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DYNAMICAL ANALYSIS OF SOLITON SOLUTIONS FOR SPACE-TIME FRACTIONAL CALOGERO-DEGASPERIS AND SHARMA-TASSO-OLVER EQUATIONS

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Abstract. The plan of this study is to construct analytical exact solutions of space-time fractional Calogero-Degasperis equation and space-time fractional Sharma-Tasso-Olver equation. Our approach stems mainly on applying the generalized $\left(\frac{G'}{G}\right)$ expansion method. Fractional complex transform, along with fractional derivative version of Jumarie's modified Riemann-Liouville method, are employed to transform fractional differential equations into the corresponding nonlinear ordinary differential equations. As a result, three types of exact analytical solutions, namely, the generalized hyperbolic function solutions, generalized trigonometric function solutions and rational solutions with free parameters are successfully furnished. A variety of the obtained results are new and reported here for the first time, to the best of our knowledge. We provide proper graphs to illustrate the obtained results. The distinct properties of the obtained solutions, such as soliton characteristics, are enumerated corresponding to the behavior of each solution.

Key words: Space-Time Fractional Calogero-Degasperis Equation; Space-Time Fractional Sharma-Tasso-Olver Equation; Generalized $\left(\frac{G'}{G}\right)$ Expansion Method; Exact Solutions; Solitons.

1. INTRODUCTION

Fractional calculus [1–4] is viewed as a novel sphere of theoretical and applied mathematics concerning, derivatives and integrals with arbitrary orders. Fractional derivatives such as the Riemann-Liouville derivative and the Caputo derivative are useful in describing the memory and hereditary properties of materials and processes, which is different from ordinary derivatives. The nonlocal property is the most essential supremacy of fractional order differential equation in mathematical community for extensive research analysis. This is due to the silent feature stating that the subsequent state of the system depends upon its current state as well as upon all of its occurred states.

It is significantly observed that in recent past years, work has been focused over various extensions and implementations of the known powerful techniques to construct the solutions to various partial differential equations [5–18], but at the same, numerous physical processes are framed as fractional partial differential equations (FPDEs) such as vibrations with fractional damping solid mechanics, optical fibers, signal processing, biomedical sciences, diffusion processes and many more. Therefore, the scientific community has established notable results in theory and applications of fractional differential equations for diverse areas of mathematical physics. There are different versions of fractional integral and differential operators. Riemann-Liouville definition is one of the well known, used extensively in various fields of engineering and science, but this definition deduce that the constant function derivative is not zero. Caputo put definitions stating that smooth and differentiable function results into zero value for fractional differentiation of a constant function [2, 19]. Furthermore, Jumarie developed definitions for differential and integral with fractional order, which are appropriately dealing with continuous and non-differentiable functions and results into differentiation of a constant function equal to zero. It is called the modified Riemann-Liouville method [20–22].

Hunting for exact solutions to nonlinear FPDEs has been flourishing day by day and attracts more researchers due to its active interest. Nowadays, a variety of many authentic methods have been developed to efficiently provide the exact solutions of FPDEs with the help of symbolic software packages. Examples of the methods used in the literature are the finite difference method [23], homotopy analysis method [24], homotopy perturbation method [25], fractional sub-equation method [26], modified simple equation method [27], first integral method [28], improved extended tanh coth method [29], the $\left(\frac{G'}{G}, \frac{1}{G}\right)$-expansion method [30], novel method [31], auxiliary equation method [32], Lie symmetry analysis [33] and other methods as well [34–36].

In this work, we aim to study the (2+1)-dimensional space-time fractional Calogero-Degasperis (CD) equation [37] given as

$$D_t^{\beta} D_x^{\alpha} u - 4 D_x^{\alpha} u D_x^{2\alpha} u - 2 D_y^{\eta} u D_x^{2\alpha} u + D_y^{\eta} D_x^{3\alpha} u = 0, \tag{1}$$

where $0 < \alpha, \beta, \eta \leq 1$ and $u = u(x,y,t),\ t > 0,\ x,\ y \in \mathbf{R}$, being representing time and space coordinates sequentially. Equation (1) illustrates the (2+1)-dimensional interfacing of a long wave along the X-axis with a Riemann wave propagating along the Y-axis. Moreover, we will also investigate another important physical model, namely space-time fractional Sharma-Tasso-Olver (STO) equation [38, 39], symbolized as

$$D_t^{\beta} u + 3\epsilon (D_x^{\alpha} u)^2 + 3\epsilon u^2 D_x^{\alpha} u + 3\epsilon u D_x^{2\alpha} u + \epsilon D_x^{3\alpha} u = 0, \tag{2}$$

where $u = u(x,t),\ 0 < \alpha, \beta \leq 1$ and $\epsilon \neq 0$ is a constant, $t > 0, x \in \mathbf{R}$ representing time and space coordinates sequentiality. Here, the space coordinate is in the direction of propagating field.

Our main concern of this study is to discover new exact solutions to the aforementioned space-time fractional differential equations by means of generalized $\left(\frac{G'}{G}\right)$ expansion method. In Sec. 2, a quick review of definitions and properties of Jumarie's modified Riemann-Liouville derivative will be discussed and then the description for solving FPDEs *via* generalized $\left(\frac{G'}{G}\right)$ expansion method is presented. The exact solutions for equations (1) and (2) by using generalized $\left(\frac{G'}{G}\right)$ expansion method are established and some graphical representations and physical interpretations of the solutions are given in Sec. 3 and Sec. 4. Lastly, some conclusions are provided at the end of the paper.

2. MATHEMATICAL PRELIMINARIES

In this Section, fundamental concepts required for obtaining analytically explicit solutions of equation (1) and (2) will be discussed. We will present the key properties of the modified Riemann-Liouville differentiation characterized by Jumarie [20–22], then the generalized $\left(\frac{G'}{G}\right)$ expansion method is implemented to obtain exact analytical solutions of nonlinear FPDEs associated with the mentioned fractional derivative.

2.1. JUMARIE'S MODIFIED RIEMANN-LIOUVILLE DIFFERENTIATION AND ITS PROPERTIES

Some important characteristics of Jumarie's modified Riemann-Liouville derivative [21, 22] of order η are summarized as follows:

$$\begin{aligned}
&D_y^{\eta}(y)^{\Im} = \left(\frac{\Gamma(1+\Im)}{\Gamma(1+\Im-\eta)}\right) y^{(\Im-\eta)}, \;\; \gamma > 0, \\
&D_y^{\eta}(ah(y)+bg(y)) = aD_y^{\eta}h(y)+bD_y^{\eta}g(y), \\
&D_y^{\eta}(h(y)g(y)) = g(y)D_y^{\eta}h(y)+h(y)D_y^{\eta}g(y), \\
&D_y^{\eta}h(g(y)) = D_g^{\eta}h(g(y))(g'_y)^{\eta}, \\
&D_y^{\eta}c = 0,
\end{aligned} \tag{3}$$

where $a,\ b$, and c are arbitrary constants. The above mentioned fundamental operations and results play a magnificent role for dealing with exact explicit solutions of fractional differential equations.

