k = 0, -1$ are always complex.

A plot of $W(x)$ that satisfies the functional equation $x = W(x)e^{W(x)}$ as a function of real x is displayed in Fig. 1.

By convention, the branch denoted by $W_0(x)$ is taken to be the principal branch, whereas the branch $W_{-1}(x)$ is known as the secondary real branch. The principal

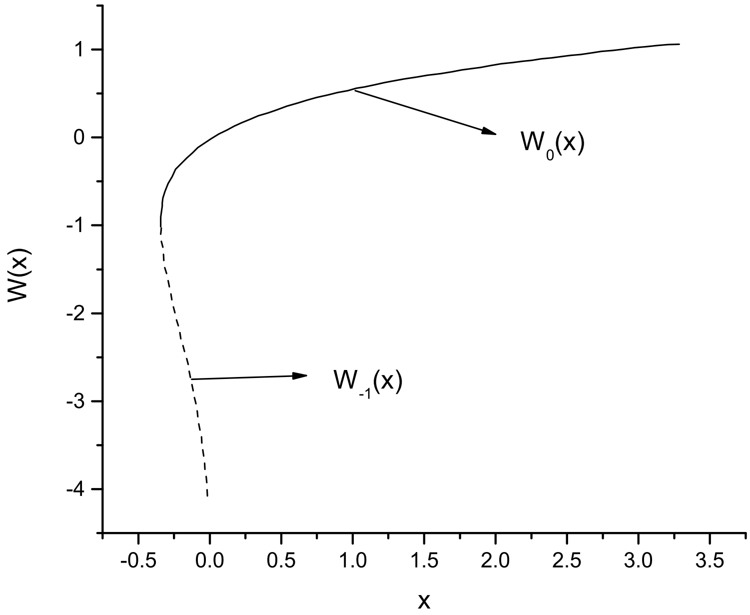

Fig. 1 – Two real branches of the W function. The solid line shows $W_0(x)$ and the dashed line shows $W_{-1}(x)$.

branch W_0 is analytic at $x = 0$ and its Taylor series expansion is given in the form:

$$W_0(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n = x\left(1 - x + \frac{3}{2}x^2 - \frac{8}{3}x^3 + \frac{125}{24}x^4 - \frac{54}{5}x^5 + \frac{16807}{720}x^6 - \cdots\right) \tag{19}$$

The $W(x)$ function can be represented by a linear-logarithmic relationship. Taking the natural logarithm of both sides of the $x = W(x)e^{W(x)}$ we obtain

$$W(x) = \ln x - \ln W(x). \tag{20}$$

The general procedure is to first transform the considered transcendental algebraic equation into the following form:

$$\phi(x)e^{\phi(x)} = y, \tag{21}$$

where $\phi(x)$ is an explicit function of x. Then, according to Eq. (18), for equation (20) to be satisfied, $\phi(x)$ must be equal to $W(y)$, *i.e.*

$$\phi(x) = W(y). \tag{22}$$

Finally, solving equation (21) for x, gives a solution to the original transcendental algebraic equation. As a illustration of the above-mentioned procedure we shall solve a typical transcendental algebraic equation stated as

$$ae^{-bx} = cx + d, \tag{23}$$

where x is the unknown and a, b, c, and d are real constants. By performing successive elementary transformations, one can put equation (22) into the form given by equation (20) as

$$\frac{a}{c} = (x + \frac{d}{c})e^{bx}, \tag{24}$$

$$\frac{ab}{c}e^{bd/c} = (bx + bd/c)e^{bx+bd/c}, \tag{25}$$

$$x = -\frac{d}{c} + \frac{1}{b}W(\frac{ab}{c}e^{bd/c}). \tag{26}$$

This procedure will be applied for determining Wien's peaks.

4. DIFFERENT FORMS OF WIEN'S DISPLACEMENT LAW

The positions of maxima in the spectral functions for blackbody radiation give the so-called Wien peaks. The wavelength λ_{max} at which W_λ has a maximum obeys Wien's displacement law (WDL) $\lambda_{max}T = b$, where b is Wien's displacement constant. This law was proposed by Wien in 1893 from general thermodynamic arguments. The WDL can be deduced and the value of b can be determined from Planck's spectral distribution. The value of λ for which the function W_λ has a maximum can be obtained by solving $\partial W_\lambda/\partial\lambda = 0$. This leads to the equation:

$$-5\exp\left(\frac{hc}{\lambda kT}\right) + 5 + \frac{hc}{\lambda kT}\exp\left(\frac{hc}{\lambda kT}\right) = 0. \tag{27}$$