2.2. ALGORITHM

This part of manuscript deals with exact solutions of equations (1) and (2) by using generalized $\left(\frac{G'}{G}\right)$ method. The foremost stages of involved technique are concisely given as follws:
Consider a nonlinear fractional partial differential equation given by

$$H(u, D_t^{\beta}u, D_x^{\alpha}u, D_y^{\eta}u, D_t^{\beta}D_x^{\alpha}u, D_y^{\eta}D_t^{\beta}u, D_x^{2\alpha}u, ...) = 0,\ 0 < \alpha, \beta, \eta < 1, \quad (4)$$

where $u = u(x, y, t)$ is an unknown function, x, y are the spatial variables, t is a temporal variable, and H represents a polynomial in $u(x, y, t)$ and its partial fractional derivatives.
Step I: The equation (4) is transformed into an ordinary differential equation (ODE)

$$Q(U, U', U'', U''', ...) = 0, \quad (5)$$

by considering the fractional complex transformation:

$$u(x,y,t) = U(\zeta),\ \ \zeta = \frac{k_1 x^{\alpha}}{\Gamma(1+\alpha)} + \frac{k_2 y^{\eta}}{\Gamma(1+\eta)} + \frac{k_3 t^{\beta}}{\Gamma(1+\beta)}, \quad (6)$$

where $k_j,\ 1 \le j \le 3$ are constants, whose values are evaluated and the prime $'$ stands for the derivative with respect to ζ.
Step II: The formal solution to ODE (5) is assumed to be expressed by a polynomial:

$$U(\zeta) = \sum_{i=-N}^{0} a_i(\phi(\zeta))^i + \sum_{i=1}^{N} b_i(\phi(\zeta))^i. \quad (7)$$

In equation (7), $a_0,\ a_i,\ b_i\ (i = 1, 2, 3, ..., N)$ are constants to be evaluated and the positive integer N is found by employing homogeneous balance rule. Also here, $\phi(\zeta) = \left(e + \frac{G'(\zeta)}{G(\zeta)}\right)$, where $G = G(\zeta)$ follows the auxiliary differential equation as:

$$PGG'' - QGG' - RG^2 - C(G')^2 = 0, \quad (8)$$

where $P,\ Q,\ R,\ C$ are real free parameters.
Step III: Substituting (7) along with (8) into (5) with possible value of N, we yield polynomials in terms of $(\phi(\zeta))^m,\ (m = 0, \pm 1, \pm 2, ...)$. Assembling all coefficients of like-power of the resulting polynomials to zero, we acquire an over-determined system of nonlinear algebraic equations for $a_0,\ a_i,\ b_i\ (i = 1, 2, ..., N), k_j (1 \le j \le 3)$, and e.
Step IV: Deciphering the algebraic equations system obtained in Step III and then inserting the values of these constants $a_0,\ a_i,\ b_i\ (1 \le i \le N), k_j (1 \le j \le 3)$, and e together with the general solutions of (8) into (7), we furnish the exact solutions of nonlinear FPDEs (4) systematically.

3. SPACE-TIME FRACTIONAL CALOGERO-DEGASPERIS EQUATION

In this Section, equation (1), which is basically a generic member of a class of partial differential equations possessing integrability properties and is associated with infinite dimensional Lie algebras [40], has been explored by means of the generalized $\left(\frac{G'}{G}\right)$ method as follows. By considering the transformation (6), equation (1) can be reduced to the following nonlinear ODE:

$$k_3U' - (2k_1^2 + k_1k_2)(U')^2 + k_1^2k_2U''' = 0,\ U' = \frac{dU}{d\zeta}. \tag{9}$$

By making the balance between the highest order derivative term and the nonlinear term for equation (9), leads to $N = 1$ and then the solution of the equation (9) takes the following ansätz:

$$U(\zeta) = a_0 + \frac{a_1}{\phi(\zeta)} + b_1\phi(\zeta), \tag{10}$$

where a_0, a_1, b_1 are constants to be evaluated. On substitution of (10) into equation (9) along with (8) and each coefficient of $(\phi(\zeta))^M$, $(M = 0, \pm1, \pm2, ...)$ is equated to zero, we obtain a system of over-determined nonlinear algebraic equations for $a_0,\ a_1,\ b_1, k_j, (j = 1,2,3)$,and e, which is further solved to derive the following non-trivial cases.

Case I: In this case, $a_0,\ k_1,\ k_2,\ C_0$,and C_1 are free parameters. The coefficients for exact solution of equation (1) are found as

$$\begin{aligned} &e = \tfrac{Q}{2(C-P)},\ a_1 = \tfrac{-3k_2k_1\left(4CR-4PR-Q^2\right)}{2P(2k_1+k_2)(C-P)},\ b_1 = \tfrac{6k_2k_1(C-P)}{P(2k_1+k_2)}, \\ &k_3 = \tfrac{4k_2{k_1}^2\left(4CR-4PR-Q^2\right)}{P^2} \end{aligned} \tag{11}$$

We, therefore, achieved the following exact solutions for equation (1), by considering the general solutions of equations (8) and (11) and then substituting in (10): for $\sigma = -4CR + 4PR + Q^2 > 0,\ Q \neq 0$, we have

$$\begin{aligned} u_{11}(x,y,t) = a_0 &+ \tfrac{-3k_1k_2\sqrt{\sigma}}{P(2k_1+k_2)} \frac{\left(C_0\cosh\left(\frac{\zeta\sqrt{\sigma}}{2P}\right) + C_1\sinh\left(\frac{\zeta\sqrt{\sigma}}{2P}\right)\right)}{\left(C_0\sinh\left(\frac{\zeta\sqrt{\sigma}}{2P}\right) + C_1\cosh\left(\frac{\zeta\sqrt{\sigma}}{2P}\right)\right)} \\ &- \tfrac{3k_1k_2\sqrt{\sigma}}{P(2k_1+k_2)} \frac{\left(C_0\sinh\left(\frac{\zeta\sqrt{\sigma}}{2P}\right) + C_1\cosh\left(\frac{\zeta\sqrt{\sigma}}{2P}\right)\right)}{\left(C_0\cosh\left(\frac{\zeta\sqrt{\sigma}}{2P}\right) + C_1\sinh\left(\frac{\zeta\sqrt{\sigma}}{2P}\right)\right)}, \end{aligned} \tag{12}$$

for $\sigma = -4CR + 4PR + Q^2 < 0,\ Q \neq 0$, we have

$$\begin{aligned} u_{12}(x,y,t) = a_0 &+ \tfrac{-3k_1k_2\sqrt{-\sigma}}{P(2k_1+k_2)} \frac{\left(C_0\cos\left(\frac{\zeta\sqrt{-\sigma}}{2P}\right) + C_1\sin\left(\frac{\zeta\sqrt{-\sigma}}{2P}\right)\right)}{\left(C_0\sin\left(\frac{\zeta\sqrt{-\sigma}}{2P}\right) - C_1\cos\left(\frac{\zeta\sqrt{-\sigma}}{2P}\right)\right)} \\ &+ \tfrac{3k_1k_2\sqrt{-\sigma}}{P(2k_1+k_2)} \frac{\left(C_0\sin\left(\frac{\zeta\sqrt{-\sigma}}{2P}\right) - C_1\cos\left(\frac{\zeta\sqrt{-\sigma}}{2P}\right)\right)}{\left(C_0\cos\left(\frac{\zeta\sqrt{-\sigma}}{2P}\right) + C_1\sin\left(\frac{\zeta\sqrt{-\sigma}}{2P}\right)\right)}, \end{aligned} \tag{13}$$

for $\sigma = -4CR+4PR+Q^2 = 0, Q \neq 0$, we have

$$u_{13}(x,y,t) = \left(\frac{2\zeta C_1 a_0 k_1+\zeta C_1 a_0 k_2+2C_0 a_0 k_1+C_0 a_0 k_2-6C_1 k_1 k_2}{(\zeta C_1+C_0)(2k_1+k_2)}\right), \tag{14}$$

for $\Delta = (P-C)R > 0,\ Q = 0$, we have

$$\begin{aligned} u_{14}(x,y,t) = a_0 + \frac{6k_1k_2}{(2k_1+k_2)P}\left(\frac{Q}{2} - \frac{\sqrt{\Delta}\left(C_0\sinh\left(\frac{\zeta\sqrt{\Delta}}{P}\right)+C_1\cosh\left(\frac{\zeta\sqrt{\Delta}}{P}\right)\right)}{\left(C_0\cosh\left(\frac{\zeta\sqrt{\Delta}}{P}\right)+C_1\sinh\left(\frac{\zeta\sqrt{\Delta}}{P}\right)\right)}\right) \\ -\frac{3k_2k_1\left(4CR-4PR-Q^2\right)}{2(2k_1+k_2)P\left(\frac{Q}{2}-\frac{\sqrt{\Delta}\left(C_0\sinh\left(\frac{\zeta\sqrt{\Delta}}{P}\right)+C_1\cosh\left(\frac{\zeta\sqrt{\Delta}}{P}\right)\right)}{\left(C_0\cosh\left(\frac{\zeta\sqrt{\Delta}}{P}\right)+C_1\sinh\left(\frac{\zeta\sqrt{\Delta}}{P}\right)\right)}\right)} \end{aligned} \tag{15}$$

for $\Delta = (P-C)R < 0,\ Q = 0$, we have

$$\begin{aligned} u_{15}(x,y,t) = a_0 + \frac{6k_1k_2}{(2k_1+k_2)P}\left(\frac{Q}{2} - \frac{\sqrt{-\Delta}\left(-C_0\sin\left(\frac{\zeta\sqrt{-\Delta}}{P}\right)+C_1\cos\left(\frac{\zeta\sqrt{-\Delta}}{P}\right)\right)}{\left(C_0\cos\left(\frac{\zeta\sqrt{-\Delta}}{P}\right)+C_1\sin\left(\frac{\zeta\sqrt{-\Delta}}{P}\right)\right)}\right) \\ -\frac{3k_2k_1\left(4CR-4PR-Q^2\right)}{2(2k_1+k_2)P\left(\frac{Q}{2}-\frac{\sqrt{-\Delta}\left(-C_0\sin\left(\frac{\zeta\sqrt{-\Delta}}{P}\right)+C_1\cos\left(\frac{\zeta\sqrt{-\Delta}}{P}\right)\right)}{\left(C_0\cos\left(\frac{\zeta\sqrt{-\Delta}}{P}\right)+C_1\sin\left(\frac{\zeta\sqrt{-\Delta}}{P}\right)\right)}\right)}, \end{aligned} \tag{16}$$

where $\zeta = \frac{k_1x^\alpha}{\Gamma(1+\alpha)} + \frac{k_2y^\eta}{\Gamma(1+\eta)} + \frac{4k_2{k_1}^2\left(4CR-4PR-Q^2\right)t^\beta}{P^2\Gamma(1+\beta)}$.

Case II: Here, $a_0,\ e,\ k_1\ k_2,\ C_0,$ and C_1 are free parameters. The coefficients for the exact solution of equation (1) are expressed as

$$b_1 = 0,\ a_1 = \frac{-6k_2k_1\left(Ce^2-Pe^2-Qe+R\right)}{P(2k_1+k_2)},\ k_3 = \frac{k_2{k_1}^2\left(4CR-4PR-Q^2\right)}{P^2} \tag{17}$$

We, therefore, obtain the following solutions for equation (1), by considering the general solutions of equations (8) and (17) and then substituting in (10): For $\sigma = -4CR+4PR+Q^2 > 0,\ Q \neq 0$, we have

$$u_{21}(x,y,t) = a_0 - \frac{6k_2k_1\left(Ce^2-Pe^2-Qe+R\right)}{P(2k_1+k_2)\left(e+\frac{Q}{2P-2C}+\frac{\sqrt{\sigma}\left(C_0\sinh\left(\frac{\zeta\sqrt{\sigma}}{2P}\right)+C_1\cosh\left(\frac{\zeta\sqrt{\sigma}}{2P}\right)\right)}{(2P-2C)\left(C_0\cosh\left(\frac{\zeta\sqrt{\sigma}}{2P}\right)+C_1\sinh\left(\frac{\zeta\sqrt{\sigma}}{2P}\right)\right)}\right)}, \tag{18}$$

for $\sigma = -4CR+4PR+Q^2 < 0,\ Q \neq 0$, we have

$$u_{22}(x,y,t) = -\frac{6k_2k_1\left(Ce^2-Pe^2-Qe+R\right)}{P(2k_1+k_2)\left(e+\frac{Q}{2P-2C}+\frac{\sqrt{-\sigma}\left(-C_0\sin\left(\frac{\zeta\sqrt{-\sigma}}{2P}\right)+C_1\cos\left(\frac{\zeta\sqrt{-\sigma}}{2P}\right)\right)}{(2P-2C)\left(C_0\cos\left(\frac{\zeta\sqrt{-\sigma}}{2P}\right)+C_1\sin\left(\frac{\zeta\sqrt{-\sigma}}{2P}\right)\right)}\right)} + a_0, \tag{19}$$

for $\sigma = -4CR + 4PR + Q^2 = 0, Q \neq 0$, we have

$$u_{23}(x,y,t) = a_0 - \frac{6k_2k_1\left(Ce^2 - Pe^2 - Qe + R\right)}{P(2k_1+k_2)}\left(e - \frac{Q}{2(C-P)} + \frac{PC_1}{(\zeta C_1 + C_0)(P-C)}\right)^{-1}, \tag{20}$$

for $\Delta = (P-C)R > 0,\ Q = 0$, we have

$$u_{24}(x,y,t) = a_0 - \frac{6k_2k_1\left(Ce^2 - Pe^2 - Qe + R\right)}{P(2k_1+k_2)\left(e + \frac{\sqrt{\Delta}\left(C_0\sinh\left(\frac{\zeta\sqrt{\Delta}}{P}\right) + C_1\cosh\left(\frac{\zeta\sqrt{\Delta}}{P}\right)\right)}{(P-C)\left(C_0\cosh\left(\frac{\zeta\sqrt{\Delta}}{P}\right) + C_1\sinh\left(\frac{\zeta\sqrt{\Delta}}{P}\right)\right)}\right)}, \tag{21}$$

for $\Delta = (P-C)R < 0,\ Q = 0$, we have

$$u_{25}(x,y,t) = a_0 - \frac{6k_2k_1\left(Ce^2 - Pe^2 - Qe + R\right)}{P(2k_1+k_2)\left(e + \frac{\sqrt{-\Delta}\left(-C_0\sin\left(\frac{\zeta\sqrt{-\Delta}}{P}\right) + C_1\cos\left(\frac{\zeta\sqrt{-\Delta}}{P}\right)\right)}{(P-C)\left(C_0\cos\left(\frac{\zeta\sqrt{-\Delta}}{P}\right) + C_1\sin\left(\frac{\zeta\sqrt{-\Delta}}{P}\right)\right)}\right)}, \tag{22}$$

where $\zeta = \frac{k_1x^\alpha}{\Gamma(1+\alpha)} + \frac{k_2y^\eta}{\Gamma(1+\eta)} + \frac{k_2{k_1}^2\left(4CR - 4PR - Q^2\right)t^\beta}{P^2\Gamma(1+\beta)}$.