Utilizing a dimensionless quantity $x = hc/(\lambda kT)$, the Eq. (26) can be written as the following transcendental equation:

$$5e^{-x} = 5 - x. \tag{28}$$

Using Eqs. (22) and (25) we can write the solution of the Eq. (27) in the form:

$$x = x_m = hc/(\lambda_{max}kT) = 5 + W_0(-5e^{-5}) = 4.965114. \tag{29}$$

It should be noted that for the secondary real branch $W_{-1}(x)$ of the Lambert function we get the trivial solution results recognizing simplification $W_{-1}(-ne^{-n}) = -n$, for $n > 1$, so that $x = x_m = hc/(\lambda_{max}kT) = 5 + W_{-1}(-5e^{-5}) = 5 - 5 = 0$.

We can check the series expansion (19) precision by comparison with the result in Eq. (28). Employing only two terms in the expansion we obtain the result

$$x = x_m = 5 + W_0(-5e^{-5}) \simeq 5 - 5e^{-5} - 5^2 e^{-2\times 5} = 4.965175 \qquad (30)$$

with the relative error 0.0012%. We can deduce that only two terms in the expansion is a very good approximation.

Using the Eq. (28) we obtain Wien's displacement law:

$$\lambda_{max} T = \frac{hc}{kx_m} = \frac{hc}{k[5 + W_0(-5e^{-5})]} = b_\lambda = 2.899777 \times 10^{-3} mK. \qquad (31)$$

In that way the Wien's displacement constant b_λ has been expressed in closed form in terms of Lambert W function.

The maximum of spectral distribution in terms of frequency W_ν from Eq. (2) can be obtained by $dW_\nu/d\nu = 0$ and it gives the following transcendental equation:

$$3e^{-y} = 3 - y, \qquad (32)$$

with solution (according to Eq. 25)

$$y = y_m = h\nu_{max}/(kT) = 3 + W_0(-3e^{-3}) = 3 - 0.178561 = 2.821439, \qquad (33)$$

or with approximate solution by using the first three terms in series expansion (19):

$$y = y_m = 3 + W_0(-3e^{-3}) = \simeq 3 - 3e^{-3} - 3^2 e^{-2\times 3} + \frac{3}{2}[-3e^{-3}]^3 = 2.823332, \qquad (34)$$

with the relative error 0.0671%. From Eq. (32) we can write:

$$\frac{\nu_{max}}{T} = \frac{k[3 + W_0(-3e^{-3})]}{h} = 5.878926 \times 10^{10} Hz/K. \qquad (35)$$

By setting $\nu_{max} = c/\lambda_{max}$ in (34) we arrive at another form of the Wien's displacement law:

$$\lambda_{max} T = \frac{hc}{k[3 + W_0(-3e^{-3})]} = b_\nu = 5.099443 \times 10^{-3} mK. \qquad (36)$$

Comparing Eqs. (35) and (28) we can see that peak locations are different as well as the shapes of spectral distributions W_λ and W_ν.

The nonlinear relationship between wavelength and frequency $\lambda = c/\nu$ has a significant effect on the shape and peak location of each spectral curve. Physically, the total area under each spectral curve at a given temperature T must remain unchanged, since $W_\lambda |d\lambda| = W_\nu |d\nu|$. This is simply the conservation of energy. The interval widths between the two scales are related by $d\nu = -(c/\lambda^2)d\lambda$ and it means that with increasing wavelength, intervals of constant $d\lambda$ therefore correspond to increasingly narrow intervals of $d\nu$. The nonconstancy in interval widths between the two scales results in the differences found in their spectral curve shapes and corresponding peak locations.

On the other hand, the constant involved in the equation (34) is not directly related to the constant in the wavelength version of Wien's equation (30). That is, a simple substitution using $\nu = c/\lambda$ does not generate the frequency version constant. The product of ν_{max} from (34) and λ_{max} from (30) equals $\nu_{max} \times \lambda_{max} = 1.704757 \times 10^8 m/s$, instead of being equal to the speed of light c, in case the maxima of both distributions coincided.

The traditional physicists' preference for the wavelength dispersion rule comes from the fact that spectrometers using diffraction gratings give experimental dispersion that approximates the linear-wavelength rule. It can be shown that the other common class of spectrometers, those using prisms, give an experimental dispersion that approximates a frequency-squared rule; that is, the spectrum is spread out as "intensity per increment in frequency-squared" [1] and the integration differential is $d(\nu^2) = 2\nu d\nu$.