Case III: In this case, $a_0,\ e,\ k_1,\ k_2,\ C_0,$ and C_1 are free parameters. The coefficients for the solution of equation (1) are found as

$$b_1 = \frac{6k_2k_1(C-P)}{P(2k_1+k_2)},\ a_1 = 0,\ k_3 = \frac{k_2{k_1}^2\left(4CR - 4PR - Q^2\right)}{P^2} \tag{23}$$

We, therefore, obtain the following solutions for equation (1), by considering the general solutions of equations (8) and (23) and then substituting in (10): For $\sigma = -4CR + 4PR + Q^2 > 0,\ Q \neq 0$, we have

$$u_{31}(x,y,t) = a_0 + \frac{6k_2k_1(C-P)}{P(2k_1+k_2)}\left(e + \frac{Q}{2P-2C} + \frac{\sqrt{\sigma}}{2P-2C}\frac{\left(C_0\sinh\left(\frac{\zeta\sqrt{\sigma}}{2P}\right) + C_1\cosh\left(\frac{\zeta\sqrt{\sigma}}{2P}\right)\right)}{\left(C_0\cosh\left(\frac{\zeta\sqrt{\sigma}}{2P}\right) + C_1\sinh\left(\frac{\zeta\sqrt{\sigma}}{2P}\right)\right)}\right), \tag{24}$$

for $\sigma = -4CR + 4PR + Q^2 < 0,\ Q \neq 0$, we have

$$u_{32}(x,y,t) = a_0 + \frac{6k_2k_1(C-P)}{P(2k_1+k_2)}\left(e + \frac{Q}{2P-2C} + \frac{\sqrt{-\sigma}}{2P-2C}\frac{\left(-C_0\sin\left(\frac{\zeta\sqrt{-\sigma}}{2P}\right) + C_1\cos\left(\frac{\zeta\sqrt{-\sigma}}{2P}\right)\right)}{\left(C_0\cos\left(\frac{\zeta\sqrt{-\sigma}}{2P}\right) + C_1\sin\left(\frac{\zeta\sqrt{-\sigma}}{2P}\right)\right)}\right), \tag{25}$$

for $\sigma = -4CR + 4PR + Q^2 = 0,\ Q \neq 0$, we have

$$u_{33}(x,y,t) = a_0 + \frac{6k_2k_1(C-P)}{P(2k_1+k_2)}\left(e - \frac{Q}{2(C-P)} + \frac{PC_1}{(\zeta C_1 + C_0)(P-C)}\right), \tag{26}$$

for $\Delta = (P-C)R > 0,\ Q = 0$, we have

$$u_{34}(x,y,t) = a_0 + \frac{6k_2k_1(C-P)}{P(2k_1+k_2)}\left(e + \frac{\sqrt{\Delta}}{P-C}\frac{\left(C_0\sinh\left(\frac{\zeta\sqrt{\Delta}}{P}\right) + C_1\cosh\left(\frac{\zeta\sqrt{\Delta}}{P}\right)\right)}{\left(C_0\cosh\left(\frac{\zeta\sqrt{\Delta}}{P}\right) + C_1\sinh\left(\frac{\zeta\sqrt{\Delta}}{P}\right)\right)}\right), \tag{27}$$

for $\Delta = (P - C)R < 0,\ Q = 0$, we have

$$u_{35}(x,y,t) = a_0 + \frac{6k_2k_1(C-P)}{P(2k_1+k_2)}\left(e + \frac{\sqrt{-\Delta}}{P-C}\,\frac{\left(-C_0\sin\left(\frac{\zeta\sqrt{-\Delta}}{P}\right)+C_1\cos\left(\frac{\zeta\sqrt{-\Delta}}{P}\right)\right)}{\left(C_0\cos\left(\frac{\zeta\sqrt{-\Delta}}{P}\right)+C_1\sin\left(\frac{\zeta\sqrt{-\Delta}}{P}\right)\right)}\right), \tag{28}$$

where $\zeta = \frac{k_1x^{\alpha}}{\Gamma(1+\alpha)} + \frac{k_2y^{\eta}}{\Gamma(1+\eta)} + \frac{k_2{k_1}^2\left(4CR-4PR-Q^2\right)t^{\beta}}{P^2\Gamma(1+\beta)}$.

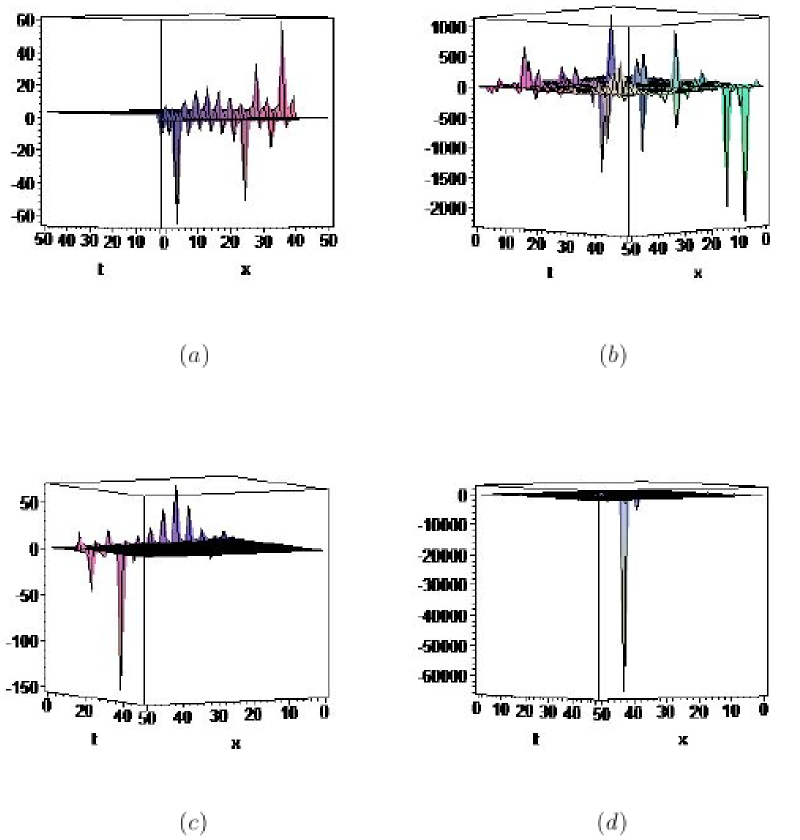

Fig. 1 – Graphical portray for solutions of equation (1): (a) singular multiple soliton solution for $u_{31}(x,y,t)$ with $C=1$, $P=2$, $Q=1$, $R=1$, $\alpha=0.8$, $\beta=0.9$, $\eta=0.5$, $y=2$, $e=1$, $\epsilon=2$, $B_0=1$, $B_1=2$, $a_0=4$, $k_1=1$, $k_2=2$, (b) multiple singular soliton solution $u_{32}(x,y,t)$ with $C=2$, $P=1$, $Q=1$, $R=1$, $\alpha=0.8$, $\beta=0.9$, $\eta=0.5$, $y=2$, $e=1$, $\epsilon=2$, $B_0=1$, $B_1=2$, $a_0=4$, $k_1=1$, $k_2=2$, (c) multiple singular soliton solution for $u_{34}(x,y,t)$ with $C=1$, $P=2$, $Q=1$, $R=1$, $\alpha=0.8$, $\beta=0.9$, $\eta=0.5$, $y=2$, $e=1$, $\epsilon=2$, $B_0=1$, $B_1=2$, $a_0=4$, $k_1=1$, $k_2=2$, (d) 1-soliton solution for $u_{35}(x,y,t)$ with $C=2$, $P=1$, $Q=1$, $R=1$, $\alpha=0.8$, $\beta=0.9$, $\eta=0.5$, $y=2$, $e=1$, $\epsilon=2$, $B_0=1$, $B_1=2$, $a_0=4$, $k_1=1$, $k_2=2$.

Remarks 1: It is noteworthy to indicate that the space-time fractional Calogero-Degasperis (CD) equation is considered very foremost time. However, the obtained solutions, with more free parameters, are newer relative results compared to the previously existing ones in literature yet. The free parameters of the solutions are carrying rich mathematical structures, which play a vital role for unfolding certain physical phenomena such as wave propagation and many more. Also, by taking $\alpha = \eta = \beta = 1$, some solutions for CD equation [37] can be recovered uniformly from solutions obtained in this Section, which simulate diverse physical situations. The modernity in obtaining of such exact solutions might pave a magnificent way out among community of researchers for further developments.
Remarks 2: The exact solutions of space-time fractional CD equation are very effective for characterizing multiple types of soliton solutions in mathematical physics as shown in Fig. 1.

4. SPACE-TIME FRACTIONAL SHARMA-TASSO-OLVER EQUATION

Now we consider general space-time fractional Sharma-Tasso-Olver (STO) equation [38, 39]:

$$D_t^{\beta} u + 3\epsilon (D_x^{\alpha} u)^2 + 3\epsilon u^2 D_x^{\alpha} u + 3\epsilon u D_x^{2\alpha} u + \epsilon D_x^{3\alpha} u = 0, \tag{29}$$

where $0 < \alpha, \beta \leq 1$ and $\epsilon \neq 0$ is a constant and the transformation

$$u(x,t) = u = U(\zeta), \ \ \zeta = \frac{k_1 x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{k_3 t^{\alpha_2}}{\Gamma(1+\alpha_2)}. \tag{30}$$

Then, equation (29) is transformed to an ODE equation as follows:

$$-k_3 U + 3\epsilon {k_1}^2 U'U + \epsilon k_1 U^3 + \epsilon {k_1}^3 U'' = 0, \text{ where } U' = \frac{dU}{d\zeta}. \tag{31}$$

Making balance between U'' and $U'U$, we get $N = 1$, therefore the formal solution of equation (31) can be identified as

$$U(\zeta) = a_0 + \frac{a_1}{\phi(\zeta)} + b_1 \phi(\zeta), \tag{32}$$

while a_0, a_1, b_1 are unknown constants. We insert (32) and (8) into equation (31) and then all same order terms of $\phi(\zeta)$ are assembled together. The left-hand side of equation (31) is reframed into a polynomial in $\phi(\zeta)^m, \ (m = 0, \pm 1, \pm 2, ...)$ and then each coefficient of this polynomial is equated to zero. As a result, a set of over-determined algebraic equations are furnished for $a_0, a_1, b_1, k_j (j = 1,2,3),$ and e. Solving these equations, different types of solutions are derived, which are enumerated in non-trivial cases subsequently.
Case I: The following is a set of values for the parameters of the solution of equation

(29)

$$a_0 = -\frac{k_1(2Ce-2Pe-Q)}{P},\ a_1 = \frac{2k_1\left(Ce^2-Pe^2-Qe+R\right)}{P},\ b_1 = 0,\\ k_3 = \frac{\left(4CR-4PR-Q^2\right)\epsilon k_1{}^3}{P^2}, \tag{33}$$

where $k_1,\ C_0,\ C_1$, and e are free parameters.

Considering the general solutions for equations (8) and (33) into (32), the solutions for equation (29) are attained as:
for $\sigma = -4CR+4PR+Q^2 > 0,\ Q \neq 0$, we have

$$u_{11}(x,t) = -\frac{k_1(2Ce-2Pe-Q)}{P} + \frac{2k_1\left(Ce^2-Pe^2-Qe+R\right)}{P\left(e+\frac{Q}{2P-2C}+\frac{\sqrt{\sigma}}{2P-2C}\frac{\left(C_0\sinh\left(\frac{\zeta\sqrt{\sigma}}{2P}\right)+C_1\cosh\left(\frac{\zeta\sqrt{\sigma}}{2P}\right)\right)}{\left(C_0\cosh\left(\frac{\zeta\sqrt{\sigma}}{2P}\right)+C_1\sinh\left(\frac{\zeta\sqrt{\sigma}}{2P}\right)\right)}\right)}, \tag{34}$$

for $\sigma = -4CR+4PR+Q^2 < 0,\ Q \neq 0$, we have

$$u_{12}(x,t) = -\frac{k_1(2Ce-2Pe-Q)}{P} + \frac{2k_1\left(Ce^2-Pe^2-Qe+R\right)}{P\left(e+\frac{Q}{2P-2C}+\frac{\sqrt{-\sigma}}{2P-2C}\frac{\left(-C_0\sin\left(\frac{\zeta\sqrt{-\sigma}}{2P}\right)+C_1\cos\left(\frac{\zeta\sqrt{-\sigma}}{2P}\right)\right)}{\left(C_0\cos\left(\frac{\zeta\sqrt{-\sigma}}{2P}\right)+C_1\sin\left(\frac{\zeta\sqrt{-\sigma}}{2P}\right)\right)}\right)}, \tag{35}$$

for $\sigma = -4CR+4PR+Q^2 = 0,\ Q \neq 0$, we have

$$u_{13}(x,t) = \frac{2k_1\left(Ce^2-Pe^2-Qe+R\right)}{P}\left(e-\frac{Q}{2(C-P)}+\frac{PC_1}{(\zeta C_1+C_0)(P-C)}\right)^{-1}\\ -\frac{k_1(2Ce-2Pe-Q)}{P}, \tag{36}$$

for $\Delta = (P-C)R > 0,\ Q = 0$, we have

$$u_{14}(x,t) = -\frac{k_1(2Ce-2Pe-Q)}{P} + \frac{2k_1\left(Ce^2-Pe^2-Qe+R\right)}{P\left(e+\frac{\sqrt{\Delta}}{P-C}\frac{\left(C_0\sinh\left(\frac{\zeta\sqrt{\Delta}}{P}\right)+C_1\cosh\left(\frac{\zeta\sqrt{\Delta}}{P}\right)\right)}{\left(C_0\cosh\left(\frac{\zeta\sqrt{\Delta}}{P}\right)+C_1\sinh\left(\frac{\zeta\sqrt{\Delta}}{P}\right)\right)}\right)}, \tag{37}$$

for $\Delta = (P-C)R < 0,\ Q = 0$, we have

$$u_{15}(x,t) = -\frac{k_1(2Ce-2Pe-Q)}{P} + \frac{2k_1\left(Ce^2-Pe^2-Qe+R\right)}{P\left(e+\frac{\sqrt{-\Delta}}{P-C}\frac{\left(-C_0\sin\left(\frac{\zeta\sqrt{-\Delta}}{P}\right)+C_1\cos\left(\frac{\zeta\sqrt{-\Delta}}{P}\right)\right)}{\left(C_0\cos\left(\frac{\zeta\sqrt{-\Delta}}{P}\right)+C_1\sin\left(\frac{\zeta\sqrt{-\Delta}}{P}\right)\right)}\right)}, \tag{38}$$

where $\zeta = \frac{k_1 x^\alpha}{\Gamma(1+\alpha)} + \frac{\left(4CR-4PR-Q^2\right)\epsilon k_1{}^3 t^\beta}{P^2\Gamma(1+\beta)}$.