The positions of maxima for wavelength-squared and frequency-squared spectral distribution W_{λ^2} and W_{ν^2} from (4) and (8) can be obtained by taking the derivative with respect to wavelength and frequency, respectively, and equating to zero. Such procedure leads to, respectively, the following transcendental equations $x = 6 - 6e^{-x}$ and $x = 2 - 2e^{-x}$ and thus to two forms of Wien's displacement law:

$$\lambda_{max} T = \frac{hc}{kx_m} = \frac{hc}{k[6 + W_0(-6e^{-6})]} = b_{\lambda^2} = 2.404011 \times 10^{-3} mK, \tag{37}$$

$$\lambda_{max} T = \frac{hc}{ky_m} = \frac{hc}{k[2 + W_0(-2e^{-2})]} = b_{\nu^2} = 9.028332 \times 10^{-3} mK. \tag{38}$$

By setting the derivative of the quantities $W_{\ln\lambda}$ and $W_{\ln\nu}$ from (5) and (9) with respect to wavelength and frequency to zero, one obtains the *same* transcendental equation $x = 4 - 4e^{-x}$, where $x_m = hc/(\lambda_{max} kT) = 4 + W_0(-4e^{-4}) = 3.920690$, for which

$$\lambda_{max} T = \frac{hc}{kx_m} = \frac{hc}{k[4 + W_0(-4e^{-4})]} = b_{\ln\lambda} = b_{\ln\nu} = 3.669703 \times 10^{-3} mK. \tag{39}$$

Hence, by introducing a logarithmic frequency or wavelength scale, a unified Wien's displacement law is obtained regardless of whether the frequency or wavelength is used as the independent variable [8, 25]. This has been called a wavelength-frequency-neutral peak [8]. Another convenient property of the quantities $W_{\ln\lambda}$ and $W_{\ln\nu}$ is that they are expressed in the same unit *i.e.* in J/m^3.

Applying the above procedure to find maxima for the quantities $W_{\sqrt{\lambda}}$ and $W_{\sqrt{\nu}}$ from (6) and (10) we arrive to the following closed forms of Wien's displacement law:

$$\lambda_{max} T = \frac{hc}{kx_m} = \frac{hc}{k[9/2 + W_0(-9/2e^{-9/2})]} = b_{\sqrt{\lambda}} = 3.235167 \times 10^{-3} mK, \tag{40}$$

$$\lambda_{max}T = \frac{hc}{ky_m} = \frac{hc}{k[7/2 + W_0(-7/2e^{-7/2})]} = b_{\sqrt{\nu}} = 4.255546 \times 10^{-3} mK. \quad (41)$$

These distributions $W_{\sqrt{\lambda}}$ and $W_{\sqrt{\nu}}$ might seem unnatural or unphysical to some people but perfectly reasonable and useful to others.

The photon distribution functions also suffer a similar non-uniqueness of spectral maxima. Namely, by differentiating the spectral photon emittance J_λ from Eq. (14) with respect to wavelength and setting it to zero, one obtains the transcendental equation $x = 4 - 4e^{-x}$, which leads to WDL given by Eq. (38). On the other hand J_ν from Eq. (15) satisfies $x = 2 - 2e^{-x}$, so that we arrive at Eq. (37).

For each versions the Wien's displacement law presented in this article one can calculate λ_{max} for the accepted surface temperature of the Sun, about 5800 K. This temperature, according to Eq. (30), corresponds to $\lambda_{max} = 500\ nm$ and this wavelength corresponds to the green part of spectrum. However, using $T = 5800\ K$, the maximum from Eq. (35) corresponds to $\lambda_{max} \simeq 880\ nm$, which is in the near infrared. Discussions about the color of the Sun related to solar maximum λ_{max} can be found in Refs. [11, 26, 27].

5. CONCLUSION

We have discussed the standard problem of physics in which the Lambert W function can be used. We have examined the position of the peaks for different spectral distributions of blackbody radiation. As seen in the case of Wien's displacement law, solution based on series expansion can be very accurate even with a few terms. As shown in the present paper, there are a number of ways of representing a density distribution function. Each one represents the function with equal mathematical validity and without loss or gain of information, even though each has a different shape. The meaning and usefulness of the chosen representation depends entirely on the intention, interest, and the convenience of the user.

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