Case II: The following is a set of coefficients for the solution of equation (29)

$$a_0 = 0,\ a_1 = \frac{k_1\left(Ce^2-Pe^2-Qe+R\right)}{P},\ b_1 = -\frac{k_1(C-P)}{P},\ k_3 = \frac{\left(4CR-4PR-Q^2\right)\epsilon k_1{}^3}{P^2}, \tag{39}$$

obtained upon using $k_1,\ C_0,\ C_1$, and e as free parameters.

Substituting the general solutions of equation (8) and (39) into (32), we obtain the following solutions for equation (29):

for $\sigma=-4CR+4PR+Q^2>0,\ Q\neq0$, we have

$$u_{21}(x,t)=-\frac{k_1(C-P)}{P}\left(e+\frac{Q}{2P-2C}+\frac{\sqrt{\sigma}}{2P-2C}\frac{\left(C_0\sinh\left(\frac{\zeta\sqrt{\sigma}}{2P}\right)+C_1\cosh\left(\frac{\zeta\sqrt{\sigma}}{2P}\right)\right)}{\left(C_0\cosh\left(\frac{\zeta\sqrt{\sigma}}{2P}\right)+C_1\sinh\left(\frac{\zeta\sqrt{\sigma}}{2P}\right)\right)}\right)$$
$$+\frac{k_1\left(Ce^2-Pe^2-Qe+R\right)}{P}\left(e+\frac{Q}{2P-2C}+\frac{\sqrt{\sigma}}{2P-2C}\frac{\left(C_0\sinh\left(\frac{\zeta\sqrt{\sigma}}{2P}\right)+C_1\cosh\left(\frac{\zeta\sqrt{\sigma}}{2P}\right)\right)}{\left(C_0\cosh\left(\frac{\zeta\sqrt{\sigma}}{2P}\right)+C_1\sinh\left(\frac{\zeta\sqrt{\sigma}}{2P}\right)\right)}\right)^{-1}, \tag{40}$$

for $\sigma=-4CR+4PR+Q^2<0,\ Q\neq0$, we have

$$u_{22}(x,t)=-\frac{k_1(C-P)}{P}\left(e+\frac{Q}{2P-2C}+\frac{\sqrt{-\sigma}}{2P-2C}\frac{\left(-C_0\sin\left(\frac{\zeta\sqrt{-\sigma}}{2P}\right)+C_1\cos\left(\frac{\zeta\sqrt{-\sigma}}{2P}\right)\right)}{\left(C_0\cos\left(\frac{\zeta\sqrt{-\sigma}}{2P}\right)+C_1\sin\left(\frac{\zeta\sqrt{-\sigma}}{2P}\right)\right)}\right)$$
$$+\frac{k_1\left(Ce^2-Pe^2-Qe+R\right)}{P}\left(e+\frac{Q}{2P-2C}+\frac{\sqrt{-\sigma}}{2P-2C}\frac{\left(-C_0\sin\left(\frac{\zeta\sqrt{-\sigma}}{2P}\right)+C_1\cos\left(\frac{\zeta\sqrt{-\sigma}}{2P}\right)\right)}{\left(C_0\cos\left(\frac{\zeta\sqrt{-\sigma}}{2P}\right)+C_1\sin\left(\frac{\zeta\sqrt{-\sigma}}{2P}\right)\right)}\right)^{-1}, \tag{41}$$

for $\sigma=-4CR+4PR+Q^2=0,\ Q\neq0$, we have

$$u_{23}(x,t)=-\frac{k_1(C-P)}{P}\left(e-\frac{Q}{2(C-P)}+\frac{PC_1}{(\zeta C_1+C_0)(P-C)}\right)+\frac{k_1\left(Ce^2-Pe^2-Qe+R\right)}{P\left(e-\frac{Q}{2(C-P)}+\frac{PC_1}{(\zeta C_1+C_0)(P-C)}\right)}, \tag{42}$$

for $\Delta=(P-C)R>0,\ Q=0$, we have

$$u_{24}(x,t)=-\frac{k_1(C-P)}{P}\left(e+\frac{\sqrt{\Delta}}{P-C}\frac{\left(C_0\sinh\left(\frac{\zeta\sqrt{\Delta}}{P}\right)+C_1\cosh\left(\frac{\zeta\sqrt{\Delta}}{P}\right)\right)}{\left(C_0\cosh\left(\frac{\zeta\sqrt{\Delta}}{P}\right)+C_1\sinh\left(\frac{\zeta\sqrt{\Delta}}{P}\right)\right)}\right)+\frac{k_1\left(Ce^2-Pe^2-Qe+R\right)}{P\left(e+\frac{\sqrt{\Delta}}{P-C}\frac{\left(C_0\sinh\left(\frac{\zeta\sqrt{\Delta}}{P}\right)+C_1\cosh\left(\frac{\zeta\sqrt{\Delta}}{P}\right)\right)}{\left(C_0\cosh\left(\frac{\zeta\sqrt{\Delta}}{P}\right)+C_1\sinh\left(\frac{\zeta\sqrt{\Delta}}{P}\right)\right)}\right)}, \tag{43}$$

for $\Delta=(P-C)R<0,\ Q=0$, we have

$$u_{25}(x,t)=-\frac{k_1(C-P)}{P}\left(e+\frac{\sqrt{-\Delta}}{P-C}\frac{\left(-C_0\sin\left(\frac{\zeta\sqrt{-\Delta}}{P}\right)+C_1\cos\left(\frac{\zeta\sqrt{-\Delta}}{P}\right)\right)}{\left(C_0\cos\left(\frac{\zeta\sqrt{-\Delta}}{P}\right)+C_1\sin\left(\frac{\zeta\sqrt{-\Delta}}{P}\right)\right)}\right)+\frac{k_1\left(Ce^2-Pe^2-Qe+R\right)}{P\left(e+\frac{\sqrt{-\Delta}}{P-C}\frac{\left(-C_0\sin\left(\frac{\zeta\sqrt{-\Delta}}{P}\right)+C_1\cos\left(\frac{\zeta\sqrt{-\Delta}}{P}\right)\right)}{\left(C_0\cos\left(\frac{\zeta\sqrt{-\Delta}}{P}\right)+C_1\sin\left(\frac{\zeta\sqrt{-\Delta}}{P}\right)\right)}\right)}, \tag{44}$$

where $\zeta=\frac{k_1x^\alpha}{\Gamma(1+\alpha)}+\frac{(4CR-4PR-Q^2)\epsilon k_1{}^3t^\beta}{P^2\Gamma(1+\beta)}$.

Case III: The following is another set of values of the coefficients derived for solution of equation (29) with $k_1,\ C_0,\ C_1$ and e as free parameters:

$$a_0=\frac{k_1(2Ce-2Pe-Q)}{2P},\ a_1=0,\ b_1=-\frac{k_1(C-P)}{P},k_3=\frac{\left(4CR-4PR-Q^2\right)\epsilon k_1{}^3}{4P^2}. \tag{45}$$

We found the following solutions for equation (29), by using the general solutions of equations (8) and (45) in (29):

for $\sigma=-4CR+4PR+Q^2>0,\ Q\neq 0$, we have

$$u_{31}(x,t)=-\frac{k_1(C-P)}{P}\left(\frac{\sqrt{\sigma}}{2P-2C}\frac{\left(C_0\sinh\left(\frac{\zeta\sqrt{\sigma}}{2P}\right)+C_1\cosh\left(\frac{\zeta\sqrt{\sigma}}{2P}\right)\right)}{\left(C_0\cosh\left(\frac{\zeta\sqrt{\sigma}}{2P}\right)+C_1\sinh\left(\frac{\zeta\sqrt{\sigma}}{2P}\right)\right)}+e+\frac{Q}{2P-2C}\right)+\frac{k_1(2Ce-2Pe-Q)}{2P}, \tag{46}$$

for $\sigma=-4CR+4PR+Q^2<0,\ Q\neq 0$, we have

$$u_{32}(x,t)=-\frac{k_1(C-P)}{P}\left(e+\frac{Q}{2P-2C}+\frac{\sqrt{-\sigma}}{2P-2C}\frac{\left(-C_0\sin\left(\frac{\zeta\sqrt{-\sigma}}{2P}\right)+C_1\cos\left(\frac{\zeta\sqrt{-\sigma}}{2P}\right)\right)}{\left(C_0\cos\left(\frac{\zeta\sqrt{-\sigma}}{2P}\right)+C_1\sin\left(\frac{\zeta\sqrt{-\sigma}}{2P}\right)\right)}\right)+\frac{k_1(2Ce-2Pe-Q)}{2P}, \tag{47}$$

for $\sigma=-4CR+4PR+Q^2=0,\ Q\neq 0$, we have

$$u_{33}(x,y,t)=\left(\frac{C_1k_1}{\zeta C_1+C_0}\right), \tag{48}$$

for $\Delta=(P-C)R>0,\ Q=0$, we have

$$u_{34}(x,t)=-\frac{k_1(C-P)}{P}\left(e+\frac{\sqrt{\Delta}}{P-C}\frac{\left(C_0\sinh\left(\frac{\zeta\sqrt{\Delta}}{P}\right)+C_1\cosh\left(\frac{\zeta\sqrt{\Delta}}{P}\right)\right)}{\left(C_0\cosh\left(\frac{\zeta\sqrt{\Delta}}{P}\right)+C_1\sinh\left(\frac{\zeta\sqrt{\Delta}}{P}\right)\right)}\right)+\frac{k_1(2Ce-2Pe-Q)}{2P}, \tag{49}$$

for $\Delta=(P-C)R<0,\ Q=0$, we have

$$u_{35}(x,t)=-\frac{k_1(C-P)}{P}\left(e+\frac{\sqrt{-\Delta}}{P-C}\frac{\left(-C_0\sin\left(\frac{\zeta\sqrt{-\Delta}}{P}\right)+C_1\cos\left(\frac{\zeta\sqrt{-\Delta}}{P}\right)\right)}{\left(C_0\cos\left(\frac{\zeta\sqrt{-\Delta}}{P}\right)+C_1\sin\left(\frac{\zeta\sqrt{-\Delta}}{P}\right)\right)}\right)+\frac{k_1(2Ce-2Pe-Q)}{2P}, \tag{50}$$

where $\zeta=\frac{k_1x^{\alpha}}{\Gamma(1+\alpha)}+\frac{\left(4CR-4PR-Q^2\right)\epsilon {k_1}^3t^{\beta}}{P^2\Gamma(1+\beta)}$.

Case IV: Here, C_0, C_1, and k_1 are free parameters. The involved coefficients for the solution of equation (1) are expressed as

$$e=\frac{Q}{2(C-P)},\ a_0=\pm\frac{\sqrt{-4CR+4PR+Q^2}k_1}{P},\ a_1=\frac{k_1\left(4CR-4PR-Q^2\right)}{4P(C-P)},$$
$$b_1=-\frac{k_1(C-P)}{P},\ k_3=\frac{4\left(4CR-4PR-Q^2\right)\epsilon {k_1}^3}{P^2}. \tag{51}$$

We, therefore, attain the subsequent solutions of equation (1), by considering the general solutions of (8) and (17) and then substituting them in (10):
for $\sigma=-4CR+4PR+Q^2>0,\ Q\neq 0$, we have

$$u_{41}(x,t)=\frac{\sqrt{\sigma}k_1}{P}+\frac{\sqrt{\sigma}k_1}{2P}\frac{\left(C_0\sinh\left(\frac{\zeta\sqrt{\sigma}}{2P}\right)+C_1\cosh\left(\frac{\zeta\sqrt{\sigma}}{2P}\right)\right)}{\left(C_0\cosh\left(\frac{zeta\sqrt{\sigma}}{2P}\right)+C_1\sinh\left(\frac{\zeta\sqrt{\sigma}}{2P}\right)\right)}-\frac{k_1\sqrt{\sigma}}{2P}\frac{\left(C_0\cosh\left(\frac{\zeta\sqrt{\sigma}}{2P}\right)+C_1\sinh\left(\frac{\zeta\sqrt{\sigma}}{2P}\right)\right)}{\left(C_0\sinh\left(\frac{\zeta\sqrt{\sigma}}{2P}\right)+C_1\cosh\left(\frac{\zeta\sqrt{\sigma}}{2P}\right)\right)}, \tag{52}$$

for $\sigma=-4CR+4PR+Q^2<0,\ Q\neq 0$, we have

$$\begin{aligned} u_{42}(x,t)=&\frac{\sqrt{-\sigma}k_1}{P}+\frac{\sqrt{-\sigma}k_1}{2P}\frac{\left(C_0\cos\left(\frac{\zeta\sqrt{-\sigma}}{2P}\right)+C_1\sin\left(\frac{\zeta\sqrt{-\sigma}}{2P}\right)\right)}{\left(C_0\sin\left(\frac{\zeta\sqrt{-\sigma}}{2P}\right)-C_1\cos\left(\frac{\zeta\sqrt{-\sigma}}{2P}\right)\right)}\\ &-\frac{\sqrt{-\sigma}k_1}{2P}\frac{\left(C_0\sin\left(\frac{\zeta\sqrt{-\sigma}}{2P}\right)-C_1\cos\left(\frac{\zeta\sqrt{-\sigma}}{2P}\right)\right)}{\left(C_0\cos\left(\frac{\zeta\sqrt{-\sigma}}{2P}\right)+C_1\sin\left(\frac{\zeta\sqrt{-\sigma}}{2P}\right)\right)}, \end{aligned}\tag{53}$$

for $\sigma=-4CR+4PR+Q^2=0,\ Q\neq 0$, we have

$$u_{43}(x,t)=\left(\frac{C_1k_1}{\zeta C_1+C_0}\right),\tag{54}$$

for $\Delta=(P-C)R>0,\ Q=0$, we got

$$\begin{aligned} u_{44}(x,t)=&-\frac{k_1}{P}\left(\frac{Q}{2}-\sqrt{\Delta}\frac{\left(C_0\sinh\left(\frac{\zeta\sqrt{\Delta}}{P}\right)+C_1\cosh\left(\frac{\zeta\sqrt{\Delta}}{P}\right)\right)}{\left(C_0\cosh\left(\frac{\zeta\sqrt{\Delta}}{P}\right)+C_1\sinh\left(\frac{\zeta\sqrt{\Delta}}{P}\right)\right)}\right)+\frac{\sqrt{-4CR+4PR+Q^2}k_1}{P}\\ &+\frac{k_1\left(4CR-4PR-Q^2\right)}{4P\left(\frac{Q}{2}-\sqrt{\Delta}\frac{\left(C_0\sinh\left(\frac{\zeta\sqrt{\Delta}}{P}\right)+C_1\cosh\left(\frac{\zeta\sqrt{\Delta}}{P}\right)\right)}{\left(C_0\cosh\left(\frac{\zeta\sqrt{\Delta}}{P}\right)+C_1\sinh\left(\frac{\zeta\sqrt{\Delta}}{P}\right)\right)}\right)}, \end{aligned}\tag{55}$$

for $\Delta=(P-C)R<0,\ Q=0$, we have

$$\begin{aligned} u_{45}(x,t)=&-\frac{k_1}{P}\left(\frac{Q}{2}-\frac{\sqrt{-\Delta}\left(-C_0\sin\left(\frac{\zeta\sqrt{-\Delta}}{P}\right)+C_1\cos\left(\frac{\zeta\sqrt{-\Delta}}{P}\right)\right)}{\left(C_0\cos\left(\frac{\zeta\sqrt{-\Delta}}{P}\right)+C_1\sin\left(\frac{\zeta\sqrt{-\Delta}}{P}\right)\right)}\right)+\frac{\sqrt{-4CR+4PR+Q^2}k_1}{P}\\ &+\frac{k_1\left(4CR-4PR-Q^2\right)}{4P\left(\frac{Q}{2}-\frac{\sqrt{-\Delta}\left(-C_0\sin\left(\frac{\zeta\sqrt{-\Delta}}{P}\right)+C_1\cos\left(\frac{\zeta\sqrt{-\Delta}}{P}\right)\right)}{\left(C_0\cos\left(\frac{\zeta\sqrt{-\Delta}}{P}\right)+C_1\sin\left(\frac{\zeta\sqrt{-\Delta}}{P}\right)\right)}\right)}, \end{aligned}\tag{56}$$

where $\zeta=\frac{k_1x^\alpha}{\Gamma(1+\alpha)}+\frac{\left(4CR-4PR-Q^2\right)k_1t^\beta}{4P(C-P)\Gamma(1+\beta)}$.

Remark 3: It is important to point out that some of the acquired solutions have outstanding resemblance with the previously established solutions [38, 39], by taking suitable values of arbitrary constants. In particular, the solutions, mentioned in [38], are similar to the hyperbolic, trigonometric, and rational function solutions obtained in Sec. 3 and Sec. 4, by setting appropriate values to the free parameters. Thus, we have found new solutions, which are more general in the sense of having much more free parameters than the previously obtained solutions. Besides these solutions, we have obtained additional exact solutions, which were not yet available before, to the best of our knowledge.

Remark 4: The graphical description of some of exact solutions of the considered equation has been given by assuming some appropriate values to the involved

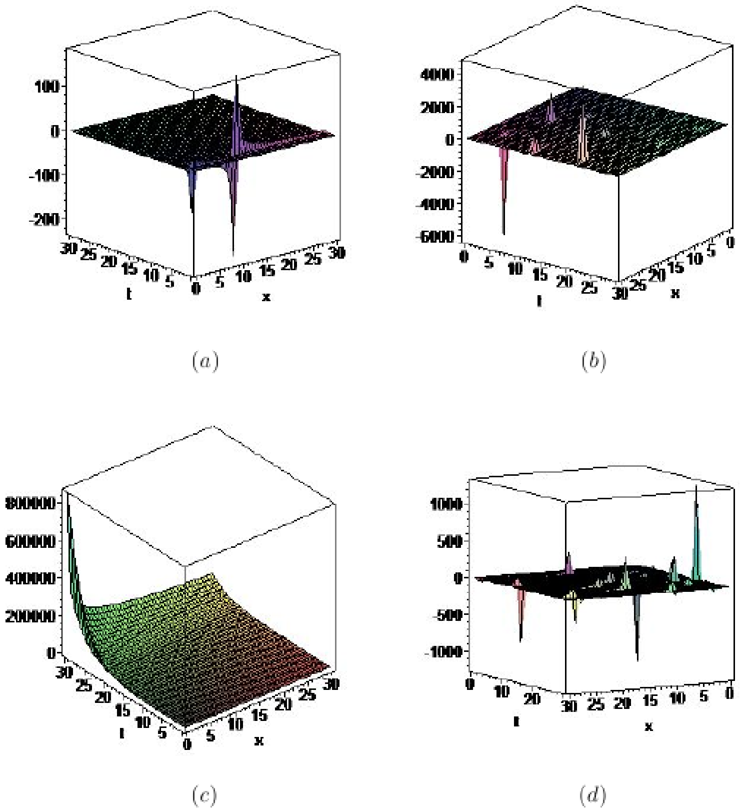

Fig. 2 – Solutions of equation (2) are graphically depicted as: (a) singular double soliton solution for $u_{41}(x,t)$ with $B_0 = 1$, $B_1 = 2$, $C = 1$, $P = 2$, $Q = 1$, $R = 1$, $\alpha = 0.2$, $\beta = 0.5$, $e = 1$, $\epsilon = 2$, $a_0 = 4$, $k_1 = 1$, $k_2 = 2$, (b) multiple singular soliton solution for $u_{42}(x,t)$ with $C = 2$, $P = 1$, $Q = 1$, $R = 1$, $\alpha = 0.2$, $\beta = 0.5$, $e = 1$, $\epsilon = 2$, $B_0 = 1$, $B_1 = 2$, $a_0 = 4$, $k_1 = 1$, $k_2 = 2$, (c) 1-soliton solution for $u_{44}(x,t)$ with $C = 1$, $P = 2$, $Q = 1$, $R = 1$, $\alpha = 0.2$, $\beta = 0.5$, $e = 1$, $\epsilon = 2$, $B_0 = 1$, $B_1 = 2$, $a_0 = 4$, $k_1 = 1$, $k_2 = 2$, (d) multiple singular soliton solution for $u_{45}(x,t)$ with $B_0 = 1$, $B_1 = 2$, $C = 2$, $P = 1$, $Q = 1$, $R = 1$, $\alpha = 0.2$, $\beta = 0.5$, $e = 1$, $\epsilon = 2$, $a_0 = 4$, $k_1 = 1$, $k_2 = 2$.

free parameters, as shown in Fig. 2. As an important output, the 1-soliton solution, the singular multiple-soliton, and the singular double soliton have been obtained, which are relevant to analyze the fission and fusion phenomena that occur in a series of physical settings.

5. CONCLUDING REMARKS

In this study, exact analytic solutions of space-time fractional order Calogero-Degasperis equation and space-time fractional order Sharma-Tasso-Olver equation have been obtained by utilizing the generalized $\left(\frac{G'}{G}\right)$ expansion method, which is one of the most efficient and important methods from both theoretical and applied point of views. Novel results have been derived by converting the considered fractional differential equations into another nonlinear ordinary differential equation by a fractional complex transformation along with the modified Riemann-Liouville derivative. Consequently, different types of exact solutions, including hyperbolic function, trigonometric function, and rational function have been obtained.

Furthermore, some of these exact solutions provided by this method have been graphically characterized into a variety of distinct physical structures such as 1-soliton solution, multiple singular soliton, and double soliton solutions. It is remarkable to mention that the involved free parameters in the produced exact solutions of both equations could be used to explain diverse physical features of the obtained solitonic waveforms.

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